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W. T. MARTIN G. B. PRICE

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INTEGRALS AND SUMMABLE TRIGONOMETRIC SERIES

R. D. JAMES

I should like to begin by expressing my thanks to Professor Zygmund, who not only interested me in the subject of trigonometric series but also suggested the possibility of defining generalized integrals which would have applications to summable trigonometric series, and then supplied helpful hints from time to time.

1. **Introduction.** One of the problems in the theory of trigonometric series

$$(1.1)\qquad \frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}a_n(x)$$

is that of suitably defining a trigonometric integral with the property that, if the series (1.1) converges everywhere to a function $f(x)$, then $f(x)$ is necessarily integrable and the coefficients, a_n and b_n, given in the usual Fourier form. It is well known that a series may converge everywhere to a function which is not Lebesgue summable nor even Denjoy integrable (completely totalisable, [3]). A simple example is that given by Fatou in which $a_n=0$, $n\geqq 0$, and $b_n=1/\log(n+1)$, $n\geqq 1$.

More generally, if $\{b_n\}$ is any monotone decreasing sequence with limit zero such that the series $\sum b_n/n$ diverges, then the sum of the everywhere convergent series $\sum b_n\sin nx$ is not completely totalisable [5, pp. 42–44].

The problem has been solved by Denjoy [4; 5], Verblunsky [14], Marcinkiewicz and Zygmund [10], Burkill [1; 2], and James [8]. In Verblunsky's paper and Burkill's first paper, additional hypotheses other than the convergence of (1.1) are made, and in all the papers a change in the form of the Fourier formulas is required. The solutions are described, mainly in the order in which they were published, in §§2–7 below.

An extension of this problem is to consider series which are not necessarily convergent, but are summable (C, k), $k\geqq 1$ (Cesàro means of order k, [6, Chaps. V and VII]). Here the situation becomes more complicated since even though a series is summable to a Lebesgue

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summable function, the coefficients may not be given by the usual formulas. In particular, there are series such as $\sum n \sin nx$ which is summable $(C, 2)$ to zero for all x, whereas the coefficients are not zero. It is difficult to conceive of a process of integration which would integrate an identically zero function and yield anything but zero. Wolf [15] has solved the problem for series summable (C, k) to a Denjoy integrable function and his results show that there may be a reducible set of exceptional values of x and that these exceptional values enter into the formulas for the coefficients. Since Wolf was not concerned with the question of a trigonometric integral, his work will not be discussed in this paper.

There is another difference between convergent and summable trigonometric series which makes the latter more difficult to deal with. The formal product ([6, Chap. X], [16], [17, Chap. XI]) of the series (1.1) and, for example, $\sin mx$ is the trigonometric series obtained by muliplying (1.1) by $\sin mx$ and replacing the products $\cos nx \sin mx$, $\sin nx \sin mx$ by sums of cosines and sines. It is easy to see that, if the series (1.1) converges to $f(x)$, then the formal product of (1.1) and $\sin mx$ converges to $f(x) \sin mx$. This statement is not true in general for summable trigonometric series. A simple example due to Rajchman [16, p. 78] is the series $\sum \sin nx$, which is summable $(C, 1)$ to $2^{-1} \cot (x/2)$ for $x \neq 0$, and to zero for $x = 0$. The formal product of this series and $\sin x$ is, however, $(1+\cos x)/2$, which is equal to 1 when $x=0$. Thus it is possible that a process of integration designed for series summable $(C, 1)$ would integrate $f(x)$ but not $f(x) \sin x$, and the usual Fourier formulas could not be valid.

The problem of suitably defining a trigonometric integral which may be applied to trigonometric series summable (C, k) can be solved if an additional condition involving the series

$$(1.2) \qquad \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) = - \sum_{n=1}^{\infty} b_n(x),$$

conjugate to (1.1), is imposed. The condition is that the series $\sum \{ b_n(x) - b_{n+1}(x) \}$ be summable (C, k) to $b_1(x)$ or, equivalently, that $b_n(x) \to 0$, (C, k). In the case of convergent series $(k=0)$, this condition is automatically satisfied. With this extra condition, it can be shown that the generalized P^{k+2}-integral [9] integrates trigonometric series summable (C, k) and that the coefficients are given by a suitable modification of the Fourier formulas. A description of the method, which has not yet been published, is given in §8 below.

2. **Totalisation of generalized second derivatives.** The process of

totalisation introduced by Denjoy goes back to 1912 and the extension to the problem of calculating a second primitive (totalisation symétrique à deux degrés) was outlined by him in a series of notes in 1921 [4]. The publication of a detailed account was begun in 1941 and, delayed by the war, completed in 1949 [5].

The application to trigonometric series is based on a fundamental result due to Riemann. Let $\theta_2(x, h) = \theta_2(F; x, h)$ be defined by

$$(2.1) \qquad \frac{h^2}{2!}\,\theta_2(x, h) = \frac{1}{2}\,\{F(x + h) + F(x - h)\} - F(x).$$

If the limit of $\theta_2(x, h)$ as $h \to 0$ exists it is called the generalized (symmetric) second derivative of $F(x)$ and written $D^2F(x)$. Riemann proved that, if the series (1.1) converges for all x to a function $f(x)$, then the series

$$(2.2) \qquad \frac{1}{2}\,a_0\,\frac{x^2}{2!} - \sum_{n=1}^{\infty} \frac{a_n(x)}{n^2}$$

(obtained by integrating (1.1) formally term-by-term twice) converges uniformly to a continuous function $F(x)$ such that $D^2F(x) = f(x)$. This suggests the problem of defining a process of integration, which, starting with the generalized second derivative of a continuous function, yields the function itself.

Denjoy extended the problem to include the case where the generalized second derivative may not exist for all x and considered generalized second derivates. The lim sup and lim inf of $\theta_2(x, h)$ as $h \to 0$ are called, respectively, the upper and lower (extreme) generalized second derivates and denoted by $\Delta^2F(x)$, $\delta^2F(x)$. It may happen that $\theta_2(x, h_n)$ tends to other limits for suitably chosen sequences $\{h_n\}$ tending to zero. Such limits are called median generalized second derivates. The problem he set is that of finding a process of totalisation for a function $f(x)$, knowing only that it is one of the generalized second derivates (extreme or median) of a continuous function with finite extreme generalized second derivates at each point of an interval (a, b). The process should determine the second primitive $F(x)$ up to an arbitrary linear function, or, equivalently, determine the second variation of $F(x)$ defined by

$$(2.3) \quad V(F; \alpha, \beta, \gamma) = (\gamma - \beta)F(\alpha) + (\alpha - \gamma)F(\beta) + (\beta - \alpha)F(\gamma)$$

for any three points, α, β, γ, of (a, b).

The starting point of the attack on the problem is the following result [5, §52b, 2, pp. 219–220]:

Suppose that the maximum of $|\theta_2(x, h)|$ *for all* h *is a finite number* $A(x)$ *for* x *belonging to a perfect set* P *contained in* (a, b). *Then, either* $A(x)$ *is bounded by a number* $A(P)$ *independent of* $x \in P$, *or the set* E *of points of* P *in the neighborhood of which* $A(x)$ *is not bounded is nondense in* P.

Since $\Delta^2 F(x)$ and $\delta^2 F(x)$ are supposed finite in (a, b) the maximum of $\theta_2(x, h)$ is also finite, and the hypothesis above is satisfied with $P=(a, b)$. In every closed sub-interval of (a, b) having no point in common with E, $A(x)$ is bounded and hence $f(x)$, being one of the generalized second derivatives, is also bounded in this sub-interval. It follows that the set $S_1 = S_1((a, b), f)$ of points of (a, b) in the neighborhood of which $f(x)$ is not Lebesgue summable is nondense in (a, b). Thus if α, β, γ are any three points in the same interval contiguous to S_1, the second variation of $F(x)$ is given by

$$V(F; \alpha, \beta, \gamma) = \int_\beta^\gamma (\beta\gamma + \alpha x) f(x)dx + \int_\gamma^\alpha (\gamma\alpha + \beta x) f(x)dx + \int_\alpha^\beta (\alpha\beta + \gamma x) f(x)dx, \tag{2.4}$$

where the integrals are taken in the Lebesgue sense.

Denjoy defines what he calls problem (U) relative to a set E in (a, b) as that of calculating $V(F; \alpha, \beta, \gamma)$ for any three points α, β, γ, of E. Problem (U) contiguous to a set E is said to be solved if problem (U) relative to each interval contiguous or semi-contiguous to E has been solved. Using this terminology, what has just been shown is that problem (U) is solved (by Lebesgue integrals) contiguous to the set S_1. (This Lebesgue integration is named operation 1 by Denjoy [5, p. 277].)

Two further operations (2 and 3) lead from the solution of problem (U) contiguous to S_1 to the solution of problem (U) contiguous to S_1', the first derivative of S_1. The two operations involve the taking of limits. In general, if μ is an ordinal number of the first kind and problem (U) contiguous to $S_1^{(\mu-1)}$, the derivative of order $\mu-1$ of S_1, has been solved, a finite number of applications of operation 3 leads to the solution contiguous to $S_1^{(\mu)}$. If μ is of the second kind and problem (U) contiguous to $S_1^{(\nu)}$ has been solved for every $\nu < \mu$, then it is also solved contiguous to $S_1^{(\mu)}$. Eventually, after at most a countable number of applications of operations 2 and 3, the solution is found contiguous to P_1, the maximal perfect set contained in S_1. The set P_1 is called by Denjoy [5, p. 91], the perfect nucleus of S_1. The conclusion now is that problem (U) contiguous to P_1 is solved.

Unlike the problem of simple (or complete) totalisation [5, §76b, 2, p. 337], it is not known a priori that P_1 contains a closed set S_2, nondense in P_1, such that problem (U) contiguous to S_2 may be solved. Hence another method must be found.

It may happen that P_1 is one of three special types of perfect sets [5, §39, pp. 121–126], and that $A(x)$ is bounded on P_1. In this case, Denjoy shows, using possibly six more operations (4, 5, 6, 7, 8, 9), that problem (U) relative to (a, b) can be solved. This involves intricate analysis and a great deal of detailed calculation. The operations themselves consist of a Lebesgue integration of $f(x)$ and $xf(x)$ over a perfect discontinuous set where $f(x)$ is summable, completed by the summation of an absolutely convergent series and the taking of various limits. There is a recapitulation of the role played by each of the nine operations in [5, §73e, pp. 317–318].

If P_1 is not one of the three special types, or if $A(x)$ is not bounded on P_1, it becomes necessary to construct a closed set $S_2 \subset P_1$ which is nondense in P_1 and such that problem (U) contiguous to S_2 may be solved [5, §74, pp. 318–325]. Then the above argument can be repeated with S_2 in place of S_1.

In this way a sequence $\{S_p, P_p\}$ of pairs of sets is constructed with S_p closed and P_p the perfect nucleus of S_p. If at any stage P_p is one of the three special types and $A(x)$ is bounded on P_p, problem (U) relative to (a, b) is solved. Otherwise, the sequence S_p of closed sets, each containing the next (equality excluded) must terminate so that all the sets S_p beyond a certain S_r are empty. Then problem (U) relative to (a, b) is again solved.

The calculation of the coefficients of a trigonometric series which converges for all x to a finite function $f(x)$ is now an easy matter. The second variation of the function $F(x)$ obtained by totalisation from $f(x)$ is the second variation of the series (2.2). This, for the particular points -2π, 0, 2π gives

$$4\pi^2 a_0 = V(F, -2\pi, 0, 2\pi). \tag{2.5}$$

The formal products of (1.1) and cos mx and of (1.1) and sin mx converge for all x to $f(x)$ cos mx and $f(x)$ sin mx, respectively. The constant terms in the new series are $a_m/2$ and $b_m/2$, respectively. Hence, if $G_m(x)$ and $H_m(x)$ denote the functions obtained by totalisation from $f(x)$ cos mx and $f(x)$ sin mx, respectively, then

$$\begin{aligned} 4\pi^2 a_m &= V(G_m, -2\pi, 0, 2\pi), \\ 4\pi^2 b_m &= V(H_m, -2\pi, 0, 2\pi). \end{aligned} \tag{2.6}$$

Thus the problem for a convergent trigonometric series is completely solved. The solution may seem unnecessarily complicated and Denjoy himself remarks [5, p. 325] "Certains de mes lecteurs me feront peut-être grief de la minutie des calculs développés dans les cinquantes pages de ce Chapitre (X)." It must be pointed out, however, that he gives examples [5, §114, pp. 585–593], of trigonometric series which converge for all x and for which the last five operations are required. In addition he shows that it is impossible that the sequence $\{S_\mu, P_\mu\}$ of pairs of sets be such that there is a unique μ_0 beyond which all the sets S_μ are empty for every convergent trigonometric series.

3. **Approximate totalisation.** Instead of using the series (2.2) Verblunsky [14] considered the series found by a single formal integration term-by-term of (1.1). In addition he restricted his attention to the complex power series

$$\sum_{n=1}^{\infty} c_n e^{inx}, \tag{3.1}$$

$$\sum_{n=1}^{\infty} c_n e^{inx}/in, \tag{3.2}$$

where $c_n = a_n - ib_n$. It is still true, however, that a process more powerful than totalisation is needed. There are series of the form (3.1) convergent for all x, such that (3.2) is not uniformly convergent [11], whereas the indefinite Denjoy integral is necessarily continuous.

Denjoy defines a class of functions which he calls *résoluble* (cf. [13, Chapter VII, particularly §9]). A function of this class has an approximate derivative almost everywhere which is (simply) totalisable. Verblunsky widens the class to approximately resoluble functions which still have an approximate derivative almost everywhere. An approximately resoluble function in (a, b) is one that is a differential coefficient and such that every perfect set in (a, b) contains a portion on which the function is resoluble.

One of the conditions imposed on a function $f(x)$ in the process of totalisation is that the set of points of (a, b) in the neighbourhood of which $f(x)$ is not Lebesgue summable is nondense in (a, b). Verblunsky replaces "not Lebesgue summable" by "not totalisable." It is then possible to define approximate totalisation, a process which integrates the approximate derivative of any approximately resoluble function.

Verblunsky's result is as follows [14, Theorem V]:

If the partial sums of the series (3.1) *are bounded for all x except possibly those belonging to a countable set, and if the series* (3.2) *con-*

verges for all x *to* $G(x)-iH(x)$, *then the approximate derivatives* $G_a(x)$, $H_a(x)$ *exist almost everywhere and*

$$\pi a_m = \int_{-\pi}^{\pi} G_a(x) \cos mx dx = -\int_{-\pi}^{\pi} H_a(x) \sin mx dx, \tag{3.3}$$

$$\pi b_m = \int_{-\pi}^{\pi} G_a(x) \sin mx dx = \int_{-\pi}^{\pi} H_a(x) \cos mx dx, \tag{3.4}$$

where the integrals are taken in the approximately totalisable sense. If, in addition, the series (3.1) *is Abel* (*or Poisson*) *summable to* $g(x)-ih(x)$ *almost everywhere, then* $G_a(x)=g(x)$, $H_a(x)=h(x)$ *almost everywhere and* a_m, b_m *are given by the usual Fourier formulas.*

4. **The integral of Marcinkiewicz and Zygmund.** One method of defining a process of integration is through the use of majorants and minorants. If $f(x)$ is defined in an interval (a, b), a majorant of $f(x)$ in the ordinary Perron sense is a function $M(x)$ satisfying the conditions

$$M(x) \text{ is continuous in } (a, b), \tag{4.1}$$

$$M(a) = 0, \tag{4.2}$$

$$D_* M(x) \geqq f(x), \text{ almost everywhere in } (a, b), \tag{4.3}$$

$$D_* M(x) \neq -\infty, \text{ except possibly at a countable set in } (a, b). \tag{4.4}$$

In this definition $D_* M(x)$ denotes the (ordinary) lower derivate of $M(x)$. A minorant $m(x)$ is similarly defined with $D^* m(x) \leqq f(x)$ and $D^* m(x) \neq \infty$ replacing (4.3) and (4.4), respectively. Let $J(b)$ denote the inf of the numbers $M(b)$ for all majorants and $j(b)$ the sup of the numbers $m(b)$ for all minorants. If $J(b)=j(b)$, the common value is the Perron integral of $f(x)$ over (a, b).

Since the Perron integral and totalisation (complete) are equivalent, it is clear that some extension of the classes of majorants and minorants is needed for applications to trigonometric series. Marcinkiewicz and Zygmund [10] altered condition (4.1) to the requirement that $M(x)$ be continuous in mean, that is, $M(x)$ is Perron integrable and

$$\lim_{h\to 0} \frac{1}{h} \int_0^h M(x+t)dt = M(x), \tag{4.5}$$

for all $x \in P$, where $P \subset (a, b)$, $a \in P$, $b \in P$, and $(a, b)-P$ is of measure zero. They also replaced the ordinary lower derivates in (4.3) and (4.4) by the lower Borel symmetric derivate, $BD_* M(x)$ defined as the lim inf as $h\to 0$ of

$$\frac{1}{h}\int_0^h \frac{M(x+t)-M(x-t)}{2t}\,dt. \tag{4.6}$$

Similar changes were made in the definition of a minorant. If $J(b) = j(b)$, the common value $F(b)$ is the MZ-integral of $f(x)$ with respect to P over (a, b) and may be written

$$F(b) = \left(a \int_P b\right) f(x)dx.$$

One of the properties of the integral is that, for $\alpha \in P$, $\beta \in P$,

$$F(\beta) - F(\alpha) = \left(\alpha \int_P \beta\right) f(x)dx.$$

The integral is not defined for all α, β, but only for almost all α, β, in (a, b). This, surprisingly enough, is an advantage, rather than a disadvantage for applications to trigonometric series. The reason is as follows. If the series (1.1) converges to $f(x)$ for all x, the series

$$\frac{1}{2} a_0 x - \sum_{n=1}^{\infty} \frac{b_n(x)}{n} \tag{4.7}$$

(obtained from (1.1) by integrating formally term-by-term) converges almost everywhere to a function $G(x)$ which is continuous in mean. If $F(x)$ denotes the sum of the series (2.2), the fundamental result of Riemann (§2) shows that $D^2F(x) = f(x)$. But, it is not difficult to show that whenever $D^2F(x)$ exists, so do $BD_*G(x)$ and $BD^*G(x)$ and that each is equal to $D^2F(x) = f(x)$. Hence, if P denotes the set of points where $G(x)$ exists and is continuous in mean, and if $\alpha \in P$, then $G(x) - G(\alpha)$ is both a (Borel) majorant and a minorant of $f(x)$. It follows that

$$G(\alpha + 2\pi) - G(\alpha) = \left(\alpha \int_P \alpha + 2\pi\right) f(x)dx. \tag{4.8}$$

But, from (4.7), the left side of (4.8) is equal to πa_0, which gives the first of the Fourier formulas. The others are found by the process of formal multiplication explained at the end of §2.

5. **The Cesàro-Perron integral.** Burkill defined a process of integration which is similar to that of Marcinkiewicz and Zygmund. He altered (4.1) in the same way although he used the term Cesàro- or C-continuous instead of continuous in mean. He replaced the ordinary lower derivate in (4.3) and (4.4) by the lower Cesàro-derivate, $CD_*M(x)$, defined as the lim inf as $h \to 0$ of

$$(5.1)\qquad \frac{2}{h^2}\int_0^h \{M(x+t) - M(x)\}dt.$$

Similar changes were made in defining a minorant. If $J(b)=j(b)$, the common value $F(b)$ is the Cesàro-Perron or CP-integral of $f(x)$ over (a, b), and may be written

$$F(b) = (\mathrm{CP})\int_a^b f(x)dx.$$

This integral also applies to the problem considered by Verblunsky (§3). It turns out [1] that the first two conditions imposed by Verblunsky imply (in the notation of §3) that $G(x)$ and $H(x)$ have finite upper and lower Cesàro derivates in (a, b) with the possible exception of a countable set. From this it follows that the upper and lower Cesàro derivates are equal almost everywhere, and, denoting the common values by $CDG(x)$, $CDH(x)$, that each of $CDG(X)$, $CDH(x)$ is CP-integrable over (a, b). Moreover, the coefficients a_m, b_m are given by (3.3) and (3.4), respectively, with $G_a(x)$, $H_a(x)$ replaced by $CDG(x)$, $CDH(x)$, respectively, and the integrals by CP-integrals.

Burkill [1] also shows that the third condition of Verblunsky that the series (3.1) is Abel summable almost everywhere is redundant, being implied by the other two.

6. The Perron second integral. James and Gage [7] defined a Perron second (or P^2-integral) which, like Denjoy's totalisation of a generalized second derivate, starts with a function $f(x)$ and goes directly to a second primitive $F(x)$ such that $D^2F(x)=f(x)$ almost everywhere.

They made no change in condition (4.1) but required $M(x)$ to vanish at both ends of the interval (a, b). They also replaced the ordinary lower derivate in (4.3) and (4.4) by the lower generalized second derivate $\delta^2M(x)$. In addition the condition of smoothness,

$$(6.1)\qquad h\theta_2(x, h)\to 0 \quad \text{as} \quad h\to 0,$$

was required to hold for all x in the countable set of exceptional points of (4.4). If c is any point of (a, b), let $-J(c)$ denote the inf of the numbers $-M(c)$ for all majorants and $-j(c)$ the sup of the numbers $-m(c)$ for all minorants. If $-J(c)=-j(c)$, the common value $-F(c)$ is the P^2-integral of $f(x)$ over $(a, b; c)$ and may be written

$$-F(c) = \int_{(a,b)}^{c} f(x)d_2x.$$

The reason for using negative signs is that $M(x)-m(x)$ is convex in (a, b) and this makes $-M(x)\geqq -m(x)$.

In a second paper, James [8] showed that the P^2-integral provides a solution of Denjoy's problem (U) relative to the interval (a, b). In fact, if $f(x)$ is P^2-integrable over $(a, b; c)$ it is also integrable over $(\alpha, \beta; \gamma)$, where $a\leqq\alpha<\gamma<\beta\leqq b$, and, in addition,

$$(\beta-\alpha)\int_{(\alpha,\beta)}^{\gamma} f(x)d_2x = V(F; \alpha, \beta, \gamma).$$

Hence, if $f(x)$ and $F(x)$ are the sums of the series (1.1) and (2.2), respectively, Denjoy's results (2.5), (2.6) show that

$$\pi^2 a_m = \int_{(-2\pi,\, 2\pi)}^{0} f(x)\cos mx\, d_2x, \qquad \pi^2 b_m = \int_{(-2\pi,\, 2\pi)}^{0} f(x)\sin mx\, d_2x.$$

Thus the P^2-integral also provides a complete solution to the problem for convergent trigonometric series.

It follows from a general result of Denjoy [5, §66] that a function with a finite generalized second derivative almost everywhere also has a derivative $F'(x)$ almost everywhere. In the case of trigonometric series, this derivative is equal to sum of the series (4.7) wherever it converges. Hence another expression for the Fourier coefficient a_0 is

$$\pi a_0 = F'(\alpha+2\pi) - F'(\alpha)$$

where $\alpha\in P$, the set of points at which the series (4.7) converges. There are similar expressions for the other coefficients using the derivatives of the functions $G_m(x)$, $H_m(x)$ defined in §2.

7. **The symmetric Cesàro-Perron integral.** Burkill [2] noted that the CP indefinite integral, being defined for all x, was not suitable for applications to trigonometric series because the series (4.7) need converge only almost everywhere even when the series (1.1) converges everywhere. He therefore modified his definition of the CP-integral in order to have one of the same type as the MZ-integral (§4).

He altered condition (4.1) to the condition (4.5) and replaced the lower Cesàro-Perron derivate by the lower symmetric Cesàro-Perron derivate, $SCD_*M(x)$, defined as the lim inf as $h\to 0$ of

$$\frac{1}{h^2}\int_0^h \{M(x+t) - M(x-t)\}dt. \tag{7.1}$$

He also required $M(x)$ to satisfy

$$\frac{1}{h}\int_0^h \{M(x+t) - M(x-t)\}\,dt \to 0 \quad \text{as} \quad h \to 0 \tag{7.2}$$

in addition to (4.5) at the countable set of points where $SCD_*M(x) = -\infty$.

If $T(x)$ is the integral of $M(x)$, then $SCD_*M(x) = \delta^2T(x)$ and condition (7.2) is simply the requirement (6.1) of smoothness. It can also be shown [10, Lemma 40] that

$$\delta^2T(x) = SCD_*M(x) \leq BD_*M(x),$$
$$BD^*m(x) \geq SCD^*m(x) = \Delta^2t(x).$$

Hence any function which is SCP-integrable is necessarily MZ- and P^2-integrable, and the formulas for the coefficients are of the same form.

As far as the integration of convergent trigonometric series is concerned, the three types of integrals are equivalent. Whether or not the MZ- or P^2-integral is actually more general than the SCP-integral is not yet known.

8. **The P^n-integral.** James [9] extended the P^2-integral by replacing generalized second derivates by generalized symmetric derivates of higher order, but required that the inequalities analogous to (4.3) and (4.4) hold without exception. In addition a condition similar to the smoothness condition (6.1) was imposed at all points of (a, b) with the possible exception of a countable set. Only the case $n=4$ will be considered here since it fully illustrates the difference between the general case and the case $n=2$.

If $M(x)$ is continuous and $D^2M(x)$ is finite in (a, b), let $\theta_4(x, h) = \theta_4(M; x, h)$ be defined by

$$\frac{h^4}{4!}\theta_4(x, h) = \frac{1}{2}\{M(x+h) + M(x-h)\} - M(x) - \frac{h^2}{2!}D^2M(x).$$

The lim sup and lim inf as $h \to 0$ of $\theta_4(x, h)$ are the upper and lower generalized symmetric derivates of order 4, $\Delta^4M(x)$, $\delta^4M(x)$, respectively. If the two are equal, their common value is the generalized symmetric derivative of order 4, $D^4M(x)$.

The main difference between the general case and the case $n=2$ is that the analogue of the theorem of Schwarz does not hold. This theorem states that, if $D^2M(x) = 0$ throughout an interval (a, b), then $M(x)$ is a linear function. It is not true, for example, that if $D^4M(x) = 0$ in (a, b), then $M(x)$ is a cubic function. A simple example given by M. Riesz [12] is the function defined by

$$M(x) = \begin{cases} x^2, & x \geqq 0, \\ -x^2, & x \leqq 0. \end{cases}$$

It is clear that $M(x)$ is continuous, that

$$D^2M(x) = \begin{cases} 2, & x > 0, \\ 0, & x = 0, \\ -2, & x < 0, \end{cases}$$

and that $D^4M(x)$ is identically zero. Obviously $M(x)$ is not a cubic function. The interesting part of the example is that $D^2M(x)$ has an ordinary discontinuity at $x=0$. If such a possibility is ruled out, the analogue of the theorem of Schwarz is true. It follows from [9, Theorem 4.2] (with $m=2$) that if $M(x)$ is continuous, $D^2M(x)$ is finite and has no ordinary discontinuities, and $D^4M(x)=0$ in (a, b), then $M(x)$ is a cubic function. In defining a majorant or minorant using generalized derivates of order 4, the condition that the generalized second derivative have no ordinary discontinuities must be imposed.

The definition of a majorant (in the P^4 sense) follows the pattern of §§4 and 6. The function $M(x)$ is required to be continuous and to vanish at four points a_i, $i=1, \cdots, 4$, such that $a=a_1<a_2<a_3<a_4=b$. Conditions (4.3) and (4.4) hold without exception with $D_*M(x)$ replaced by $\delta^4M(x)$, and the smoothness condition (6.1) becomes $h\theta_4(x, h)\to 0$ as $h\to 0$ for all x in (a, b) with the possible exception of a countable set. In addition, $D^2M(x)$ has no ordinary discontinuities in (a, b). A minorant is defined in a similar way.

With each majorant and minorant there is associated the function defined by

$$M^*(x) = (-1)^rM(x), \quad m^*(x) = (-1)^rm(x), \quad \text{for} \quad a_r \leqq x \leqq a_{r+1}, \quad r = 1, 2, 3.$$

The point of the last definition is that $M(x)-m(x)$ is 4-convex [9, §2] and that then $M^*(x)\geqq m^*(x)$. If c is any point of (a, b), let $J^*(c)$ denote the inf of the numbers $M^*(c)$ for all majorants and $j^*(c)$ the sup of the numbers $m^*(c)$ for all minorants. If $J^*(c)=j^*(c)$, the function $f(x)$ is P^4-integrable over $(a_i; c)=(a_1, a_2, a_3, a_4; c)$. In this case the function defined by $(-1)^sF(c)=J^*(c)=j^*(c)$ when $a_s<c<a_{s+1}$, $s=1, 2, 3$, is the P^4-integral and may be written

$$(-1)^sF(c) = \int_{(a_i)}^{c} f(x)d_4x.$$

Since $J^*(a_i)=j^*(a_i)=0$, the integral is zero if $c=a_i$, $i=1, \cdots, 4$.

The P^4-integral integrates the generalized derivative of order 4 of any continuous function $G(x)$ provided that $D^2G(x)$ has no ordinary discontinuities. In fact,

$$(8.1) \qquad (-1)^4 \int_{(a_i)}^{c} D^4G(x)d_4x = G(c) - \sum_{i=1}^{4} \lambda(c;\, a_i)G(a_i),$$

where $\lambda(x;\, a_i) = \prod_{j\neq i} (c-a_j)/(a_i-a_j)$ is a polynomial of the third degree in c.

The application of the P^4-integral to summable trigonometric series is based on an extension ([15, Theorem B] and [17, §10.42]) of Riemann's result for the generalized second derivative. For $n=4$ the extension is as follows:

If the trigonometric series (1.1) *is summable* $(C, 2)$ *for all* x *to a function* $g(x)$, *then the series*

$$(8.2) \qquad \frac{1}{2} a_0 \frac{x^4}{4!} + \sum_{n=1}^{\infty} \frac{a_n(x)}{n^4}$$

(*obtained by integrating* (1.1) *formally term-by-term four times*) *converges uniformly to a function* $G(x)$ *such that* $D^4G(x)=g(x)$.

It also follows, of course, that $D^2G(x)$ is the sum of the convergent series (2.2), but it is not true, in general, that $D^2G(x)$ has no ordinary discontinuities. For example, the series $\sum n \sin nx$ is summable $(C, 2)$ to zero for all x, but the series $-\sum (\sin nx)/n$ converges to $-(\pi+x)/2$ if $-2\pi<x<0$, to zero if $x=0$, and to $-(\pi-x)/2$ if $0<x<2\pi$. Thus there is an ordinary discontinuity at $x=0$.

The difficulty is overcome by imposing a condition on the coefficients of the series (4.7) obtained by integrating (1.1) formally term-by-term. The condition is that $b_n(x)/n\to 0$ $(C, 1)$, which is equivalent to requiring that the series $\sum\{b_n(x)/n-b_{n+1}(x)/(n+1)\}$ be summable $(C, 1)$ to $b_1(x)$. It then follows by a slight modification of Zygmund's argument [17, §10.42] that $G'(x)$ exists and that

$$D^2G(x) = \lim_{h\to 0} \frac{G(x+h) - G(x) - hG'(x)}{h^2/2!} \cdot$$

Hence $D^2G(x)$ is equal to $G_{(2)}(x)$, the second differential coefficient of $G(x)$, which by [9, Lemma 8.1] cannot have an ordinary discontinuity. In the example above, $b_n(x)/n=\cos nx$, which for $x=0$, the point of discontinuity of $D^2G(x)$, does not tend to zero $(C, 1)$.

The condition $b_n(x)/n\to 0$ $(C, 1)$ is not strong enough to deal with formal multiplication of summable trigonometric series. However, if

$b_n(x)\to 0$ $(C, 2)$, then the formal products of (1.1) and $\cos mx$ and $\sin mx$ are summable $(C, 2)$ to $g(x)\cos mx$ and $g(x)\sin mx$ whenever (1.1) is summable $(C, 2)$ to $g(x)$. Zygmund proves this for Abel summability and indicates in a footnote [16, p. 79] that analogous results hold for Cesàro summability.

Since $b_n(x)\to 0$ $(C, 2)$ implies that $b_n(x)/n\to 0$ $(C, 1)$, the stronger condition is used in the final result which may be summed up as follows:

If the trigonometric series (1.1) *is summable* $(C, 2)$ *to a function* $g(x)$ *and if* $b_n(x)\to 0$ $(C, 2)$ *for all* x, *then* $g(x)$, $g(x)\cos mx$, $g(x)\sin mx$ *are* P^4*-integrable. It also follows from* (8.1) *that the coefficients of* (1.1) *are given by*

$$\frac{4\pi^4}{3}a_m = \int_{(a_i)}^{0} g(x)\cos mx\,d_4x, \tag{8.3}$$

$$\frac{4\pi^4}{3}b_m = \int_{(a_i)}^{0} g(x)\sin mx\,d_4x, \tag{8.4}$$

where $(a_i) = (-4\pi, -2\pi, 2\pi, 4\pi)$.

The general result takes the same form with summability $(C, 2)$ replaced by summability (C, k), $k\geqq 1$, and the P^4-integral by the P^{k+2}-integral. The numerical coefficients of a_m and b_m in (8.3) and (8.4) and the set (a_i) have different forms according as k is even or odd. If $k=2r-2$, the numerical coefficient is $2^{2r-1}\pi^{2r}(r!)^2/(2r)!$ and

$$(a_i) = (-2r\pi, \cdots, -2\pi, 2\pi, \cdots, 2r\pi).$$

If $k=2r-1$, the numerical coefficient is $2^{2r}\pi^{2r+1}r!(r+1)!/(2r+1)!$ and

$$(a_i) = (-(2r+1)\pi, \cdots, -3\pi, -\pi, 3\pi, \cdots, (2r+1)\pi)$$

and the 0 in the integral becomes π.

Thus, if a trigonometric series is summable (C, k) and the coefficients of the conjugate series tend to zero (C, k) for all x, the P^{k+2}-integral provides a complete solution to the problem of expressing the coefficients in terms of integrals analogous to the usual Fourier formulas.

REFERENCES

1. J. C. Burkill, *The expression of trigonometrical series in Fourier form*, J. London Math. Soc. vol. 11 (1936) pp. 43–48.

2. ———, *Integrals and trigonometric series*, Proc. London Math. Soc. (3) vol. 1 (1951) pp. 46–57.

3. A. Denjoy, *La totalisation des nombres dérivés non sommables*, Ann. École Norm. (3) vol. 33 (1916) pp. 127–222; vol. 34 (1917) pp. 181–236.

4. ———, *Calcul des coefficients d'une série trigonométrique partout convergente*, C. R. Acad. Sci. Paris vol. 172 (1921) pp. 653–655, 833–835, 903–906, 1218–1221; vol. 173 (1921) pp. 127–129.

5. ———, *Leçons sur le calcul des coefficients d'une série trigonométrique*, Paris, 1941 and 1949.

6. G. H. Hardy, *Divergent series*, Oxford, 1949.

7. R. D. James and Walter H. Gage, *A generalized integral*, Trans. Roy. Soc. Canada, Sect. III vol. 40 (1946) pp. 25–35.

8. R. D. James, *A generalized integral* II, Canadian Journal of Mathematics vol. 2 (1950) pp. 297–306.

9. ———, *Generalized nth primitives*, Trans. Amer. Math. Soc. vol. 76 (1954) pp. 149–176.

10. J. Marcinkiewicz and A. Zygmund, *On the differentiability of functions and summability of trigonometrical series*, Fund. Math. vol. 26 (1936) pp. 1–43.

11. L. Neder, *Zur Theorie der trigonometrischen Reihen*, Math. Ann. vol. 84 (1921) pp. 117–136.

12. M. Riesz, *Über summierbare trigonometrische Reihen*, Math. Ann. vol. 71 (1912) pp. 54–75.

13. S. Saks, *Theory of the integral*, Warsaw, 1937.

14. S. Verblunsky, *On the theory of trigonometric series* VII, Fund. Math. vol. 23 (1934) pp. 193–235.

15. F. Wolf, *Summable trigonometric series: an extension of uniqueness theorems*, Proc. London Math. Soc. (2) vol. 45 (1939) pp. 328–356.

16. A. Zygmund, *Sur la théorie riemannienne des séries trigonométriques*, Math. Zeit. vol. 24 (1926) pp. 47–104.

17. ———, *Trigonometrical series*, Warsaw, 1935.

UNIVERSITY OF BRITISH COLUMBIA

SEMI-GROUPS OF OPERATORS

R. S. PHILLIPS[1]

1. **Introduction.** I should like to present some of the recent developments in the theory of one-parameter semi-groups of linear bounded operators. Actually this field is relatively new, having its origin in the work of M. H. Stone [16] on one-parameter groups of unitary operators in Hilbert space which appeared in 1930. By now the subject has an extensive literature, the definitive work being E. Hille's [4] *Functional analysis and semi-groups*, published in 1948. An adequate bibliography for the material in this field published prior to 1948 can be found in Hille's treatise. I shall limit my remarks to developments since 1948, emphasizing perhaps unduly my own contributions. Further I shall not discuss in any detail the applications to partial differential equations of parabolic type, although this has been one of the most gratifying aspects of the theory; this material can be found in papers by W. Feller [1], E. Hille [7], and K. Yosida [20; 21].

Let $\mathfrak{S} = [T(\xi); \xi > 0]$ be a one-parameter family of linear bounded operators defined on a complex Banach space $\mathfrak{X}$ to itself and satisfying the product law

$$T(\xi_1 + \xi_2) = T(\xi_1)T(\xi_2), \qquad \xi_1, \xi_2 > 0. \tag{1.1}$$

We shall further assume that $T(\xi)x$ is a continuous function of ξ for $\xi > 0$ and each $x \in \mathfrak{X}$, that is, we assume that $\mathfrak{S}$ is continuous in the strong operator topology. The *infinitesimal operator* is defined as the limit in norm

$$\lim_{\eta \to 0+} \frac{T(\eta) - I}{\eta} x = A_0 x \tag{1.2}$$

wherever this limit exists, the domain of A_0 (in symbols $\mathfrak{D}(A_0)$) being the set of elements for which the limit exists. It is easy to see that A_0 is a linear operator. However, A_0 need not be bounded nor even closed; its least closed extension, when it exists, will be called the *infinitesimal generator of* $\mathfrak{S}$ and will be designated by A. The latter operator plays a basic role in the theory. For $x \in \mathfrak{D}(A_0)$ we have

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$$(1.3) \qquad \frac{dT(\xi)x}{d\xi} = A_0T(\xi)x = T(\xi)A_0x, \qquad \xi > 0.$$

Moreover, denoting the range space for $\mathfrak{S}$ by

$$(1.4) \qquad \mathfrak{X}_0 = \bigcup_{\xi>0} T(\xi)[\mathfrak{X}],$$

it can be shown that

$$[\mathfrak{D}(A_0)]^- = [\mathfrak{X}_0]^-.$$

In applications it is usually required that the semi-group satisfy an initial condition of the sort

$$(C_0) \qquad \lim_{\xi\to 0+} T(\xi)x = x, \qquad x \in \mathfrak{X}.$$

Now $\lim_{\xi\to 0+} T(\xi)x = x$ for each $x \in \mathfrak{X}_0$ whereas this relation is never satisfied for $x \notin [\mathfrak{X}_0]^-$. This suggests that we restrict the operators $T(\xi)$ to the subspace $\mathfrak{Y} = [\mathfrak{X}_0]^-$, in which case we obtain a semi-group of operators on $\mathfrak{Y}$ to itself. The range space for the restricted semi-group is again $\mathfrak{X}_0$ so that the above initial condition is now satisfied at least by a set of vectors dense in the space. Instead of considering the restricted operators, it is simpler to assume at the outset that $\mathfrak{X}_0$ is dense in $\mathfrak{X}$. If in addition $\|T(\xi)\|$ is bounded near the origin, then the condition (C_0) is satisfied in its entirety and we say that $\mathfrak{S}$ *is of class* (C_0). For such semi-groups A_0 is itself closed so that $A = A_0$.

Many problems in classical analysis are subsumed under the theory of semi-groups of linear bounded operators. This is in particular true of Cauchy's problem which arises in the theory of partial differential equations. Abstracting from this, E. Hille [6; 7] has introduced an *abstract Cauchy problem*, a modification of which is

ACP. *Given a closed linear operator U with domain and range in $\mathfrak{X}$, and given an element $y_0 \in \mathfrak{X}$, find a function $y(\xi) = y(\xi; y_0)$ on $[0, \infty)$ to $\mathfrak{X}$ such that*

(1) $y(\xi)$ *is continuously differentiable on* $[0, \infty)$,

(2) $y(\xi) \in \mathfrak{D}(U)$ *and* $U[y(\xi)] = y'(\xi)$ *for each* $\xi > 0$,

(3) $\lim_{\xi\to 0+} y(\xi; y_0) = y_0$.

As an example of the above consider the classical diffusion problem in the Banach space $C[-\infty, \infty]$:

$$(1.5) \qquad \begin{aligned} \partial u/\partial\xi &= \partial^2 u/\partial x^2, && -\infty < x < \infty, \quad \xi > 0; \\ \lim_{\xi\to 0+} u(\xi, x) &= y_0(x), && -\infty < x < \infty. \end{aligned}$$

Here one defines the closed linear operator U by

$$(1.6)\quad \begin{aligned}&[Uy](x) = y''(x),\\ &\mathfrak{D}(U) = [y;\ y(x),\ y'(x) \text{ absolutely continuous}\\ &\qquad\qquad\qquad\qquad \text{and } y,\ y'' \in C[-\infty, \infty]].\end{aligned}$$

If $y(\xi) = y(\xi;\ y_0)$ satisfies the ACP and has the realization $\phi(\xi, x)$ so that $y(\xi) = \phi(\xi, \cdot)$, then condition (1) means that $\phi(\xi, x)$ is continuously differentiable with respect to ξ, $\xi > 0$, uniformly with respect to x; condition (2) implies that $\phi(\xi, \cdot) \in \mathfrak{D}(U)$ for each $\xi > 0$ and that $u = \phi(\xi, x)$ satisfies the first relation in (1.5); and finally condition (3) requires that $u = \phi(\xi, x)$ satisfies the second relation in (1.5) uniformly with respect to x.[3]

There is a close connection between the ACP and the theory of semi-groups. Indeed, if $\mathfrak{S} = [T(\xi)]$ is a semi-group of class (C_0) and $y_0 \in \mathfrak{D}(A)$, A being the infinitesimal generator of $\mathfrak{S}$, then the relation (1.3) shows that $y(\xi;\ y_0) = T(\xi)y_0$ is continuously differentiable in ξ for $\xi \geqq 0$ and that $A[y(\xi)] = y'(\xi)$, whereas condition (C_0) implies that $\lim_{\xi \to 0+} y(\xi;\ y_0) = y_0$. Thus $y(\xi;\ y_0) = T(\xi)y_0$ satisfies the ACP for each $y_0 \in \mathfrak{D}(A)$. Conversely we have (see R. S. Phillips [14])

THEOREM 1.1. *Let U be a closed linear operator with dense domain and a nonempty resolvent set. Suppose for each $y_0 \in \mathfrak{D}(U)$ that there exists a unique solution $y(\xi;\ y_0)$ to the* ACP. *Then U is the infinitesimal generator of a semi-group $[T(\xi)]$ of class* (C_0) *such that $T(\xi)y_0 = y(\xi;\ y_0)$ for each $y_0 \in \mathfrak{D}(U)$.*

We remark that the semi-group property of the solution to the ACP is an easy consequence of the uniqueness assumption. However, the surprising feature in the above assertion is that $y(\xi;\ y_0)$ is a continuous function of the initial data or, otherwise expressed, that the solution is stable relative to the initial data; this follows from the fact that the semi-group operators $T(\xi)$ are bounded.

2. **Generation of semi-groups.** In view of the above connection between the ACP and semi-groups of operators, the question of when a closed linear operator U generates a semi-group is of particular interest. To begin with one can ask whether such a problem is *well set* in the sense of Hadamard, that is, one can ask whether

(1) *there exists at most one solution,*

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[3] If $\mathfrak{X}$ is a Lebesgue space rather than $C[-\infty, \infty]$, then the interpretation of conditions (1), (2), and (3) of ACP is quite different since the limits are then limits in the mean. However, it is possible to construct a realization $\phi(\xi, x)$ of $y(\xi;\ y_0)$ such that $\phi(\xi, x)$ is absolutely continuous in ξ, $\xi \geqq 0$, for each x, $\partial\phi(\xi, x)/\partial\xi$ exists almost everywhere in the product space, and for almost all ξ this function is a representation of $U[y(\xi)]$.

(2) *there exists at least one solution,*

(3) *the solution is stable, that is, the semi-group is a continuous function of the infinitesimal generator.*

We shall take these conditions up in order. As for (1), it can be shown rather easily and under quite general conditions that a closed linear operator generates at most one semi-group.[3] On the other hand the existence question is fairly deep and permits many kinds of answers. Indeed, Theorem 1.1 provides an answer of a sort for this problem. However, for many applications it is more convenient to have a criteria which depends more directly on the properties of U itself. Condition (3) will be discussed in the next section.

The most satisfactory answer to (2) is given in terms of the resolvent of U, namely, $R(\lambda; U)$. The motivation here is as follows. For a semi-group $\mathfrak{S} = [T(\xi)]$ continuous in the strong operator topology, the

$$(2.1) \qquad \lim_{\xi\to\infty} \xi^{-1} \log \|T(\xi)\| = \omega_0$$

exists; the quantity ω_0, called the *type* of $\mathfrak{S}$, is finite or $-\infty$, but never $+\infty$. Consequently if $\mathfrak{S}$ is of class (C_0), it is possible to take the Laplace transform of $T(\xi)x$. It turns out that

$$(2.2) \qquad R(\lambda; A)x = \int_0^\infty e^{-\lambda\xi} T(\xi)x d\xi, \qquad \Re(\lambda) > \omega_0;$$

here the right member is obtained as an abstract Bochner integral in $\mathfrak{X}$. Thus our problem is in essence one of determining when $R(\lambda; U)$ is the Laplace transform of an operator-valued function. Phrased in this way, the problem has its classical analogue (see D. V. Widder [18, Chap. VII]), and the criteria which we shall obtain for $R(\lambda; U)$ are the prototypes of the classical criteria. However, the proofs required in the present problem differ radically from the classical case in that they no longer depend on compactness arguments but rather on the properties of the resolvent function.

The first result in this direction, and for many purposes the most useful, was obtained independently by E. Hille [4; 5] and K. Yosida [19] in 1948. A complete characterization of the infinitesimal generator for semi-groups of class (C_0) was not found until 1953 when the following theorem was published almost simultaneously by W. Feller [3], I. Miyadera [1], and R. S. Phillips [12].

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[3] More precisely, a closed linear operator can generate at most one semi-group of class (A) (see §4).

THEOREM 2.1. *A necessary and sufficient condition that a closed linear operator U generate a semi-group $[T(\xi)]$ of class* (C_0) *is that* $\mathfrak{D}(U)$ *be dense in* $\mathfrak{X}$ *and that there exist real constants M and ω such that*

$$\|[R(\lambda;U)]^n\| \leqq M(\lambda-\omega)^{-n} \tag{2.3}$$

for all $\lambda>\omega$ *and* $n=1, 2, 3, \cdots$. *In this case* $\|T(\xi)\| \leqq M \exp(\omega\xi)$.

We obtain the Hille-Yosida theorem as a corollary, viz.

COROLLARY. *If U is a closed linear operator with dense domain, if* $R(\lambda; U)$ *exists for* $\lambda>0$, *and if*

$$\lambda\|R(\lambda;U)\| \leqq 1 \tag{2.4}$$

for all $\lambda>0$, *then U is the infinitesimal generator of a semi-group* $[T(\xi)]$ *of class* (C_0) *such that* $\|T(\xi)\| \leqq 1$ *for all* $\xi>0$.

In all of the present applications of semi-group theory to diffusion theory, the above corollary has been used to establish the existence of a semi-group solution. In order to see what is involved here, we shall again consider the differential equation (1.5), the associated operator U being given by (1.6). It is clear that $\mathfrak{D}(U)$ is dense in $C[-\infty, \infty]$. Further $R(\lambda; U)f$ is obtained as the solution to

$$(\lambda I - U)v = \lambda v - v'' = f \tag{2.5}$$

with $v \in \mathfrak{D}(U)$ so that

$$[R(\lambda;U)f](x) = (2\lambda^{1/2})^{-1}\int_{-\infty}^{\infty} \exp(-\lambda^{1/2}|x-y|)f(y)dy, \qquad \lambda>0. \tag{2.6}$$

An elementary calculation shows that

$$\begin{aligned}\|R(\lambda;U)f\| &\leqq \sup_x (2\lambda^{1/2})^{-1}\int_{-\infty}^{\infty}\exp(-\lambda^{1/2}|x-y|)dy\cdot\|f\| \\ &= \lambda^{-1}\|f\|.\end{aligned} \tag{2.7}$$

The conditions of the corollary are therefore satisfied and it follows that (1.5) has a semi-group solution of class (C_0).

The advantage obtained in using the criterion (2.4) rather than (2.3) clearly lies in the fact that the former requires a bound only for the first power of the resolvent operator, whereas the latter requires a bound on all of the positive integral powers of the resolvent. In order to make such conditions on U less stringent we are led to consider more general classes of semi-groups.

If we wish to keep the theory within the present framework, there is only one condition which can be relaxed and that is the behavior of the semi-group near the origin. Indeed, assuming strong measurability instead of strong continuity for $\xi>0$ does not in fact broaden the class of semi-groups since strong measurability can be shown to imply strong continuity (see R. S. Phillips [9]). The assumption of weak measurability, on the other hand, does not imply strong continuity and leads to a considerably more general theory as W. Feller [2] has shown. However, the added generality obtained by so weakening the hypotheses does not, in our opinion, compensate for the concomitant loss of detail in the theory.

A useful class of semi-groups can be defined as

DEFINITION. *A semi-group of linear bounded operators* $[T(\xi)]$ *is said to be of class* (1, A) *if it satisfies the conditions*

(1) $T(\xi)$ *is continuous in the strong operator topology for* $\xi>0$,

(2) $\int_0^1\|T(\xi)\|d\xi<\infty$,

(3) $\lim_{\lambda\to\infty}\lambda\int_0^\infty e^{-\lambda\xi}T(\xi)x\,d\xi=x$ *for each* $x\in\mathfrak{X}$.

Condition (2) in this definition allows $\|T(\xi)\|$ to become unbounded as $\xi\to 0+$, whereas condition (3) replaces (C_0) by strong Abel summability to the identity at $\xi=0$. For semi-groups of class (1, A) the infinitesimal operator A_0 need not be closed; however, the least closed extension of A_0, namely the infinitesimal generator A, again satisfies the relation (2.2) in the half-plane $\mathfrak{R}(\lambda)>\omega_0$. This shows, incidentally, that the resolvent set for A contains a right half-plane. Moreover, it is easy to see that $\|R(\lambda; A)\|$ is bounded in every half-plane $\mathfrak{R}(\lambda)\geqq\omega>\omega_0$. We now have (R. S. Phillips [13])

THEOREM 2.2. *A necessary and sufficient condition that a closed linear operator* U *be the infinitesimal generator of a semi-group* $[T(\xi)]$ *of class* (1, A) *is that*

(1) $\mathfrak{D}(U)$ *is dense in* $\mathfrak{X}$;

(2) $\|R(\lambda;U)\|=O(1/\lambda)$ *as* $\lambda\to\infty$;

(3) *there exist real constants* M *and* ω *such that*

$$\int_0^\infty \lambda^{n-1}\|R^{(n)}(\lambda+\omega;U)\|d\lambda\leqq(n-1)!M \tag{2.8}$$

for $n=1, 2, 3, \cdots$.

Condition (3) *can be replaced by*

(3′) *there exists a non-negative measurable function* $\phi(\xi)$ *and a constant* ω *such that* $\int_0^\infty e^{-\omega\xi}\phi(\xi)d\xi<\infty$ *and*

$$\|R^{(n)}(\lambda;U)\|\leqq\int_0^\infty e^{-\lambda\xi}\xi^n\phi(\xi)d\xi \tag{2.9}$$

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for all real $\lambda>\omega$ *and integers* $n\geqq 0$. *If condition* (3′) *is verified, then* $\|T(\xi)\|\leqq\phi(\xi)$ *almost everywhere and if in addition* $\phi(\xi)$ *is bounded near the origin, then* $[T(\xi)]$ *is of class* (C_0).

In the numerical case, condition (2.8) is a necessary and sufficient condition that $R(\lambda;\ U)$ be a Laplace-Stieltjes transform. However, (2.8) together with the properties of the resolvent operator (and (1) and (2)) suffice to make $R(\lambda;\ U)$ a Laplace transform.

The following corollary generalizes a theorem due to E. Hille [6].

COROLLARY. *A necessary and sufficient condition that a closed linear operator* U *be the infinitesimal generator of a semi-group of class* (1, A) *is that*

(1) $\mathfrak{D}(U)$ *is dense in* $\mathfrak{X}$;

(2) $\|R(\lambda;\ U)\|=O(1/\lambda)$ *as* $\lambda\to\infty$;

(3) *there exists a family* $[T(\xi);\ \xi>0]$ *of linear bounded operators strongly continuous for* $\xi>0$ *with* $\int_0^\infty e^{-\omega\xi}\|T(\xi)\|d\xi<\infty$ *for some real* ω *such that*

$$R(\lambda;\ U)x=\int_0^\infty e^{-\lambda\xi}T(\xi)x\,d\xi \tag{2.10}$$

holds for all $\lambda>\omega$ *and all* $x\in\mathfrak{X}$.

Theorem 2.2 can be used to obtain the following sufficiency criterion: *A closed linear operator* U *with dense domain generates a semi-group of class* (1, A) *if*

$$\|R(\sigma+i\tau;\ U)\|\leqq M/(\sigma-\omega+|\tau|^{\beta}),\qquad \sigma>\omega, \tag{2.11}$$

providing $\beta>1/2$.

Condition (2.11) has the virtue of requiring a bound only on $\|R(\lambda;\ U)\|$ and it has some advantage over the Hille-Yosida criterion in that M is not restricted to the value one. The above criterion with $\beta=0$ is not sufficient that U be the infinitesimal generator of a strongly continuous semi-group of linear bounded operators (see R. S. Phillips [13]). As we shall see in §4, if $0<\beta\leqq 1/2$ then the operator U does generate a semi-group of linear bounded operators; however, in this case the semi-group may belong to a more general class than (1, A).

3. Perturbation theory. Returning to the generation problem, we recall that stability was the third property needed in order that this problem be well set. We shall establish stability as a by-product of a perturbation theory for semi-groups of class (1, A). In what follows we shall perturb the infinitesimal generator by adding to it certain linear operators (cf. R. S. Phillips [12; 13]). One would expect that those semi-group properties which persist under such perturbations are the more basic semi-group properties and that the significant theorems of the subject should evolve about these properties.

We shall consider only semi-groups of class (1, A). If A is the infinitesimal generator of such a semi-group, we denote the corresponding semi-group by $[T(\xi; A)]$. A linear operator B will be said to belong to the perturbing class $\mathfrak{P}(A)$ if (i) $\mathfrak{D}(B) = \mathfrak{D}(A)$, (ii) $BR(\lambda; A)$ is a bounded linear operator for some λ in the resolvent set of A, and (iii) $\int_0^1 \|BT(\xi; A)\|_A d\xi < \infty$, where the subscript A means that the norm is taken relative to the subspace $\mathfrak{D}(A)$. We remark that an operator $B \in \mathfrak{P}(A)$ may be unbounded and need not even have a closed extension. However, such a B will always have a uniquely defined extension $\bar{B}$ such that $\bar{B}T(\xi; A)$ is linear and bounded on $\mathfrak{X}$ for each $\xi > 0$. If B does possess a closed extension, then condition (ii) is automatically satisfied by the closed graph theorem.

In order to illustrate the above conditions we again consider the diffusion equation (1.5) whose solution is given by a semi-group $[T(\xi)]$ of class (C_0) with infinitesimal generator U defined by (1.6). As is well known, the Weierstrass-Gauss kernel furnishes a representation for this semi-group, namely,

$$(3.1) \quad [T(\xi)f](x) = \frac{1}{2}(\pi\xi)^{-1/2} \int_{-\infty}^{\infty} \exp\left[-(x-y)^2/4\xi\right] f(y)dy, \quad \xi > 0.$$

We now define the linear operator B with domain $\mathfrak{D}(U)$ by

$$(3.2) \qquad [Bf](x) = b(x)f'(x), \qquad f \in \mathfrak{D}(U),$$

where $b \in C[-\infty, \infty]$. According to the relation (2.6)

$$(3.3) \quad \begin{aligned} &[BR(\lambda; U)f](x) \\ &= \frac{1}{2} b(x) \int_{-\infty}^{\infty} \operatorname{sgn}(y-x) \exp\left[-\lambda^{1/2} |x-y|\right] f(y)dy \end{aligned}$$

so that the operator $BR(\lambda; U)$ is bounded for $\Re(\lambda) > 0$. In the present case $\mathfrak{D}(U) \supset \mathfrak{X}_0$ and

$$(3.4) \quad \begin{aligned} &[BT(\xi)f](x) \\ &= \frac{1}{4} b(x)(\pi\xi^3)^{-1/2} \int_{-\infty}^{\infty} (y-x) \exp\left[-(x-y)^2/4\xi\right] f(y)dy. \end{aligned}$$

It is easy to see from (3.4) that

$$(3.5) \qquad \|BT(\xi)\| \leq \|b\|(\pi\xi)^{-1/2}, \qquad \xi > 0,$$

so that $\int_0^1\|BT(\xi)\|d\xi<\infty$. It follows that $B\in\mathfrak{P}(U)$. Thus B is an unbounded perturbing operator for the semi-group (3.1).

Our basic perturbation theorem is

THEOREM 3.1. *Let* $[T(\xi;A)]$ *be a semi-group of class* (1, A) *and suppose that* $B\in\mathfrak{P}(A)$. *Then* $A_1=A+B$ *with domain* $\mathfrak{D}(A_1)=\mathfrak{D}(A)$ *is again the infinitesimal generator of a semi-group of class* (1, A) *and*

$$T(\xi;A_1)=\sum_{n=0}^{\infty}S_n(\xi),\qquad \xi>0, \tag{3.6}$$

where $S_0(\xi)=T(\xi;A)$ *and* $S_n(\xi)x=\int_0^\xi T(\xi-\sigma;A)\bar{B}S_{n-1}(\sigma)x\,d\sigma$; *the series converges absolutely, uniformly with respect to* ξ *in each interval of the form* $(\epsilon,1/\epsilon)$, $0<\epsilon<1$.

The series expansion (3.6) is essentially a power series expansion in B. The interpretation of the terms of this series is of some interest. The zero power term consists of the original semi-group operator at time ξ. The first power term consists of the original semi-group operator up to the time σ, an interaction with B at this time, and this followed by the original semi-group operator acting on the so-perturbed development for the remaining time $\xi-\sigma$; finally the resulting operators are averaged over all times σ between 0 and ξ. The second power term consists of two interactions with B and these averaged over all possible times of occurrence between 0 and ξ; etc. Theorem 3.1 can be proved by means of the corollary to Theorem 2.2.

As a consequence of the above result and our previous calculations, we can now state that the equation

$$\begin{aligned} &\partial u/\partial\xi=\partial^2u/\partial x^2+b(x)\partial u/\partial x, && -\infty<x<\infty,\ \xi>0;\\ &\lim_{\xi\to0+}u(\xi;x)=y_0(x), && y_0\in\mathfrak{D}(U), \end{aligned}\tag{3.7}$$

has a semi-group solution of class (1, A) (actually of class (C_0) as we shall see below); here $b\in C[-\infty,\infty]$.

We shall say that A_1 is *closely related* to A_0 and write $A_1\sim A_0$, if $\mathfrak{D}(A_1)=\mathfrak{D}(A_0)$ and $B=A_1-A_0\in\mathfrak{P}(A_0)$. It can be shown that this is an equivalence relation. Consequently if $[T(\xi;A_1)]$ can be obtained from $[T(\xi;A_0)]$ by a perturbation of the above kind, then $[T(\xi;A_0)]$ can likewise be obtained from $[T(\xi;A_1)]$ by such a perturbation. Moreover, any semi-group attained by a finite sequence of such perturbations can be obtained by a single perturbation of this kind.

Under this equivalence relation the set of all infinitesimal generators of semi-groups of class (1, A) splits up into equivalence classes $[\mathfrak{C}_\alpha]$. *The following stable properties are shared by all members of an equivalence class if they belong to any member of this class.*

(1) *A is a bounded linear operator;*

(2) *A generates a semi-group of class* (C_0);

(3) *A generates a group of linear bounded operators on* $(-\infty, \infty)$;

(4) *$T(\xi; A)$ is continuous in the uniform operator topology for* $\xi > 0$.

Within each equivalence class it is possible to introduce a metric in such a way that the class becomes a complete metric space. To this end we choose a fixed $A_\alpha \in \mathfrak{C}_\alpha$ and set

$$d_\alpha(A_1, A_2) \equiv \int_0^1 \|(A_1 - A_2)T(\xi; A_\alpha)\|_{A_\alpha} d\xi \tag{3.8}$$

for all pairs $A_1, A_2 \in \mathfrak{C}_\alpha$. It can be shown that this function is a suitable metric function and that the resulting topology in $\mathfrak{C}_\alpha$ is actually independent of the particular choice of A_α in $\mathfrak{C}_\alpha$. One can also define an equivalent topology for $\mathfrak{C}_\alpha$ in terms of $R(\lambda; A_\alpha)$ rather than $[T(\xi; A_\alpha)]$ by means of the metric function

$$\Delta_\alpha(A_1, A_2) \equiv \sup_{n \geq 1} \int_0^\infty \frac{\lambda^{n-1}}{(n-1)!} \|(A_1 - A_2)R^{(n)}(\lambda + \omega; A_\alpha)\| d\lambda \tag{3.9}$$

where ω is any fixed number greater than the type of $[T(\xi; A_\alpha)]$.

We can now state the sense in which the generation problem is stable.

THEOREM 3.2. *For the infinitesimal generators of a single equivalence class $\mathfrak{C}_\alpha$, we have*

$$\lim_{A \to A_0} \|T(\xi; A) - T(\xi; A_0)\| = 0$$

uniformly with respect to ξ in each interval of the form $(\epsilon, 1/\epsilon)$, $0 < \epsilon < 1$.

4. **Semi-groups of class** (A). It is of course desirable to obtain criteria for generating semi-groups more general than those of class (1, A). The most far reaching result in this direction is due to W. Feller [3], who has found necessary and sufficient conditions that an operator generate a strongly continuous semi-group of linear bounded operators with range set $\mathfrak{X}_0$ dense in $\mathfrak{X}$; no assumption is made concerning the behavior of the semi-group in the neighborhood of $\xi = 0$. For such semi-groups the operator which corresponds to our infinitesimal generator may have an empty resolvent set. For this and other reasons Feller's criteria can be verified only with difficulty. As a compromise between such extreme generality and semi-groups of

class (1, A) we have proposed the following class.

DEFINITION 4.1. *A semi-group* $\mathfrak{S} = [T(\xi);\ \xi > 0]$ *of linear bounded operators will be said to be of class* (A) *if*

(1) $T(\xi)$ *is continuous in the strong operator topology for* $\xi > 0$;

(2) $\mathfrak{X}_0$ *is dense in* $\mathfrak{X}$;

(3) *there exists a constant* $\omega_1 > \omega_0$ *and a family of linear bounded operators* $[R(\lambda);\ \Re(\lambda) > \omega_1]$ *such that*

(a) $\|R(\lambda)\|$ *is bounded in the half-plane* $\Re(\lambda) > \omega_1$,

(b) $\|R(\lambda)\| = O(1/\lambda)$ *as* $\lambda \to \infty$,

(c) *for each* $x \in \mathfrak{X}_0$

$$R(\lambda)x = \int_0^\infty e^{-\lambda\xi} T(\xi) x\, d\xi, \qquad \Re(\lambda) > \omega_1. \tag{4.1}$$

It follows from the properties already ascribed to semi-groups of class (1, A) that such semi-groups are also of class (A). On the other hand there exist semi-groups of class (A) which are not of class (1, A).[4]

Semi-groups of class (A) satisfy a generalized initial value condition. In fact it can be shown that

$$\lim_{\lambda \to \infty} \lambda R(\lambda) x = x, \qquad x \in \mathfrak{X}. \tag{4.2}$$

Thus $\mathfrak{S}$ is strongly Abel summable to the identity at $\xi = 0$ in this extended sense. The infinitesimal operator A_0 again has a least closed extension A and it turns out that $R(\lambda) = R(\lambda; A)$ for all λ with $\Re(\lambda) > \omega_1$; further $\mathfrak{D}(A) \supset \mathfrak{D}(A_0) \supset \mathfrak{D}(A^2)$. If $x \in \mathfrak{D}(A)$, then

$$dT(\xi)x/d\xi = A_0 T(\xi) x = T(\xi) A x, \qquad \xi > 0; \tag{4.3}$$

and if $x \in \mathfrak{D}(A^2)$, then $\lim_{\xi \to 0+} T(\xi) x = x$. The relations (4.2) and (4.3) show that $T(\xi)y_0$ for $y_0 \in \mathfrak{D}(A)$ satisfies an abstract Cauchy problem. This problem is somewhat more general than that designated earlier as ACP inasmuch as $T(\xi)Ay_0 = y'(\xi; y_0)$ need not be continuous for $\xi \geqq 0$, in fact it need not even be integrable on $(0, 1)$.

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[4] The following semi-group is of class (A) but not of class (1, A). Let $\mathfrak{X}$ consist of all sequence pairs $x = \{(\chi_n, \eta_n);\ n = 1, 2, 3, \cdots\}$ such that $\lim_{n\to\infty} \chi_n = 0$ and $\sum_{n=1}^\infty n^2 |\eta_n|^2 < \infty$ with norm $\|x\| = \sup_n |\chi_n| + [\sum_{n=1}^\infty n^2 |\eta_n|^2]^{1/2}$. The operator $T(\xi)x = x' = \{(\chi_n', \eta_n')\}$ is defined by $\chi_n' = \exp[-(n + ie^n)\xi](\chi_n \cos n\xi - \eta_n \sin n\xi)$, $\eta_n' = \exp[-(n + ie^n)\xi](\chi_n \sin n\xi + \eta_n \cos n\xi)$. It is easy to show that $[T(\xi)]$ is a semi-group of linear bounded operators continuous in the strong operator topology for $\xi > 0$. However, for $x = \{(1/n, 0)\}$ we have $\|T(\xi)x\| \geq c/\xi$ for $0 < \xi < \pi/2$ and hence $[T(\xi)]$ is not of class (1, A). On the other hand, the ultimately zero vectors belong to $\mathfrak{X}_0$ and since these are dense in $\mathfrak{X}$ the same is true of $\mathfrak{X}_0$. Further, for $x \in \mathfrak{X}_0$, the operator $R(\lambda)x = x' = \{(\chi_n', \eta_n')\}$ is given by $\chi_n' = \alpha_n(\lambda)\chi_n - \beta_n(\lambda)\eta_n$, $\eta_n' = \beta_n(\lambda)\chi_n + \alpha_n(\lambda)\eta_n$, where $\alpha_n(\lambda) = [\lambda + n + ie^n][(\lambda + n + ie^n)^2 + n^2]^{-1}$, $\beta_n(\lambda) = n[(\lambda + n + ie^n)^2 + n^2]^{-1}$. From this one shows that $|\alpha_n(\sigma + i\tau)|$, $|\beta_n(\sigma + i\tau)| \leq 1/\sigma$ and $|\beta_n(\sigma + i\tau)|^2 \leq n^2[\sigma^2(\sigma^2 + (\tau + e^n)^2)]^{-1}$ for $\sigma > 0$. These estimates can now be used to obtain (cf. R. S. Phillips [13, p. 349]) $\sum_{n=1}^\infty n^2 |\beta_n(\sigma + i\tau)|^2 \leq c_1\sigma^{-2} + c_2\sigma^{-3}$. Finally a straightforward calculation shows that $\|R(\sigma + i\tau)\| \leq M/\sigma$ for $\sigma > 1$. This proves that $[T(\xi)]$ is of class (A).

For semi-groups of class (A) we have the following generation theorem.

THEOREM 4.1. *A necessary and sufficient condition that a closed linear operator U be the infinitesimal generator of a semi-group of class* (A) *with $\omega_0 < \omega_1$ is that*

(1) *$\mathfrak{D}(U)$ is dense in $\mathfrak{X}$;*

(2) *$\|R(\lambda; U)\| = O(1/\lambda)$ as $\lambda \to \infty$;*

(3) *$\|R(\lambda; U)\|$ is bounded for $\Re(\lambda) > \omega_1$;*

(4) *there exists a non-negative nonincreasing function $\phi(\xi)e^{-\omega_1\xi}$ with* $\limsup_{\xi\to\infty} \xi^{-1} \log \phi(\xi) < \omega_1$ *which satisfies either of the following:*

(a) *for each $x \in \mathfrak{D}(U^2)$ there is a non-negative measurable function $\phi(\xi; x) \leqq \phi(\xi)\|x\|$ such that $\int_0^\infty e^{-\omega_1\xi}\phi(\xi; x)d\xi < \infty$ and $\|R^{(n)}(\lambda; U)x\| \leqq \int_0^\infty e^{-\lambda\xi}\xi^n\phi(\xi; x)d\xi$ for all real $\lambda > \omega_1$ and integers $n \geqq 0$;*

(b) *let $x \in \mathfrak{D}(U^2)$ and $0 < \delta' < \delta$; for each $\epsilon > 0$ there exists a $\lambda_0 = \lambda_0(x; \epsilon, \delta', \delta)$ such that if $\lambda > \lambda_0$ and $n \geqq \lambda\delta$, then*

$$\|(\lambda - \omega_1)^n[R(\lambda; U)]^n x\| \leqq \{e^{-\omega_1\delta'}\phi(\delta') + \epsilon\}\|x\|.$$

Moreover $\|T(\xi)\| \leqq \phi(\xi)$ for all $\xi > 0$.

Condition (4-b) of the above theorem was first used in connection with generation theorems by W. Feller [3]. Actually both (4-a) and (4-b) are difficult to verify in practice. The next theorem gives a much more workable criterion.

THEOREM 4.2. *Let $\psi(\tau)$ be a positive-valued function defined and continuous on $(-\infty, \infty)$ such that*

(a) *$\psi(\tau) \to \infty$ as $|\tau| \to \infty$,*

(b) *$\psi'(\tau)$ is bounded,*

(c) *$\int_{-\infty}^{\infty} \exp[-\xi\psi(\tau)]d\tau < \infty$ for each $\xi > 0$.*

Let ω be a given real constant. If U is a closed linear operator with dense domain such that $R(\lambda; U)$ exists for $\Re(\lambda) \geqq \omega$ and satisfies

(1) *$\|R(\lambda; U)\|$ is bounded for $\Re(\lambda) \geqq \omega$,*

(2) *$\|R(\lambda; U)\| = O(1/\lambda)$ as $\lambda \to \infty$,*

(3) *$\|R(\omega + i\tau; U)\| \leqq M/\psi(\tau)$,*

then U generates a semi-group $[T(\xi)]$ of class (A) *which is differentiable in the uniform operator topology for $\xi > 0$.*

The proof of Theorem 4.2 makes use of contour integration methods developed by E. Hille [4, Theorem 12.5.1].

As a special case of the above theorem we obtain the following criterion. *Let U be a closed linear operator with dense domain. If there exist real constants ω and M such that*

$$(4.4)\qquad \|R(\sigma + i\tau;\, U)\| \leqq M(\sigma - \omega + |\tau|^{\beta})^{-1}, \qquad \sigma > \omega,$$

where $\beta > 0$, then U generates a semi-group of class (A).

5. **Adjoint theory.** The collection $\mathfrak{X}^* = [x^*]$ of complex-valued linear bounded functionals on $\mathfrak{X}$ forms a Banach space in which the vector operations are defined by $(\alpha x^* + \beta y^*)(x) = \alpha x^*(x) + \beta y^*(x)$ and the norm is given by $\|x^*\| = \sup\,[|x^*(x)|;\, \|x\| \leqq 1]$. Likewise the linear bounded functionals on $\mathfrak{X}^*$ form a second adjoint space $\mathfrak{X}^{**}$ and in this way a hierarchy of adjoint spaces can be constructed from the given Banach space $\mathfrak{X}$.

If U is a closed linear operator on $\mathfrak{X}$ to itself with dense domain, an adjoint operator U^* can be defined as follows: The domain $\mathfrak{D}(U^*)$ consists of the set of all $x^* \in \mathfrak{X}^*$ for which there exists a $y^* \in \mathfrak{X}^*$ such that $x^*[Ux] = y^*(x)$ for all $x \in \mathfrak{D}(U)$; we set $U^*x^* = y^*$. It can be shown that U^* is again a closed linear operator. If U is bounded with $\mathfrak{D}(U) = \mathfrak{X}$, then U^* is also bounded with $\mathfrak{D}(U^*) = \mathfrak{X}^*$; in this case a second adjoint operator U^{**} is definable as above and this process can be repeated indefinitely. However, if U is merely closed with $\mathfrak{D}(U)$ dense in $\mathfrak{X}$, $\mathfrak{D}(U^*)$ need not be dense in $\mathfrak{X}^*$ and because of this fact the hierarchy of adjoint operators corresponding to U will in general extend no further than U^*. By way of avoiding this situation, one may limit the adjoint space to $[\mathfrak{D}(U^*)]^-$, which we denote by $\mathfrak{X}^{\odot}$, and take $U^{\odot}$ to be the restriction of U^* having domain $\mathfrak{D}(U^{\odot}) = [x^*; x^* \in \mathfrak{D}(U^*),\, U^*x^* \in \mathfrak{X}^{\odot}]$. Then, supposing $\mathfrak{D}(U^{\odot})$ to be dense in $\mathfrak{X}^{\odot}$, one may proceed to define a second "adjoint" operator $U^{\odot\odot}$ on the subspace $\mathfrak{X}^{\odot\odot} = \mathfrak{D}[(U^{\odot})^*]^-$ of $(\mathfrak{X}^{\odot})^*$. Now if U is such that $\mathfrak{D}(U)$ is dense in $\mathfrak{X}$ and $\|R(\lambda;\, U)\| = O(1/\lambda)$ as $\lambda \to \infty$ then it can be shown that this process can be continued indefinitely and one obtains an adjoint theory relative to U which is an analogue of the usual adjoint theory.

There are two rather natural ways in which this adjoint theory can be applied to semi-groups. Given a semi-group $[T(\xi)]$ of class (A) and with infinitesimal generator A, we can either take the adjoint semi-group to be the semi-group generated by $A^{\odot}$ having as its domain $\mathfrak{X}^{\odot} = [\mathfrak{D}(A^*)]^-$, or we can take it to be the semi-group of adjoint operators $[T^*(\xi)]$ with domain $\mathfrak{X}^*$. When $\mathfrak{X}^{\odot}$ is a proper subspace of $\mathfrak{X}^*$ these two notions are distinct. We have chosen to treat the infinitesimal generator as the primary entity, and from this point of view the proper domain of the adjoint semi-group is $\mathfrak{X}^{\odot}$.

It turns out that the semi-group generated by $A^{\odot}$ is again of class (A) and that the member operators are just the restrictions of the adjoint operators $T^*(\xi)$ to the domain $\mathfrak{X}^{\odot}$. As to the other alternative, while it is true that the adjoint operators $[T^*(\xi)]$ with domain $\mathfrak{X}^*$ form a semi-group, this semi-group will in general not be continuous in the strong operator topology and hence our basic theory would not apply. Actually $\mathfrak{X}^{\odot}$ is the largest closed subspace of $\mathfrak{X}^*$ on which $[T^*(\xi)]$ is strongly continuous and in which the corresponding range space is dense.

The importance of the adjoint semi-group appears to have been first recognized by W. Feller [1] in his treatise on the parabolic differential equation. In the special problem with which he was dealing, he was able to obtain the full adjoint semi-group without employing a precise notion for the adjoint to an unbounded operator. The general treatment sketched above is due to R. S. Phillips [15].

6. **Operational calculus and spectral theory.** The extension of the operational calculus to closed linear operators with nonempty resolvent sets goes back to A. E. Taylor [17], who discovered the basic integral representation for the associated mapping. This calculus can be described as follows:

Let Δ *be an open subset of the extended complex plane not containing the point* α_0, $\alpha_0 \neq \infty$. *Let* $\mathfrak{H}(\Delta)$ *be the complex algebra of all functions* $f(\lambda)$ *locally holomorphic in* Δ *with the sequence topology*: $f_n \to f$ *denoting that* $f_n(\lambda)$ *converges pointwise to* $f(\lambda)$, *uniformly in each compact subset of* Δ. *Further let* $\mathfrak{B}(\Delta)$ *be the algebra of functions* $f(A)$ *with domain* $\mathfrak{G}(\Delta) = [A;\ \sigma_e(A) \subset \Delta]$ *and range* $\mathfrak{E}(\mathfrak{X})$.[5] *Then there exists an isomorphic mapping*: $f(\lambda) \to f(A)$ *of* $\mathfrak{H}(\Delta)$ *into* $\mathfrak{B}(\Delta)$ *such that* (i) $1 \to I$, (ii) $(\alpha_0 - \lambda)^{-1} \to R(\alpha_0;\ A)$, *and* (iii) $f_n \to f$ *implies that* $\|f_n(A) - f(A)\| \to 0$ *for each* $A \in \mathfrak{G}(\Delta)$. *This mapping is unique and is represented by*

$$f(A) = \delta f(\infty) I + \frac{1}{2\pi i} \int_{\Gamma_A} f(\zeta) R(\zeta;\ A)\, d\zeta \tag{6.1}$$

where Γ_A *is an oriented envelope of* $\sigma_e(A)$ *contained in* Δ *and* $\delta = 1$ *or* 0 *according as* Γ_A *contains* $\zeta = \infty$ *in its interior or not.*

By suitably defining a notion of analyticity, it is possible to restrict $\mathfrak{B}(\Delta)$ in the above theorem to functions $f(A)$ which are locally analytic on $\mathfrak{G}(\Delta)$ to $\mathfrak{E}(\mathfrak{X})$.

If we limit our consideration to operators which are infinitesimal generators of semi-groups of class (A), we can devise an operational calculus for this restricted class of operators which, so to speak, permits $\sigma_e(A)$ to have a point on the boundary of Δ. Actually the defining entity, instead of being the domain Δ, is now a submultiplicative function[6] $\phi(\xi)$ of type

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[5] We denote the spectrum of A by $\sigma(A)$ and the extended spectrum of A by $\sigma_e(A)$. If A is bounded $\sigma_e(A) = \sigma(A)$ whereas if A is unbounded $\sigma_e(A) = \sigma(A) \cup \infty$. $\mathfrak{E}(\mathfrak{X})$ denotes the algebra of all linear bounded operators on $\mathfrak{X}$ to itself.

$$\omega_0 = \lim_{\xi\to\infty} \xi^{-1} \log \phi(\xi) > -\infty. \tag{6.2}$$

Each function $f(\lambda)$ is required to be the Laplace-Stieltjes transform of a set function belonging to the algebra $\mathfrak{S}(\phi)$ which is defined as follows:

DEFINITION 6.1. *Let $S(\phi)$ be the family of countably additive complex-valued set functions on the Borel subsets of $[0, \infty)$ such that $\int_0^\infty \phi(\xi)|da| < \infty$. For such a set function*

$$\psi(a; \lambda) = \int_0^\infty e^{\lambda\xi} da \tag{6.3}$$

exists for all λ with $\Re(\lambda) \leqq \omega_0$. Let $Q(\omega_0)$ consist of the elements in $S(e^{\omega_0\xi})$ such that $\psi(a; \omega_0+i\tau) \in L(-\infty, \infty)$ as a function of τ; and let α_0 be a fixed complex number with $\Re(\alpha_0) > \omega_0$. Then

$$\mathfrak{S}(\phi) = S(\phi) + Q(\omega_0) + S(\phi)*r_{\alpha_0} \tag{6.4}$$

where $r_\alpha(E) = \int_E e^{-\alpha\xi} d\xi$ and $*$ denotes convolution.

On introducing a product in $\mathfrak{S}(\phi)$ defined by convolution, $\mathfrak{S}(\phi)$ becomes a subalgebra of $S(e^{\omega_0\xi})$. Further $\mathfrak{S}(\phi)$ is independent of our particular choice of α_0. We note that $S(e^{\omega_0\xi})$ is the adjoint space to the space of continuous functions $f(\xi)$, $\xi \geqq 0$, with $\lim_{\xi\to\infty} e^{-\omega_0\xi}f(\xi) = 0$; the norm being given by $\|f\| = \sup_{\xi\geqq 0}|e^{-\omega_0\xi}f(\xi)|$. Consequently $S(e^{\omega_0\xi})$ has a well defined weak* topology.

The domain of the operational calculus is limited by $\phi(\xi)$ in the following manner.

DEFINITION 6.2. *Let $[T(\xi; A)]$ be a semi-group of class* (A) *and type*

$$\omega_0(A) = \lim_{\xi\to\infty} \xi^{-1} \log \|T(\xi; A)\|, \tag{6.5}$$

and set

$$\omega_1(A) = \inf [\omega; \|R(\lambda; A)\| \text{ bounded for } \Re(\lambda) > \omega > \omega_0(A)]. \tag{6.6}$$

Then if $\|T(\xi; A)\| \leqq \phi(\xi)$ and if $\omega_1(A) < \omega_0$ we say that A is strictly majorized by $\phi(\xi)$ and we denote this relation by $A \prec \phi$. Finally we set $\mathfrak{A}_0(\phi) = [A; A \prec \phi]$.

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[6] We suppose that $\phi(\xi)$ is a real-valued Borel measurable function defined on $[0, \infty)$ such that $0 < \phi(\xi_1+\xi_2) \leqq \phi(\xi_1)\phi(\xi_2)$ for all $\xi_1, \xi_2 > 0$, with $\phi(0) = 1$.

The manner in which the elements of $\mathfrak{S}(\phi)$ operate on A is given by

DEFINITION 6.3. *Let* $\mathfrak{X}_1(A) = [x; \lim_{\xi\to 0+} T(\xi; A)x = x]$. *For* $a \in \mathfrak{S}(\phi)$, $A \in \mathfrak{A}_0(\phi)$, *and* $x \in \mathfrak{X}_1(A)$ *we define*

$$(6.7) \qquad \Psi(a; A)x = \int_0^\infty T(\xi; A)x\,da$$

or, as an alternative,

$$(6.8) \quad \Psi(a; A)x = \frac{1}{2\pi i}(C, 1) - \int_{\gamma - i\infty}^{\gamma + i\infty} \psi(a; \zeta)R(\zeta; A)x\,d\zeta + \frac{1}{2}a([0])x,$$

where $\omega_1(A) < \gamma \leqq \omega_0$.

It can be shown that the relations (6.7) and (6.8) define the same linear bounded operator on $\mathfrak{X}$. We note that in general $\psi(a; \zeta)$ is singular at infinity and that infinity belongs to $\sigma_e(A)$; this can not occur in the calculus associated with the relation (6.1).

We can now state our principal result.

THEOREM 6.1. *Let* $\phi(\xi)$ *be a submultiplicative function of type* $\omega_0 > -\infty$ *and suppose* α_0 *is such that* $\Re(\alpha_0) > \omega_0$. *Let* $\mathfrak{S}(\phi)$ *be the algebra of set functions described in Definition* 6.1 *with a sequence topology*: $a_n \to a$ *denoting that the sequence* $\{a_n\}$ *converges to* a *in the weak* topology of* $S(e^{\omega\xi})$. *Further let* $\mathfrak{B}(\phi)$ *be the complex algebra of functions* $f(A)$ *defined on* $\mathfrak{A}_0(\phi)$ *and having values in* $\mathfrak{E}(\mathfrak{X})$, *the arithmetic operations being defined as in* $\mathfrak{E}(\mathfrak{X})$.

There exists an isomorphic mapping: $a \to \Psi(a; A)$ *of* $\mathfrak{S}(\phi)$ *onto a subalgebra of* $\mathfrak{B}(\phi)$ *such that* (i) *the unit* $e_0 \in \mathfrak{S}(\phi) \to I$; (ii) $r_{\alpha_0} \to R(\alpha_0; A)$; *and* (iii) $a_n \to a$ *implies that* $\Psi(a_n; A)x \to \Psi(a; A)x$ *for each* $x \in \mathfrak{X}_1(A)$. *This mapping is unique and is defined by* (6.7) *or, alternately, by* (6.8) *for each* $x \in \mathfrak{X}_1(A)$.

By limiting the above assertion to the set $\mathfrak{A}_1$ of all infinitesimal generators of semi-groups of class (1, A), rather than of class (A), a notion of analyticity can be introduced so as to make the functions $\Psi(a; A) \subset \mathfrak{B}(\phi)$ locally analytic in $\mathfrak{A}_1(\phi) = \operatorname{Int}[\mathfrak{A}_0(\phi) \cap \mathfrak{A}_1]$.

The extension of the operational calculus to unbounded operator-valued functions appears to be a very promising field for further research. The first results in this direction are to be found in a paper by R. S. Phillips [11]; a more systematic treatment has been given by A. V. Balakrishnan (University of Southern California thesis, 1954). A somewhat different approach to the subject employing the theory of distributions has also been developed by L. Schwartz (unpublished).

Having developed an operational calculus for semi-groups of class (A), we now consider the relation between the spectrum of $\Psi(a;\, A)$ and that of A, where $a \in \mathcal{S}(\phi)$ and $A \prec \phi$. Our results, although more general, follow the same pattern as those previously obtained by E. Hille [4] and R. S. Phillips [10] both as to content and method of proof.

It can be shown for $a \in \mathcal{S}(\phi)$ and $A \prec \phi$ that

$$[\psi(a;\, \sigma(A))]^{-} \subset \sigma[\Psi(a;\, A)], \tag{6.9}$$

where the inclusion can be proper. If the set function or the infinitesimal generator is sufficiently specialized, a more precise spectral mapping theorem can be proved. For instance, if a is an absolutely continuous element of $\mathcal{S}(\phi)$ and $A \prec \phi$ is otherwise arbitrary, then $\psi(a;\, \sigma(A)) \cup 0 = \sigma[\Psi(a;\, A)]$. Likewise if $a \in \mathcal{S}(\phi)$, but is otherwise not restricted, and if $A \prec \phi$ is the infinitesimal generator of a semi-group continuous in the uniform operator topology for $\xi \geqq \gamma > 0$, then $\psi(a;\, \sigma(A)) \cup a(\{0\}) = \sigma[\Psi(a;\, A)]$. In both of these cases even the fine structure of the spectrum is preserved under the mapping.

Of special interest is the relation between the spectrum of $T(\xi;\, A)$ and that of A. In view of the above results one would expect that $\sigma[T(\xi;\, A)]$ would essentially coincide with $\exp\,[\xi\sigma(A)]$ with possibly a $\lambda = 0$ in $\sigma[T(\xi;\, A)]$ being associated with a $\lambda = \infty$ in $\sigma_e(A)$. Actually, aside from the latter possibility, the correspondence for the point and residual spectra is very good. However, the continuous spectrum of $T(\xi;\, A)$ comes in part from the continuous spectrum of A and in part from the limit points of the set $\exp\,[\xi\sigma(A)]$ which are not otherwise accounted for. The continuous spectrum of $T(\xi;\, A)$ may also contain points which have no relation whatever with the points of $\sigma_e(A)$. For example, there exists a strongly continuous group of linear bounded operators $[T(\xi;\, A);\ -\infty < \xi < \infty]$ such that $\sigma(A)$ is empty, $\sigma_e(A)$ containing only the point at infinity; on the other hand $\sigma[T(\xi;\, A)]$ which is obviously nonempty does not contain the only point, namely $\lambda = 0$, that could reasonably correspond to $\sigma_e(A)$.

REFERENCES

1. W. Feller, *The parabolic differential equations and the associated semi-groups of transformations*, Ann. of Math. vol. 55 (1952) pp. 468–519.

2. ———, *Semi-groups of transformations in general weak topologies*, Ann. of Math. vol. 57 (1953) pp. 287–308.

3. ———, *On the generation of unbounded semi-groups of bounded linear operators*, Ann. of Math. vol. 58 (1953) pp. 166–174.

4. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, New York, 1948.

5. ———, *On the generation of semi-groups and the theory of conjugate functions*, Kungl. Fysiografiska Sällskapets I Lund Förhandlingar vol. 21 (14) (1952) pp. 1–13.

6. ———, *Le problème abstrait de Cauchy*, Università di Torino, Rendiconti del Seminario Mat. vol. 12 (1953) pp. 95–103.

7. ———, *The abstract Cauchy problem and Cauchy's problem for parabolic differential equations*, Journal d'Analyse Mathématiques vol. 3 (1954) pp. 83–196.

8. I. Miyadera, *Generation of a strongly continuous semi-group of operators*, Tôhoku Math. J. vol. 4 (2) (1952) pp. 109–114.

9. R. S. Phillips, *On one parameter semi-groups of linear transformations*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 234–237.

10. ———, *Spectral theory for semi-groups of linear operators*, Trans. Amer. Math. Soc. vol. 71 (1951) pp. 393–415.

11. ———, *On the generation of semi-groups of linear operators*, Pacific Journal of Mathematics vol. 2 (1952) pp. 343–369.

12. ———, *Perturbation theory for semi-groups of linear operators*, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 199–221.

13. ———, *An inversion formula for Laplace transforms and semi-groups of linear operators*, Ann. of Math. vol. 59 (1954) pp. 325–356.

14. ———, *A note on the abstract Cauchy problem*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 244–248.

15. ———, *The adjoint semi-group*, to appear in the Pacific Journal of Mathematics.

16. M. H. Stone, *Linear transformations in Hilbert space* III, Proc. Nat. Acad. Sci. U.S.A. vol. 16 (1930) pp. 172–175.

17. A. E. Taylor, *Spectral theory of closed distributive operators*, Acta Math. vol. 84 (1950) pp. 189–224.

18. D. V. Widder, *The Laplace transform*, Princeton University Press, 1941.

19. K. Yosida, *On the differentiability and the representation of one-parameter semi-groups of linear operators*, J. Math. Soc. of Japan vol. 1 (1948) pp. 15–21.

20. ———, *Integration of Fokker-Planck's equation with a boundary condition*, J. Math. Soc. of Japan vol. 3 (1951) pp. 69–73.

21. ———, *On the integration of diffusion equations in Riemann spaces*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 864–873.

The University of Southern California and
Stanford University

THE OCTOBER MEETING IN CAMBRIDGE

The five hundred sixth meeting of the American Mathematical Society was held at the Massachusetts Institute of Technology on Saturday, October 30. There were about 250 persons in attendance including 209 members of the Society.

An address entitled *On the Lebesgue-Stieltjes integral* was presented at the General Session by Professor Arne Beurling of The Institute for Advanced Study by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings. Professor B. O. Koopman presided.

Sessions for contributed papers were held in the morning and afternoon. Professors Eugene Lukacs, J. H. Roberts and Norbert Wiener presided at the morning sessions. Professors J. J. Gergen and R. M. Thrall presided at the afternoon sessions.

Abstracts of the papers presented follow. Where a paper has more than one author, that author whose name is followed by "(p)" presented it. Those papers with "*t*" following their numbers were presented by title. Mr. Mishael Zadek was introduced by Professor Garrett Birkhoff, Mr. D. S. Carter by Professor Joseph Lehner, and Mr. E. H. Brown, Jr., by Professor G. W. Whitehead.

ALGEBRA AND THEORY OF NUMBERS

1. R. A. Beaumont: *On the construction of R-modules.*

Let $S_1, S_2, \cdots, S_k$ be ideals (not necessarily distinct) in an integral domain R with characteristic zero and let G be the direct sum $S_1 \oplus S_2 \oplus \cdots \oplus S_k$ of the additive groups of these ideals. A mapping f of $R \times G$ into G is given by a set of mappings f_i, $i=1, 2, \cdots, k$, of $R \times S_1 \times S_2 \times \cdots \times S_k$ into S_i where $f(r, g) = f(r; s_1, s_2, \cdots, s_k) = (f_1(r; s_1, s_2, \cdots, s_k), \cdots, f_k(r; s_1, s_2, \cdots, s_k))$ for $r \in R$ and $g = (s_1, s_2, \cdots, s_k) \in G$. The following theorem is proved: Let f_i, $i=1, 2, \cdots, k$, be a polynomial function with coefficients in S_i. Then G is a left R-module with respect to the operator multiplication defined by the f_i if, and only if, each f_i is defined by $f_i(r; s_1, s_2, \cdots, s_k) = \sum_{j=1}^{k} a_j^{(i)} r s_j$, $a_j^{(i)} \in S_i$, and the matrix $A = (a_j^{(i)})$, $i, j = 1, 2, \cdots, k$, is idempotent. (Received September 16, 1954.)

2*t*. H. W. Becker: *Desboves transforms for Pythagorean tetrahedrons: I, II.*

A PT has $u^2+s^2=v^2+x^2=t^2$, $u^2-x^2=v^2-s^2=y^2$; $u, x, y, v, s, y, t = il \pm jk, ik \pm jl, 2(ijkl)^{1/2}, [(i^2+j^2)(k^2+l^2)]^{1/2}$. Dickson's *History* II, p. 632, Desboves[60] (6) has a misprint: the exponent 2 in $Zx'^4y'^2$ should be deleted. Adapting this to PT, the transforms D_{I} and D_{II} are: $i, j; k, l \to (ks^2-lw)^2/i$, $(ls^2-kw)^2/j$; $(is^2-jw)^2/k$, $(js^2-iw)^2/l$, and $(lx^2-kw)^2/i$, $(kx^2-lw)^2/j$; $(jx^2-iw)^2/k$, $(ix^2-jw)^2/l$, where $w=uv+ty$, and i, j and k, l correspond to Desboves' x^2, y^2. Thus 169, 4; 4, 1→262238², 220169²; 859², 1186², and 12134², 2297²; 43², 38². Since i, j and k, l and their transforms mutually tetradize, each $D_{\rm I}$, $D_{\rm II}$ pair yields 14 new PT. Thus the tetrad $43^2, 38^2; 4, l$ gives the 18th known sub-miniature PT ($t<10^4$): 7625, 3927, 8840, 5952, 6536, 9673. A. Cunningham (ibid. p. 634) "listed all $a^4+b^4=c\square<10^7$." But $43^4+38^4=17\cdot569^2$, not in his list. Put $2s[wij/(k^2+l^2)]^{1/2}=r$, $2s[wkl/(i^2+j^2)]^{1/2}=s$, then $i^2+j^2\to(i^2+j^2)[(ks^2+lw+js)^2/i^2+s^2]$, under $D_{\rm I}$, and $k^2+l^2\to[(ir^2+jw+lr)^2/k^2+r^2]\cdot(k^2+l^2)$. There are analogous transforms $\Delta_{\rm I,II}$ for the parameters such that $t, u, v, y, z, x=\iota\kappa\pm\theta\lambda, \iota\lambda\pm\theta\kappa, 2(\iota\theta\kappa\lambda)^{1/2}, [(\iota^2-\theta^2)(\kappa^2-\lambda^2)]^{1/2}$, viz. $\iota, \theta; \kappa, \lambda\to(\kappa\sigma^2+\lambda\omega)^2/\iota, (\lambda\sigma^2-\kappa\omega)^2/\theta; (\iota\sigma^2+\theta\omega)^2/\kappa, (\theta\sigma^2-\iota\omega)^2/\lambda$ etc., $\omega=uv+xz$. Thus 49, 32; 9, $8\to31^2, 8\cdot2^4; 9\cdot11^4, 8\cdot19^2$, tetradizable also. The Lenhart[46] PT (ibid. p. 506–507) is a transform of the Desboves[63] PT (p. 632); misprint, $c=81a^2x^6-\cdots$ should be $81a^2x^8$. A third misprint on this page is in Pepin[67]: the last term should be by^2 instead of bz^2. (Received September 14, 1954.)

3t. H. W. Becker: *Desboves transforms for Pythagorean tetrahedrons*: III, IV. Preliminary report.

A PT has $u, x, v, z, y, t=ef(g^2\pm h^2), gh(e^2\pm f^2), 2efgh, [(e^2g^2+f^2h^2)(e^2h^2+f^2g^2)]^{1/2}$. Desboves[123] (Dickson's *History* II, p. 637) gave a transform on the general binary quartic without odd powers $=c\square$. Changing the notation, and adapting to the above quartic $=t^2$, one gets the transform $D_{\rm III}$: $e, f, g, h, t\to(t^2y^2f^2-v^2z^2e^2)/e, (t^2y^2e^2-v^2z^2f^2)/f, g, h, t\{4u^2v^2x^2z^2-(u^2v^2+t^2z^2)^2\}$. Similarly, under $D_{\rm IV}$: $e, f, g, h, t\to e, f, (t^2y^2h^2-u^2x^2g^2)/g$ $(t^2y^2g^2-u^2x^2h^2)/h, t\{(2uvxz)^2-(u^2v^2+t^2x^2)\}$. Cancelling g^2h^2, under $D_{\rm III}$ $e, f\to e\{4f^4t^2-vz(e^4-f^4)\}, f\{4e^4t^2-vz(e^4-f^4)\}$, so each magnitude of a PT divides its transform. In a Carmichael vector, the above factors of t^2 are $\boxplus$. The smallest example has $e, f, g, h, t=10, 6, 7, 4, 1073\to10\cdot35254193, 6\cdot661577169, 7, 4, (29\cdot479689681)(37\cdot217141681)$ under $D_{\rm III}$, which in this instance transforms a CV into a CV. In terms of the Lebesgue transforms, this Bulletin vol. 59 (1953) p. 148, $t\to(X_\Lambda^2u^2-T_\Lambda^2t^2)/t$ or $(Z_L^2v^2-T_L^2t^2)/t$ under $D_{\rm III}$ or $D_{\rm IV}$. A PT also has $v, y, t, u, z, x=\epsilon\zeta(\eta^2\pm\theta^2), \eta\theta(\epsilon^2\pm\zeta^2), 2\epsilon\zeta\eta\theta, [(\epsilon^2\eta^2-\zeta^2\theta^2)(\epsilon^2\theta^2-\zeta^2\eta^2)]^{1/2}$. There are analogous transforms $\Delta_{\rm III,IV}$, transliterating as ordered. (Received September 14, 1954.)

4t. H. W. Becker: *Desboves transforms for Pythagorean tetrahedrons*: V. Preliminary report.

The Desboves[79] (4), the Desboves[123] ($d=0$), the Carmichael[163], and Rignaux[164] transforms on quartics are all the same (refs. to Dickson's *History* II, chap. XXII). The identities are $I=i\{4i^4-3(i^2+j^2)^2\}^2=i\{(i^2-j^2)^2-4j^2(i^2+j^2)\}^2=i(i^4-6i^2j^2-3j^4)^2$. They underlie the PT of Euler[304] (ii), which is no. V of Rignaux, L'Int. des Math. vol. 26 (1919) pp. 55–57. The transform $D_{\rm V}$ on PT is $i, j; k, l\to I$ above, $J=j(3i^4+6i^2j^2-j^4)^2$; $K=k(k^4-6k^2l^2-3l^4)^2$, $L=l(3k^4+6k^2l^2-3l^4)^2$. Then $t\to T=t\prod_{i,j;k,l}(m^8+28m^6n^2+6m^4n^4+28m^2n^6+n^8)$, and $y\to Y=y\prod_{i,j;k,l}(m^4-6m^2n^2-3n^4)(3m^4+6m^2n^2-n^4)$. If T' and T'' are the homogeneous tetrads between t and T, that is the longest hypotenuses of PT with parameters $i, j; I, J$ and $k, l; K, L$, then $T=T'T''/t$; similarly $Y=Y'Y''/y$. Also x, z, u, v divide their respective transforms X, Z, U, V. Simplest example: $13^2, 4; 4, 1\to13^2\cdot812988097^2, 4\cdot2449933763^2; 4\cdot157^2, 863^2$. Under the dual transform $\Delta_{\rm V}$: $\iota, \theta, \kappa, \lambda\to\iota(\iota^4+6\iota^2\theta^2-3\theta^4)^2$, etc. and $x\to X=X'X''/x=x\prod_{\iota,\theta;\kappa,\lambda}(\mu^8-28\mu^6\nu^2+6\mu^4\nu^4-28\mu^2\nu^6+\nu^8)$, $z\to Z=Z'Z''/z=z\prod_{\iota,\theta;\kappa,\lambda}(\mu^4+6\mu^2\nu^2-3\nu^4)(3\mu^4-6\mu^2\nu^2-\nu^4)$. A Carmichael vector depends on $abcd(a^2-b^2)(c^2-d^2)=\square=y^2/16$. Call a Carmichael transform $C_{\rm V}$: $a, b; c, d\to a(a^4+6a^2b^2-3b^4)^2, b(3a^4-6a^2b^2-b^4)^2$; etc. Then applied to a given CV, $C_{\rm V}=P\Delta_{\rm V}$, where P denotes a Petrus transform. Simplest example: $5, 2; 6, 1\to5\cdot1177^2, 2\cdot1259^2; 6\cdot1509^2, 3671^2$. (Received September 14, 1954.)

5*t*. Eckford Cohen: *The quadratic singular sum.*

Let A be an arbitrary *odd* ideal of a finite extension F of the rational field. Denote by $r_s(\rho)$ the number of solutions of the congruence $\rho \equiv \alpha_1 X_1^2 + \cdots + \alpha_s X_s^2 \pmod{A}$, where $\rho, \alpha_1, \cdots, \alpha_s$ are integers of F, $(\alpha_i, A) = 1$, $s \geqq 1$. In this paper, the function $r_s(\rho)$ is expressed in terms of a "singular sum" involving Hecke exponentials, and is then evaluated completely in terms of numerical functions over F. A number of special results are deduced; for example, if A is square-free, then $r_{2m+1}(\rho, A) = N^{2m}(A) \sum_{D|A} (-\alpha\rho/D)1/N^m(D)$, where $N(D)$ denotes the norm of D, (β/D) is the Jacobi symbol in F, and $\alpha = (-1)^{m+1}\alpha_1 \cdots \alpha_{2m+1}$. The results obtained generalize theorems proved by the author in the rational case (Duke Math. J. vol. 21 (1954) pp. 9–28) and in the case of congruences to a prime-power ideal modulus (Trans. Amer. Math. Soc. vol. 75 (1953) pp. 444–470). (Received September 16, 1954.)

6. C. W. Curtis: *Lie algebras of algebraic linear transformations.*

Unless otherwise stated, it is assumed that all fields considered have characteristic zero. It is proved first that every locally algebraic linear transformation (l.t.) X in a vector space M of countable dimension over a perfect field K can be expressed uniquely in the form $S+N$, where S is locally algebraic and semi-simple, and N is locally nilpotent. Both S and N coincide, on finite-dimensional subspaces of M, with polynomials in X. Now let L be a locally finite Lie algebra of algebraic l.t. in an arbitrary vector space M whose enveloping algebra contains no nonzero nil ideals. Then L can be expressed as a direct sum of its center C and an ideal L_1 containing $[L, L]$, and possessing no nonzero solvable ideals. The l.t. in C are all semi-simple. This result can be used to prove a generalization of Lie's theorem, namely that every solvable Lie algebra of algebraic l.t. whose enveloping algebra contains no nonzero nil ideals is commutative. It is proved also that every solvable algebra of algebraic l.t. is locally finite. Finally the concept of a linearly splittable solvable algebra of algebraic l.t. is introduced, making use of the decomposition theorem for algebraic l.t. stated above for locally algebraic l.t., and some theorems are proved generalizing certain results obtained by Malcev (*Solvable Lie algebras*, Izvest. Akad. Nauk SSSR, Ser. mat. vol. 9 (1945) pp. 329–352). (Received September 16, 1954.)

7. Walter Feit: *On a conjecture of Frobenius.*

Let $\mathfrak{G}$ be a group of order $g = mq$, with $(m, q) = 1$, and let $\mathfrak{M}$ be the set of all elements in $\mathfrak{G}$ whose order divides m. Frobenius conjectured that if $\mathfrak{M}$ contains exactly m elements it is a normal subgroup of $\mathfrak{G}$. He was able to prove this under the additional assumptions that $\mathfrak{G}$ contains a subgroup $\mathfrak{Q}$ of order q which is disjoint from all its conjugates and is its own normalizer. The theorem is proved by the author without the assumption that $\mathfrak{Q}$ is its own normalizer. The proof is very different from the proof Frobenius gave of his theorem; the basic tool is R. Brauer's theorem on the characterization of characters. With the help of this a representation of $\mathfrak{G}$ is constructed whose kernel is $\mathfrak{M}$. If one assumes that $\mathfrak{Q}$ is solvable, then it is possible to prove the theorem without the use of character theory. (Received September 15, 1954.)

8. N. J. Fine and Bertram Kostant (p): *The group of formal power series under iteration.*

The authors study the topological group, under iteration, of formal power series $a_1x + a_2x^2 + \cdots$, a_n complex, $a_1 \neq 0$ (component-wise convergence), principally by means of its "Lie algebra" $\mathfrak{g}$. The elements of $\mathfrak{g}$ are represented as operators $\phi(x)d/dx$ (cf. S. A. Jennings, Canadian Journal of Mathematics vol. 6 (1954)), and correspond under exponentiation to one-parameter groups. Following are some results: (1) Classification of all one-parameter groups (real or complex), and of elements on and off them. (2) Determination of centralizers for elements and one-parameter groups. (3) Determination of all ideals in $\mathfrak{g}$ and of all normal subgroups of G. (4) There exists a 1-1 correspondence between closed connected subgroups of G and closed subalgebras of $\mathfrak{g}$. (5) Complete solution of conjugacy problem, generalizing the Schroeder equation (cf. C. L. Siegel, Ann. of Math. vol. 43 (1942)). (6) For any representation of G into a Lie group (there exist sufficiently many), closed connected subgroups go into analytic groups. (7) Every connected Lie group imbedded in G is isomorphic to a subgroup of the complex $ax+b$ group. (8) Determination of all nth roots of any element. (9) Determination of all homomorphisms of G into itself; the automorphism group is generated by inner automorphisms and complex conjugation. (10) If $X=\phi(x)d/dx$, ϕ with positive radius of convergence, then $\exp tX$ also has positive radius of convergence. (Received September 15, 1954.)

9. I. N. Herstein: *A theorem concerning three fields.*

The following theorem is proved: let $L \supset K \supset F$ be three fields where the containing relations are all proper; suppose further that for every x in L there exists a nontrivial polynomial $f_x(t)$ in the variable t with coefficients coming from F and which depend on x, such that $f_x(x)$ is in K. Then either L is purely inseparable over K, or L, and so K, is algebraic over F. This is closely related to recent results of Herstein, Kaplansky, Krasner and Nagata, Nakayama, and Tuzuku. (Received August 26, 1954.)

10t. Bjarni Jónsson: *Distributive sublattices of a modular lattice.*

If X is a nonempty subset of a modular lattice A, then each of the following conditions implies that X generates a distributive sublattice of A: 1. $(\sum_{i=1}^{m} x_i)\prod_{j=1}^{n} y_j = \sum_{i=1}^{m} (x_i \prod_{j=1}^{n} y_j)$ whenever m and n are positive integers and $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n \in X$. 2. X consists of four elements x, y, z, and u such that $(y+z)u = yu+zu$, $(z+u)x = zx+ux$, $(u+x)y = uy+xy$, $(x+y)z = xz+yz$, $(x+y+z)u = xu+yu+zu$, $xyz+u = (x+u)(y+u)(z+u)$. 3. X is the union of nonempty chains $X_1, X_2, \cdots, X_p$ such that, for every $x_1 \in X_1$, $x_2 \in X_2, \cdots, x_p \in X_p$, the set $\{x_1, x_2, \cdots, x_p\}$ generates a distributive lattice. 4. $X = B \cup C$, where B and C are distributive sublattices of A such that $(b_1+b_2)c = b_1c+b_2c$ and $(c_1+c_2)b = c_1b+c_2b$ whenever b, b_1, $b_2 \in B$ and c, c_1, $c_2 \in C$. (Received September 9, 1954.)

11t. Joachim Lambek: *Initial segments of positive semigroups.*

In a recent paper [*What is an angle?* Amer. Math. Monthly vol. 61 (1954) pp. 369–378] H. Zassenhaus abstractly characterized the additive system of all reals and all integers whose absolute value does not exceed a given bound. In the present paper, necessary and sufficient conditions are obtained, under which a set M with a partial binary operation $+$ is isomorphic with an initial segment of the positive reals or positive integers. To fix matters, assume that this segment has a largest element. The conditions then are: P1. $(x+y)+z = x+(y+z)$ if either is defined. P2. $x \neq y$ if and only if $x \in M+y$ or $y \in x+M$. P3. If $X \subseteq M$ then $X+M = M$ or $= x+M$ for some $x \in M$. P1 suffices to embed M in a semigroup S; P2 is used to extend this to a simply ordered group G with positive part S; P3 shows that G is complete. (Received September 15, 1954.)

12*t*. Joachim Lambek and Leo Moser: *On integers n relatively prime to $f(n)$.*

It is well known that, if m and n are integers chosen at random, the probability that they are relatively prime is $6/\pi^2$. This result may still hold if m and n are functionally related. Given a sequence $f(n)$ of non-negative integers, let $Q_f(x)$ be the number of $n \leq x$ with $(n, f(n)) = 1$. G. L. Watson has recently shown that (1) $\lim_{x\to\infty} Q_f(x)/x = 6/\pi^2$, when $f(n) = [\alpha n]$ for irrational α [Canadian Journal of Mathematics vol. 5 (1953) pp. 451–455]. Let $f^*(n)$ be the number of m for which $f(m) = n$, and suppose f and f^* are nondecreasing. It is then proved that (2) $Q_f(x) = 6x/\pi^2 + O((f(x) + f^*(f(x))) \log f(x) + x/f(x))$. Thus, for example, if $f(x) = [x^{1/2}]$, then $Q_f(x) = 6x/\pi^2 + O(x^{1/2} \log x)$. Conditions on f are obtained which imply (1), and their necessity and independence is discussed. (Received July 26, 1954.)

13*t*. R. C. Lyndon: *Representation of relation algebras*. II.

The investigation begun in RRA (Lyndon, Ann. of Math. vol. 51 (1950) pp. 707–729) is extended, and an error detected by Alfred Tarski and Dana Scott is corrected. Contrary to Theorem IV of RRA, Tarski has shown that the class of representable relation algebras is definable by universal sentences. His proof provides no construction for such a set of axioms. An extension of the method of RRA yields explicitly such a set of axioms. The mistaken result of RRA is reinterpreted correctly in terms of strong representations. The chief new tool is the consideration of divisible properties, dual to local properties, and exploitation of the known fact that the Stone-Jónsson-Tarski completion $\bar{A}$ is compact under the topology defined by A. (Received July 30, 1954.)

14. Irving Reiner: *Maximal sets of involutions.*

Let U_n denote the unimodular group; a maximal size abelian subgroup of involutions in U_n is called briefly a "maximal set." Then every maximal set has 2^n elements, given by the collection $\{MDM^{-1}\}$, where D ranges over all 2^n diagonal matrices having ± 1's as diagonal elements, and where M is a certain integral matrix. Maximal sets generated by M and M_1 are conjugate in U_n if and only if there is a relation of the form $M_1 = AMB$, where $A \in U_n$ and where B permutes columns, possibly changing their signs. A canonical form for M under $M \to AMB$ is obtained; this leads to a discussion of the nonconjugate maximal sets, and permits a group-theoretic characterization of the elements $\pm[(-1) + I^{(n-1)}] \in U_n$ up to conjugacy. The basic tool used is the connection between an involution W and the matrix whose first p columns form an integral basis for the lattice vectors in $W^+ = \{x : Wx = x\}$, and whose remaining q columns form such a basis for $W^- = \{x : Wx = -x\}$. (Received September 15, 1954.)

15. W. F. Reynolds: *On finite groups related to permutation groups of prime degree.*

Let $\mathfrak{G}$ be a finite group such that (1) $\mathfrak{G}$ contains an element of prime order p which commutes only with its own powers, and (2) $\mathfrak{G}$ is its own commutator group. Then the order g of $\mathfrak{G}$ is of the form $g = pq(np+1)$, where $np+1$ is the number of p-Sylow subgroups of $\mathfrak{G}$, and q divides $p-1$. In Ann. of Math. vol. 44 (1943) pp. 57–79, R. Brauer listed all such groups for which $n < (p+3)/2$; in unpublished work he has extended this enumeration to $n \leq p+2$. It is now shown that no further such groups are encountered for $p+2<n<2p-3$; while for $n=2p-3$, the group $LF(2, 2p-1)$ occurs whenever $2p-1$ is a prime power. (This is an abstract of the writer's doctoral dissertation, written at Harvard University under the direction of Professor Richard Brauer.) (Received September 13, 1954.)

16. R. D. Schafer: *Noncommutative Jordan algebras of characteristic* 0.

The algebras satisfying $(xa)x=x(ax)$ and $(x^2a)x=x^2(ax)$ are here called noncommutative Jordan algebras. They are identical (for characteristic $\neq 2$) with flexible Jordan-admissible algebras, and they include such well-known algebras as Jordan, alternative, quasiassociative, and—trivially—Lie algebras. For characteristic 0 they are proved trace-admissible, and it follows from Albert, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 317–322, that any semisimple algebra (that is, an algebra having maximal nilideal $=0$) is a direct sum of simple ideals which are (a) (commutative) Jordan algebras, (b) flexible algebras of degree two, or (c) quasiassociative algebras. (Received August 18, 1954.)

17. I. M. Singer: *Automorphisms of finite factors.*

Let M be a finite factor of the type constructed by Murray and von Neumann (Ann. of Math. vol. 37 (1935) pp. 192–209) when a discrete group G acts ergodically on a measure space X. The subgroup of the automorphism group of M leaving the maximal abelian algebra of multiplications setwise invariant is analyzed. This subgroup decomposes into a semidirect product of two groups K and S. The normal subgroup K can be described as follows: The action of G on X induces an action of G on the group of measurable functions on X of absolute value 1. K is the group of 1-cocycles relative to this action, and K modulo the inner automorphisms in K is the first cohomology group. (See Eilenberg and MacLane, Ann. of Math. vol. 48 (1947) pp. 51–78.) The group S can be identified with a subgroup of the group of all measure preserving transformations on X and contains the normalizer of G. The above analysis exhibits many outer automorphisms. (Received September 15, 1954.)

18*t*. Michio Suzuki: *On finite groups with cyclic Sylow subgroups for all odd primes.*

Let G be a finite group with cyclic Sylow subgroups for all odd primes. The following results are proved. If G is simple, and if 2-Sylow subgroups are of dihedral type, then G is isomorphic with $LF(2, p)$. If G is nonsolvable and if 2-Sylow subgroups are generalized quaternion groups, then G contains a normal subgroup of index 1 or 2, which is a direct product of $SL(2, p)$ and a solvable group. Together with the results of Zassenhaus (Abh. Math. Sem. Hamburgischen Univ. (1936)), we have a complete classification of finite groups, all of whose abelian subgroups are cyclic. The theory of group characters is used in the proof. (Received September 16, 1954.)

19. George Whaples: *The generality of local class field theory.*

Call a field an r.r.f. if it satisfies the axioms for a residue class field of a field over which local class field theory holds—i.e. if (1) it has no inseparable extension and (2) it has for each n exactly one extension of degree n (in its algebraic closure). It is proved: 1. Every field of prime characteristic is contained in an r.r.f. 2. Every absolutely algebraic field of prime characteristic is maximal absolutely algebraic subfield of some r.r.f. (Received September 27, 1954.)

ANALYSIS

20. Joseph Andrushkiw: *Probability of roots of a real quadratic equation whose absolute value is less than a positive constant c.*

A real quadratic equation $xz^2+wz+y=0$ has real roots in the interval $(-c, c)$, if and only if the coefficients x, y, w are coordinates of a point which lies outside or in the surface of the cone $w^2-4xy=0$ and between the tangent planes $c^2x+cw+y=0$ and $c^2x-cw+y=0$. Denoting by v the volume of the portion of the cube $-k\leqq x\leqq k$, $-k\leqq y\leqq k$, $-k\leqq w\leqq k$ which lies outside the cone and between the above tangent planes, the author defines the probability that the quadratic equation has real roots in $(-c, c)$ by $P_r=v/8k^3$. Thus one obtains: 1. $0\leqq c\leqq 1/2$, $P_r=c^3/9$; 2. $1/2\leqq c\leqq 1$, $P_r=(11+6\ln 2c)/144+(c-c^2)/4$; 3. $1\leqq c\leqq (5^{1/2}+1)/2$, $P_r=(11+6\ln 2c)/144-(c^6-3c^4+3c^2-1)/12c^3$; 4. $(5^{1/2}+1)/2\leqq c\leqq 2$, $P_r=(71+6\ln 2c)/144-(c+1)/4c^3$; 5. $2\leqq c<\infty$, $P_r=(41+6\ln 2)/72-(9c^2+c)/18c^3$. The roots are imaginary and their absolute value is less than c if and only if the coefficients x, y, w are coordinates of a point which lies inside the cone between the plane $y=c^2x$ and the cone. Defining the probability in similar way one finds: 1. $0\leqq c\leqq 1/2$, $P_i=(2c^3)/9$; 2. $1/2\leqq c\leqq 1$, $P_i=(36c^2-5-6\ln 2c)/144$; 3. $1\leqq c\leqq 2$, $P_i=(67-6\ln 2c)/144-14c^2$; 4. $2\leqq c<\infty$, $P_i=(31-6\ln 2)/72-2/9c^3$. The probability P of roots of a real quadratic equation whose absolute value is less than c is given by $P=P_r+P_i$. (Received August 31, 1954.)

21. D. G. Austin: *On* $\lim_{h_n\to 0}(f(x+h_n)-f(x))/h_n$ *for approximately derivable functions.*

A result of Auerbach (Fund. Math. vol. 11 (1928) p. 197) is generalized to apply to difference quotients. It is shown that if $f(x)$ has a finite approximate derivative almost everywhere on a measurable set K, then there is a sequence of positive numbers δ_n with $\lim_n \delta_n=0$ such that for any sequence h_n with $|h_n|\leqq\delta_n$ one has $\lim_n (f(x+h_n)-f(x))/h_n$ equal to the approximate derivative of $f(x)$ for almost all x in K. (Received September 16, 1954.)

22. W. G. Bade: *On completeness of Boolean algebras of projections in Banach spaces.*

Let X be a real or complex Banach space and $\mathfrak{B}$ be a Boolean algebra (B.A.) of projections in X. $\mathfrak{B}$ will be called *complete* (*σ-complete*) if for each family (sequence) $\{E_\alpha\}\subseteq\mathfrak{B}$, $X=\mathfrak{M}\oplus\mathfrak{N}$ where $\mathfrak{M}=\text{clm}\,\{E_\alpha X\}$, $\mathfrak{N}=\bigcap_\alpha(I-E_\alpha)X$, and the resulting projecting is in $\mathfrak{B}$. $\mathfrak{B}$ may be complete (σ-complete) as an abstract B.A. but not in the stronger sense above. Results: (1) if $\mathfrak{B}$ is σ-complete as an abstract B.A., then $\mathfrak{B}$ is bounded, i.e. $|E|\leqq M$, $E\in\mathfrak{B}$. (2) If $\mathfrak{B}$ is σ-complete it may be imbedded in a smallest complete B.A., the closure of $\mathfrak{B}$ in the strong operator topology. (3) The algebra generated in the uniform topology by a complete B.A. of projections is weakly closed. (Received September 14, 1954.)

23. R. G. Bartle (p), Nelson Dunford, and J. T. Schwartz: *Weak compactness and vector measures.*

Let $ca(\Sigma)$ be the Banach space of all finite countably additive (scalar) measures on a σ-field Σ, with the total variation as norm. (1) The following are equivalent for bounded subsets of $ca(\Sigma)$: (a) conditional weak*compactness, (b) uniform countable additivity, (c) uniform absolute continuity with respect to some positive measure.

A function $\mu:\Sigma\rightarrow X$, a Banach space, is a *vector measure* if $x^*\mu\in ca(\Sigma)$, $x^*\in X^*$. (2) A vector measure maps into a weakly compact set. (3) A Lebesgue theory of integration of (unbounded) scalar functions with respect to a vector measure is developed. (4) If S is compact Hausdorff, an operator $T:C(S)\rightarrow X$ is weakly compact iff there is a vector measure μ on the Borel sets of S to X such that $T_f=\int f(s)\mu(ds)$. (5) T is compact iff μ maps into a compact set. Simple measure-theoretic arguments are used to derive some recent theorems of A. Grothendieck. (Received August 30, 1954.)

24. F. E. Browder: *Asymptotic distribution of eigenvalues and eigenfunctions for the general self-adjoint elliptic boundary value problem.*

Let K be a self-adjoint linear elliptic differential operator of order $2m$ on a bounded open set D of Euclidean n-space with an m-times differentiably smooth boundary. Let $a(x, \xi)$ be the characteristic form of K, $\rho(x)=\int_{a(x,\xi)<1}d\xi$. It is established for a general class of boundary value problems previously defined by the author which includes the classical Neumann and Robin problems for second-order equations, that if $\{\phi_k\}$ is a complete orthonormal sequence of eigenfunctions of $(-1)^mK$ arranged in nondecreasing order of the corresponding (real) eigenvalues $\{\lambda_k\}$, then $N(t)=\sum_{\lambda_k<t}1$ $\sim(2\pi)^{-n}(\int_D\rho(x)dx)t^{n/2m}$, while $N^{-1}\sum_{k=1}^{N}\phi_k(x)\phi_k(y)\rightarrow\delta_{x,y}\rho(x)(\int_D\rho(x)dx)^{-1}$ as $N\rightarrow\infty$ for x, y in D. Analogous results are obtained for strongly elliptic systems of differential equations. (Received September 16, 1954.)

25. D. S. Carter: *On a class of minimum-maximum problems.*

Let Y be the class of absolutely continuous real n-component functions $y(t)$ defined on an interval I, which assume given values at both end points of I, and such that a given n-component differential expression $F(t, y, \dot{y})$ has a finite maximum-norm $\langle F\rangle=\sup_{i=1}^{n}$ (ess. $\sup_{t\in I}|F_i|$). The problem concerns the existence and uniqueness of a $y_0\in Y$ which minimizes $\langle F\rangle$. The chief result is for the linear case $F=A(\dot{y}+By+c)$. Under rather weak summability conditions on the matrices A, B and the vector c, there exists a minimizing function $y_0(t)$, which can be found by maximizing an auxiliary function of $n-1$ variables. The components of $F_0=F(t, y_0, \dot{y}_0)$ are uniquely determined, and equal in absolute value to $\langle F_0\rangle$, on certain subsets J_{i0} of I, at least one of which is of positive measure. Similar results are obtained in case the class Y is restricted by requiring the components of F to vanish a.e. on given subsets of I. These facts shed some light on the solution of the nonlinear case. (Received September 16, 1954.)

26*t*. Sarvadaman Chowla: *Concerning the zeros of zeta functions.*

Solving a problem suggested by Apostol, the author obtains an infinite series for the smallest complex zero of Riemann's zeta function $\zeta(s)$. (Received September 17, 1954.)

27*t*. R. B. Davis: *The fourth boundary value problem for third order composite equations.*

Previous work on the fourth boundary value problem has depended upon classical results concerning the behavior of the derivatives of Green's functions near the boundary of the region. The validity of these results is open to question. In the present paper, the fourth boundary value problem is established for a special third order composite equation, for the case where the region is a circle. In this special case it is possible to avoid the use of the doubtful results just mentioned. Moreover, for such a special case one can give reasonably strong uniqueness theorems. The method used is a modification of a method due to Lichtenstein (Math. Ann. vol. 67 (1909) pp. 559–575). (Received September 15, 1954.)

28. R. E. Gomory: *Trajectories tending to a critical point in 3-space.*

Consider the system $dv/dt = F(v)$ where F is a real vector. Suppose a solution $v(t)$ tends to a critical point P as $t \to \infty$. If v is a 2-vector it is well known that as $v(t)$ approaches P it either spirals infinitely often around P, or else tends to P asymptotically tangent to some line through P. However, if v is a 3-vector the behavior can be much more varied. This behavior is investigated by projecting the motion $v(t)$ onto a unit sphere D around P and studying the limit sets of the projected motion. These limit sets are not arbitrary point sets, but turn out to be the union of trajectories of a certain autonomous 2-dimensional system defined on D. By analysing this 2-dimensional system a series of results is obtained about $v(t)$. For example it is proved that the limit set of the projected motion either contains critical points of the 2-dimensional system, or else is a closed curve. This means for $v(t)$ itself that either it returns infinitely often arbitrarily close to certain special lines through P, or else it tends to P spiraling asymptotically to a cone. (Received September 10, 1954.)

29t. R. E. Gomory and Felix Haas: *Trajectories near limit cycles in three-space.*

An autonomous system of three differential equations with analytic right-hand side is studied. The manner in which a solution curve can tend toward a limit cycle of such a system is investigated. It is shown that under fairly general hypothesis either of two cases holds: (1) The solution curve approaches a limiting direction in each surface of section to the limit cycle. (2) All directions in every surface of section are approached by the solution curve arbitrarily late in its path and hence arbitrarily close to the limit cycle. The main difficulty which had to be overcome was that the shadow of the solution curve on a torus surrounding the limit cycle corresponded to a non-unique trajectory, which prevented a straight application of the Denjoy theory. However, another torus was constructed by expanding the limit cycle on which there were unique trajectories. It is shown that the limit set of the shadow on the first torus is the same as the limit set which the solution curve approaches on the second torus. This second limit set is then studied. (Received September 8, 1954.)

30. P. E. Guenther: *Weak solutions of difference equations.* Preliminary report.

The functional equations $f(x+h)-f(x)=g(x)$ and $f(qx)-f(x)=g(x)$ and their respective, generally divergent, formal series solutions are studied in the real domain by the methods of distribution theory. Conditions on the given function $g(x)$ are determined for each type of difference operation in order that the respective equation have weak solutions in the sense of Mikusinski. Properties of the resulting "weak sums" are compared with those of more classical solutions of summation problems. (Received September 15, 1954.)

31t. Felix Haas: *Morse type inequalities for the limit sets of an ordinary differential equation.*

A two-dimensional closed manifold and a vector field V defining a differential equation are studied. Certain inequalities for the trajectories of the resulting differential equation are proved. It is shown that if the limit set consists of singular points, limit cycles, and limiting configurations (closed curves with a preferred side made up of singular points and trajectories) only, then the first Betti number of the manifold minus one is smaller than or equal to the number of hyperbolic sectors plus the number of stable limit cycles plus the number of unstable limit cycles plus two times the number of stable-unstable limit cycles plus the minimum of (the number of stable limiting configurations, the number of unstable limiting configurations) minus the number of sector points. If, in addition, directed closed curves made up of singular points and trajectories are ruled out, then there exists at least one stable singular point or center point or stable limit cycle or stable-unstable limit cycle. (The important point here is that stable sectors are not sufficient.) The proof involves a deformation process and repeated use of the Meyer-Vietoris sequence. (Received September 8, 1954.)

32. H. G. Haefeli: *Runge's theorem and the correspondence between linear quaternion functionals and their indicatrices.*

A right analytic quaternion function w, regular in the open, connected set M, is the limit of a sequence of rational functions w_i. This sequence converges uniformly on each closed set in M. If $\complement M$ is connected and not finite, w is the limit of a sequence of polynomials. The local left analytic function (u, N) is orthogonal to the local right analytic function (w, M) with $M \supset \complement N$, if $\int_C w(x)dXu(x)=0$, where C is a 3-dimensional closed surface in $M \cap N$, separating $\complement N$ from $\complement M$. If (u, N) is orthogonal to every (w, M) of a linear region (A) [i.e. $M \supset A$, $A=\bar{A}$], then $u(x)=0$ for all $x \subset \complement A \subset N$. Hence a linear functional $\mathcal{L}$ has a unique indicatrix $u(x)$ such that $\mathcal{L}[w]=\int_C w(x)dXu(x)$. The function (u, N) is orthogonal to the linear subregion $(A)_C$ of all complex analytic functions of two complex variables regular in $A\,[\infty \subset \complement A]$, if the coefficients of the Laurant expansion around infinity satisfy the conditions $\sum_{k=0}^{n-n_1} (i_1)^k c_{n_1, n-n_1-k, k}=0$ for all n and $n_1<n$. Two linear functionals $\mathcal{L}_1$ and $\mathcal{L}_2$ with the indicatrices u_1 and u_2, which map $(A)_C$ onto the complex plane, generate the same complex functional if u_1-u_2 is orthogonal to each function in $(A)_C$. (Received September 17, 1954.)

33. A. O. Huber: *On an inequality of Fejér and Riesz.*

Let $u(z)$ be defined on the closed unit circle $|z| \leq 1$ as a difference of two subharmonic functions $u(z)=u_1(z)-u_2(z)$. Suppose that $\mu_2(|z|<1)=\alpha<1$, μ_2 denoting the measure associated with u_2. Then the inequality $\int_{-\pi}^{+\pi} \exp\{u(e^{it})\}dt \geq 2\cos(\pi\alpha/2) \cdot \int_{-1}^{+1} \exp\{u(\rho e^{i\theta})\}d\rho$ holds for any θ $(0 \leq \theta < 2\pi)$. Equality takes place if and only if $0<\alpha<1$ and $u=\log|F'(z)F(z))^{-\alpha}(1-F^2(z))^{\alpha-1}|+\text{const.}$, where F is an arbitrary conformal transformation of $|z|<1$ onto itself which maps the diameter $\arg z=\theta, \theta+\pi$ on the real axis. This generalizes a result of L. Fejér and F. Riesz (Math. Zeit. vol. 11 (1921) pp. 305–314) and its extension by E. F. Beckenbach (J. London Math. Soc. vol. 13 (1938) pp. 82–86). The theorem can be extended to more general regions by conformal mapping and it implies a new inequality for meromorphic functions. This work was supported by the United States Air Force through the Office of Scientific Research. (Received September 13, 1954.)

34. R. C. James: *Projections in the space (m).*

Any separable Banach space can be embedded in the space (m) of all bounded sequences. R. S. Phillips has shown that the space (c) is not complemented in (m), while A. Sobczyk used this to show that a separable Banach space is not complemented if it has a subspace isomorphic with (c_0). These results are extended to include all separable, nonreflexive Banach spaces with an unconditionally convergent basis. It is also shown that no projection of (m) onto a separable subspace can be of norm 1. (Received September 15, 1954.)

35*t*. Bjarni Jónsson: *A unique decomposition theorem for binary relations.*

For the notion of a sum of binary relations (ordered systems) over a binary relation, see e.g. Day, *Arithmetic of ordered systems*, Trans. Amer. Math. Soc. vol. 58 (1945) pp. 1–43. Consider a nonempty class Φ of reflexive binary relations such that $R\in\Phi$ is implied by each of the following conditions: 1. R is isomorphic to a subrelation of a member of Φ. 2. R is a sum whose index system and terms belong to Φ. 3. Every finite subrelation of R belongs to Φ. A binary relation R is called Φ indecomposable if $R\cong\sum_{t,s} R_t$, $s\in\Phi$, implies that all but one of the R_t's are null. The author shows that every binary relation has, up to isomorphism, a unique representation as a sum over a relation in Φ of Φ indecomposable relations. This includes known results on unordered sums, linearly ordered sums (A. Tarski and the author) and sums over squares (C. C. Chang and A. Tarski). Every decomposition $R=\sum_{t,s} R_t$ induces a partitioning of the field A of R, hence an equivalence relation over A. These equivalence relations form a complete lattice $\mathcal{A}$. Decompositions with $s\in\Phi$ correspond to a principal dual ideal of $\mathcal{A}$. (Received September 9, 1954.)

36*t*. R. V. Kadison: *Isomorphisms of factors of type* II$_\infty$.

It is shown that there are *-automorphisms of factors of type II$_\infty$ with II$_1$ commutants which are not unitarily induced. In fact, the following theorem is proved. The group of unitarily induced automorphisms of a factor of type II$_\infty$ with a II$_1$ commutant is a normal subgroup of the group of all *-automorphisms of the factor, and the quotient group is (canonically) isomorphic to the fundamental group of the factor (cf. F. J. Murray and J. von Neumann, *On rings of operators.* IV, Ann. of Math. vol. 44 (1943) pp. 716–808, see, especially, pp. 740–742). A linking operator is defined for a *-isomorphism between two rings of type II$_\infty$ with II$_1$ commutants, and it is shown that a *-isomorphism is implemented by a unitary transformation if and only if its linking operator is the identity. This completes our information concerning the question of when an isomorphism between rings of operators arises spatially (cf. pp. 742–756 of paper noted above and Chapter III of E. L. Griffin, *Contributions to the theory of rings of operators*, Trans. Amer. Math. Soc. vol. 75 (1953) pp. 471–504). (Received September 17, 1954.)

37*t*. Bertram Kostant: *Representations of a Lie algebra on Hilbert space.* Preliminary report.

Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, and U_s a (strongly continuous) unitary representation of G on the Hilbert space $\mathfrak{H}$. By Stone's theorem, if $\mathfrak{D}=[\xi\in\mathfrak{H}\mid\lim_{t\to 0}\,(U_t-1)/t)\xi=\eta$ exists$]$ and U_t is such a representation of the additive real numbers R then $\mathfrak{D}$ is dense and the operator A on $\mathfrak{D}$ where $A(\xi)=\eta$ is skew-adjoint. When $U_t=U_{\exp tX}$ where $X\in\mathfrak{g}$, let $\mathfrak{D}_X=A$ and $\pi(X)=A$. Define $\mathfrak{D}^1=\bigcap_{X\in\mathfrak{g}}\mathfrak{D}_X$. $\mathfrak{D}^1$ is dense. If $E_1,\cdots,E_m$ is a basis of $\mathfrak{g}$, $\mathfrak{D}^1=\bigcap_{i=1}^m\mathfrak{D}_{E_i}$, $\pi(\sum_{i=1}^m\alpha_iE_i)=\sum_{i=1}^m\alpha_i\pi(E_i)$ on $\mathfrak{D}^1$. The author shows $\xi\in\mathfrak{D}^1$ if and only if the function $(U_s\xi,\xi)$

is of class C^1 on G. Moreover if $g(t)$ is any curve (class C^1) on G defining X at e for $t=0$, then $((U_{g(t)}-1)/t)\xi \to \pi(x)\xi$ and this characterizes $\mathfrak{D}^1$. Let $\mathfrak{D}^\infty=[\xi\in\mathfrak{H} \mid \pi(X_1)\pi(X_2)\cdots\pi(X_k)\xi$ is defined for any finite sequence $X_1, X_2, \cdots, X_k\in\mathfrak{g}]$. $\xi\in\mathfrak{D}^\infty$ if and only if $(U_g\xi, \xi)$ is of class C^∞. $\xi\in\mathfrak{D}^\infty$ is well behaved (sense of Harish-Chandra) if $\sum_{n=0}^{\infty}(\pi(X)^n/n!)\xi$ converges for X in a neighborhood of zero. (Received September 15, 1954.)

38*t*. G. R. MacLane: *On the Peano curves associated with some conformal maps.*

Salem and Zygmund (Duke Math. J. vol. 12) proved that there exists a function $f(z)$, holomorphic in $|z|<1$, continuous in $|z|\leq 1$, such that the curve $w=f(e^{it})$, $0\leq t<2\pi$, fills some square. Further examples have been given by Piranian, Titus, and Young (Michigan Math. J. vol. 1) and Schaeffer (Duke Math. J. vol. 21). All the given proofs are arithmetical. The present paper proves a similar theorem by constructing a suitable Riemann surface onto which $|z|<1$ is mapped by $w=f(z)$, thus defining $f(z)$. In this construction it is easy to see precisely what the Peano curve obtained is. (Received August 6, 1954.)

39*t*. Dorothy Maharam: *On kernel representation of linear operators.*

This paper continues the author's study of "F'-integrals" (Trans. Amer. Math. Soc. vol. 75 (1953) pp. 154–184), that is (roughly) of linear, countably additive order-preserving mappings of one function space F (satisfying the countable chain condition) in another. An F'-integral ψ has a "kernel representation" in terms of another, ϕ, if $\psi(f)=\phi(kf)$ identically, the "kernel" k being a fixed element of F. If ϕ is "full-valued" (loc. cit.) it is shown that such a k exists if and only if, for each positive f, the support of $\psi(f)$ is contained in that of $\phi(f)$. Necessary and sufficient conditions are also found for the existence of a kernel representation with the kernel drawn from a larger function space. As an application, it is shown that an "F'-operator of bounded variation" (the difference between two F'-integrals) can always be represented in the form $\psi(f)=g'$ where $g'(x)=\int k(x, y)f(x, y)dy$, after a suitable isomorphic imbedding of F in a product space; conversely, any operator so represented is of bounded variation. More generally, countably many such operators can be represented in this form simultaneously. (Received September 2, 1954.)

40*t*. M. D. Marcus: *Asymptotic behavior of linear systems.*

Consider the vector-matrix differential equation (1) $\dot{x}=(A+B(t))x$, A a constant n-square matrix, $B(t)=U(t)+iV(t)$ a continuous complex valued n-square matrix defined on $[0, \infty)$, and x a complex n-vector. Let $A^*=$conjugate transpose of A, $U'=$transpose of U. Results: (i) Assume that maximum eigenvalue $(A+A^*)=\omega$; there exists T such that $t\geq T$ implies either (a) $(1/t)\int_0^t \max_i U_{ii}(s)ds\leq -\omega/2$, $\lim_{t\to\infty}\int_0^t(|(U(s)+U'(s))_{ij}|+|(V(s)-V'(s))_{ij}|)ds<\infty$, $i\neq j$, or (b) $(1/t)\int_0^t(\max_i U_{ii}(s)+(1/2)\sum_{i\neq j}|(U(s)+U'(s))_{ij}|+|(V(s)-V'(s))_{ij}|)ds\leq -\omega/2$, then in either case (a) or (b) every solution of (1) is uniformly bounded as $t\to\infty$. If in (a) or (b) the integrals are bounded strictly below $-\omega/2$, then every solution of (1) converges to 0 as $t\to\infty$. (ii) Assume minimum eigenvalue $(A+A^*)=\delta$ and $\liminf_{t\to\infty}(1/t)\cdot\int_0^t(\min_i U_{ii}(s)-(1/2)\sum_{i\neq j}|(U(s)+U'(s))_{ij}|+|(V(s)-V'(s))_{ij}|)ds>-\delta/2$, then every solution of (1) diverges to ∞ as $t\to\infty$. Special results for A triangular and for a single nth order homogeneous linear differential equation reducible to type (1) are given as well. (Received September 17, 1954.)

41*t*. Emanuel Parzen: *A generalization of Bernstein's probabilistic proof of Weierstrass's theorem*. Preliminary Report.

The E_r denote an r-dimensional Euclidean space, and let T be an index set. For each t in T, let $\mathbf{X}_n(t)$ for $n=1, 2, \cdots$ be independent r-dimensional random variables with common distribution function $G(\mathbf{x}, t)$, whose means $m(t)=\int_{E_r}\mathbf{x}dG(\mathbf{x}, t)$ satisfy the condition (1) $\int_{\|\mathbf{x}\|>M}\|\mathbf{x}\|dG(\mathbf{x}, t)\to 0$ as $M\to\infty$ uniformly in t. Let $G_n(\mathbf{x}, t)$ be the distribution function of $(1/n)\sum_{i=1}^{n}\mathbf{X}_i(t)$. Let $g(\mathbf{x})$ be a bounded Borel function on E_r such that (2) $|g(\mathbf{x})-g(m(t))|\to 0$ uniformly in t as $\|\mathbf{x}-m(t)\|\to 0$. Using recent results of the author on the uniform convergence of families of sequences of distribution functions (University of California Publications in Statistics vol. 2 (1954) pp. 23–53), it is easy to show that $\int_{E_r}g(\mathbf{x})dG_n(\mathbf{x}, t)\to g(m(t))$ uniformly in t. Bernstein's proof of Weierstrass's theorem on the uniform approximation of continuous functions by polynomials follows from the foregoing theorem by taking T to be the closed unit cube in E_r, $g(\mathbf{x})$ to be a function continuous on T, and $\mathbf{X}(t)$ to be the Cartesian product of r independent binomial random variables $X^j(t_j)$ with parameter t_j. Other generalizations and applications to analysis are also indicated. (Received September 15, 1954.)

42. H. O. Pollak (p), G. H. Wannier, and D. J. Dickinson: *On a class of polynomials orthogonal over a denumerable set*.

If a set of polynomials $\Phi_n^r(x)$ $(r, n=0, 1, 2, \cdots, \Phi_0^r(x)=1, \Phi_1^r(x)=x)$ satisfies $\Phi_{n+1}^r(x)=\Phi_n^r(x)-\lambda_{n+r}\Phi_{n-1}^r(x)$ with $\lambda_n<B^r/(n^{1+\epsilon})$ for positive constants B and ϵ, then as $n\to\infty$, $x^n\Phi_n^r(x^{-1})\to E^r(x)$, where $E^r(x)$ is an entire function whose zeros are real and simple. Also, $\sum \Phi_n^r(x)\Phi_m^r(x)E^{r+1}(x^{-1})[dxE^r(x^{-1})/dx]^{-1}=\delta_{m,n}\prod_{i=1}^{n}\lambda_{i+r}$ where the summation is taken over those x that are zeros of $E^r(x^{-1})$. (Received September 15, 1954.)

43. R. A. Raimi: *Compact transformations and the k-topology in Hilbert space*.

Let H be a Hilbert space (complete but not necessarily separable), S the unit ball $\{x|\|x\|\leqq 1\}$, and K any compact subset of H. Then there exists a compact (completely continuous) transformation $T:H\to H$ such that $T(S)\supset K$. It follows from this lemma that the following two topologies for H are identical: (a) The *k-topology* of Arens (R. Arens, *Duality in linear spaces*, Duke Math. J. vol. 14 (1947) pp. 747–794), in which a basis set of neighborhoods of the identity is formed of the sets $\{x\in H||(x, y)|\leqq 1$ for all $y\in K$, K any norm-compact set$\}$, and (b) The topology generated by a basis of sets of the form $\{x\in H|\|T_i(x)\|\leqq 1$, T_i any compact transformation, $i=1, 2, \cdots, n\}$. (Received August 6, 1954.)

44. Jenny E. Rosenthal: *A new form for the solution of Laplace's equation in cylindrical coordinates*.

An extension of complex potential methods to the solution of potential problems with axial symmetry was reported previously by the author (Physical Review vol. 95 (1954) p. 633). The general class of functions obtained by this extension process has the form $W=\int f(K)[(z-iK)^2+r^2]^{-1/2}dK$. This is now studied in some detail, and its relationship is shown to standard forms of the solution of Laplace's equation. The suitability of this expression for W as a solution of boundary value problems is discussed, in particular its use in cases where standard methods fail, such as in problems with discontinuous boundaries. (Received September 15, 1954.)

45. L. A. Rubel: *Entire functions and Ostrowski sequences.*

A set S of positive integers is "an Ostrowski sequence of block ratio at least λ," where $\lambda > 1$, provided that there exist sequences $\{m_k\}$ and $\{n_k\}$, $k = 0, 1, 2, \cdots$, of positive integers satisfying (i) $\lambda m_k < n_k < m_{k+1}$, and (ii) if $m_k < n \leqq n_k$, then n is contained in S. It is shown that if an entire function of exponential type t less than π is not identically zero, yet vanishes over an Ostrowski sequence of block ratio at least λ, then t must exceed a certain function $t(\lambda)$ of λ. Two proofs of this theorem are offered, one utilizing the methods of harmonic measure, and the other using Carleman's Theorem, each of which yields an estimate of the restricting function $t(\lambda)$. Entire functions of type less than π, which vanish over specified Ostrowski sequences, are constructed, their rates of growth are estimated, and are compared with the theorems restricting them. (Received August 31, 1954.)

46. Walter Rudin: *On a problem of Collingwood and Cartwright.*

If f is meromorphic in the interior U of the unit circle, the cluster set of f, in the large, is the set $C(f)$ which consists of all w (including ∞) for which there is a sequence $\{z_n\}$ such that $z_n \in U$, $|z_n| \to 1$, $f(z_n) \to w$ as $n \to \infty$. The set of all values which f assumes infinitely many times in U is denoted by $R(f)$. Every $C(f)$ is a continuum; Collingwood and Cartwright (Acta Math. vol. 87 (1952) p. 123) have raised the question whether the converse is true. The question is answered negatively by constructing a plane continuum which is not a $C(f)$; the problem of finding necessary and sufficient conditions under which a continuum is a $C(f)$ remains open. The sets $R(f)$ are characterized as follows: E is an $R(f)$ if and only if E is the intersection of a countable sequence of *connected* open subsets of the Riemann sphere, each contained in the preceding one. In particular, every closed set is an $R(f)$, but there are open sets which are not. (Received August 25, 1954.)

47. V. L. Shapiro: *Cantor-type uniqueness of multiple trigonometric integrals.* II.

Let $c(u)$ be a complex-valued function integrable on every bounded domain in n-dimensional euclidean space E_n $(n \geqq 2)$. Suppose that the integral $\int_{E_n} e^{i(x,u)} c(u)du$ is spherically summable $(C, 1)$ to zero almost everywhere and that the $(C, 1)$ spherical mean of this integral of rank R, $\sigma_R^{(1)}(x)$, is such that $\limsup_{R \to \infty} |\sigma_R^{(1)}(x)| < \infty$ in $E_n - Z$ where Z is a closed set of vanishing capacity. Suppose, further, that either (a) $c(u)(|u|^2+1)^{-1}$ is in L_1 on E_n or (b) $c(u)(|u|^2+1)^{-1}$ is in L_2 on E_n and $\int_{E_n - D_n(0,1)} e^{i(x,u)} c(u)|u|^{-2}du$ converges spherically to a continuous function in E_n where $D_n(0, 1)$ is the n-dimensional unit sphere. It is then shown in this paper that $c(u)$ vanishes almost everywhere. This theorem extends results previously obtained by the author. The essential innovation, which enables one to obtain these stronger results, is a theorem on the differentiability of Fourier transforms. This theorem is in turn based on a result of Bochner concerning the Fourier transform of spherical harmonics. Some special theorems concerning uniqueness are also obtained for the plane. (Received September 1, 1954.)

48. Domina E. Spencer: *On the classification of Bôcher equations.*

The ordinary second-order differential equations, occurring when the Laplace and Helmholtz equations are solved by separation of variables, can be classified in terms of Bôcher equations (M. Bôcher, *Über die Reihenentwickelungen der Potentialtheorie*, Leipzig, 1894). These equations are of interest in applied mathematics. The paper suggests a new method of classification of Bôcher equations and points out difficulties in the classification employed by Ince (E. L. Ince, *Ordinary differential equations*, London, 1927, Chap. XX). The effect of transformations of variables and the problem of making the specification unique are handled. (Received September 8, 1954.)

49t. W. R. Wasow: *On the convergence of an approximation method of M. J. Lighthill.*

In the Philosophical Magazine (7) vol. 40 (1949), Lighthill introduced a perturbation technique that is useful for the solution of certain differential equations near their singular points. In the present paper the mathematical validity of this method is investigated for the differential equation $(x+\alpha u)du/dx+q(x)u=r(x)$. If $q(0)>0$ and a few simple conditions are satisfied, then the solution for which $u(1)=b$ is shown to permit a parametric representation of the form $u=\sum_{s=0}^{\infty} u_s(\xi)\alpha^s$, $x=\xi+\sum_{s=1}^{\infty} x_s(\xi)\alpha^s$, where m is arbitrary, $u_0(1)=b$, $x_s(1)=u_s(1)=0$, $s>0$. The series for x converges if $|\alpha| \leq \gamma_m \xi^{q(0)}$, $0 \leq \xi \leq 1$, where γ_m is a constant. The $x_s(\xi)$, $u_s(\xi)$ are found by following essentially Lighthill's scheme. If $q(0)<0$ a variant of Lighthill's procedure using the system $\xi dx/d\xi=x+\alpha u$, $\xi du/d\xi=r(x)-q(x)u$ has certain advantages. If $r(0)=0$, which according to Lighthill entails no loss of generality, it leads to a series solution $x=\xi+\sum_{s=1}^{\infty} x_s(\xi)\alpha^s$, $u=\sum_{s=0}^{\infty} u_s(\xi)\alpha^s$ convergent for $|\alpha\xi^{\rho}| \leq \gamma$, where $\rho=\min(-q(0), 1)$. The proofs use the method of dominating series. (Received August 17, 1954.)

50t. W. R. Wasow: *Singular perturbations of boundary value problems for nonlinear differential equations of the second order.*

Let the differential equation $\epsilon y''=F_1(x, y, \epsilon)y'+F_2(x, y, \epsilon)$ satisfy the following conditions. (A) The "reduced differential equation" $F_1(x, y, 0)y'+F_2(x, y, 0)=0$ possesses a solution $y=u(x)$ for which $u(\beta)=l_2$ and $F_1(x, u(x), 0)<0$ in $\alpha \leq x \leq \beta$. (B) The functions $F_j(x, y, \epsilon)$, $j=1, 2$, are regular analytic with respect to y and ϵ and of class C^2 with respect to x in a region R of the (x, y, ϵ)-space that contains in its interior all points $y=u(x)$, $\alpha \leq x \leq \beta$, $\epsilon=0$. It is shown that if ϵ and $u(\alpha)-l_1$ are sufficiently small, the full differential equation possesses a solution satisfying the boundary conditions $y(\alpha)=l_1$, $y(\beta)=l_2$ and representable by a convergent series of the form $y(x, \epsilon)=u(x)+\sum_{r=1}^{\infty} u_r(x, \epsilon)\epsilon^r$ whose coefficients satisfy certain linear differential equations and can approximately, for small ϵ, be calculated by quadratures. If the differential equation is analytic in x also, this series solution can be rearranged into the form $\sum_{m=0}^{\infty} \exp\{m\int_{\alpha}^{x} F_1(t, u(t), 0)dt/\epsilon\}\, a_m(x, l_1, \epsilon)$ where the $a_m(x, l_1, \epsilon)$ are analytic in x and l_1 and possess asymptotic expansion in powers of ϵ. These results differ from related ones by Coddington and Levinson [Proc. Amer. Math. Soc. vol. 3 (1952) pp. 73–81] and by N. I. Brish [Doklady Akad. Nauk SSSR vol. 95 (1952) pp. 429–432] in that they permit an effective construction of the solution. (Received June 28, 1954.)

51. John Wermer: *Maximal subalgebras and Riemann surfaces.*

Let B be a commutative Banach algebra and M a proper closed subalgebra of B. M is called *maximal* if for every closed subalgebra M' with $M \subseteq M'$, either $M'=M$ or $M'=B$. Let $\mathfrak{F}$ be a Riemann surface, $\mathfrak{M}$ a region on $\mathfrak{F}$ bounded by a simple closed analytic curve γ such that $\mathfrak{M}+\gamma$ is compact. Let C be the Banach algebra of all continuous complex-valued functions on γ. Let $\mathfrak{A}$ be the subalgebra of C consisting of those f in C which may be continued into $\mathfrak{M}$ to be analytic in $\mathfrak{M}$. It is easy to see that $\mathfrak{A}$ is a proper closed subalgebra of C, containing a unit and separating points on γ. *Theorem*: $\mathfrak{A}$ is a maximal subalgebra of C. This generalizes a result of the author (Bull. Amer. Math. Soc. Abstract 59-3-306) on boundary values of functions analytic in the unit disk. (Received September 16, 1954.)

52. Albert Wilansky (p) and Karl Zeller: *Summation of bounded divergent sequences, topological methods.*

A is a summability matrix; $c_A=\{x \mid Ax \text{ is convergent}\}$; c is the space of convergent sequences. Assume $c_A \supset c$, hence $m_A \supset m$, where $m_A=\{x \mid Ax \text{ is bounded}\}$, m is the space of bounded sequences. c_A, m_A are given (F) metrics, e.g. if A is a triangle, $\|x\| = \sup_n \left|\sum_k a_{nk}x_k\right|$. *Theorem* I. *The following conditions are equivalent*: i. *c is closed in c_A*, ii. *m is closed in m_A*, iii. *A sums no bounded divergent sequences*. Tropper's result (Mathematical Reviews vol. 15, p. 118) follows. Close to known results is *Theorem* II. *If* $\lim_n \inf (|a_{nn}| - \sum(k\neq n)|a_{nk}|)>0$, *$A$ sums no bounded divergent sequence, and c_A is the smallest linear space containing c and $\{x \mid Ax=0\}$. ($c_A=c$ if A is reversible.)* Results of Mazur and of Petersen of the type $c_A=c\oplus$ one sequence follows. *Theorem* III. *There exists a regular row-finite A with c_A the smallest space including c and n preassigned sequences (independent over m). There exists a regular row-finite matrix which sums c and a pressaigned sequence of sequences (independent over m), and no bounded divergent sequence.* This improves results of Tolba (Mathematical Reviews vol. 14, p. 369) and Darevsky. (Received September 14, 1954.)

53. Mishael Zedek: *Fejér's theorem on the zeros of extremal polynomials generalised.* Preliminary report.

J. L. Walsh suggested the study of generalized Chebyshev polynomials useful for approximation of given polynomials by polynomials of lower degree and to be defined as follows: Let E be a compact point set in the complex z-plane. Of all polynomials of degree n and of the form $p_n^s(z)=z^n+A_1z^{n-1}+\cdots+A_sz^{n-s}+a_{s+1}z^{n-s-1}+\cdots+a_n$ having the first $s+1$ coefficients $1, A_1, \cdots, A_s$ given, the author denotes by $T_n^s(z)$ one solving the extremal problem $\{\text{Max } |p_n^s(z)|,\ z \text{ on } E\}=\text{minimum}$. $T_n^s(z)$ are known as Chebyshev polynomials. Fejér proved (Math. Ann. vol 85 (1922) pp. 41–48) that the zeros of $T_n^0(z)$ lie in the convex hull of E. The main result for the zeros of $T_n^s(z)$ states that the sum of the angles subtended by E at any group of $s+1$ of them is $\geq \pi$. (Received September 15, 1954.)

APPLIED MATHEMATICS

54t. Peter Henrici: *An eigenvalue problem in the theory of viscous flow.*

L. Collatz and H. Goertler (Z. Angew. Math. Physik vol. 5 (1954) pp. 95–110) reduce the study of the stationary, slightly rotational, axisymmetric flow of a viscous fluid through a straight circular tube to the following eigenvalue problem for a function $G(s)$ proportional to the angular velocity component: (1) $s^2G''+sG'-G = -16\beta^2s^4(1-s^2)G$, $0\leq s\leq 1$; (2) $G(0)=G(1)=0$. They determine the first few eigenvalues and eigenfunctions by numerical methods and give for the large eigenvalues the approximation $\beta_n \sim n$. The auther shows that (1) can be solved in terms of the confluent hypergeometric function. This leads to the equation ${}_1F_1(1-\beta;\ 2;\ 4\beta)=0$ for the eigenvalues and, by application of Watson's Lemma to a suitable contour integral, to asymptotic formulas for the large eigenvalues and the corresponding eigenfunctions. In particular, $\beta_n = m + Am^{-1/3} + Bm^{-5/3} + O(m^{-10/3})$, where $m = n + 1/6$ and A and B are certain constants. Additional nonasymptotic information about the eigenfunctions is obtained from a theorem of Pólya and Sonine (see Szegö, *Orthogonal polynomials*, p. 161). Numerical results are given. The work was supported by the Office of Naval Research. (Received September 13, 1954.)

55. Peter Henrici: *Application of two methods of numerical analysis to the computation of the reflected radiation of a point source.*

The integral $f(x, y) = \int_0^\infty dr \int_0^{2\pi} (1 + x^2 + r^2 - 2xr \cos \phi)^{-3/2}(r^2 + y^2)^{-2} r\, d\phi$, which arises in the theory of radiation and which was to be computed numerically for many values of x and y, is reduced to a finite simple integral involving the hypergeometric function $F(3/4, 5/4, 1, Z)$ where (*) $0 \leq Z < 1$. In the first part of the paper the effect of Aitken's δ^2-method (Proc. Roy. Soc. Edinburgh vol. 57 (1936–37) pp. 269–304) upon the summation of the hypergeometric series is studied. It is shown that if (*) holds the application of the method reduces the truncation error after n terms by a factor k, where $1 \leq 8n^2(1-Z)^2 k \leq 2^{-3/2}\pi$. The number of terms necessary for a given accuracy is reduced similarly, and the "ratio of practical convergence" (remainder over first neglected term) is not affected adversely by the method. In the second part the error induced by numerical quadrature of the simple integral is estimated by a method due to P. Davis and P. Rabinowitz (see Journal of Rational Mechanics and Analysis vol. 2 (1953) pp. 303–313, and NBS Report No. 3270). (Received September 13, 1954.)

56t. Peter Henrici: *On helical springs of finite thickness.*

The problem of determining the stress distribution in a statically loaded, closely coiled helical spring, whose (circular) cross-section is not necessarily small in comparison with the diameter of the helix, has been solved approximately by an iterative method by Goehner (Ing. Arch. vol. 1 (1930) pp. 619–644) and exactly in terms of a series of appropriate Legendre functions by Freiberger (Aust. Journ. of Scient. Res. A vol. 2 (1949) pp. 354–375). The author shows that the exact values of the stress concentration factor as well as some other significant quantities of the problem can be developed in terms of a certain parameter (namely the reciprocal of the spring index). Thus Goehner's approximations are recovered and further approximations of the same type are obtained. The analytical work involves an application of Clausen's identity (J. Reine Angew. Math. vol. 3 (1828) pp. 89–95) to the computation of the Taylor expansion of the square of a Legendre function. The work was supported by the Office of Naval Research. (Received September 13, 1954.)

57t. R. S. Ledley: *A general method for introducing constraints on the generating propositions of a propositional calculus of symbolic logic.*

Digital computational methods for the propositional calculus of symbolic logic, or the analogous free Boolean algebra, were given by the author (National Bureau of Standards Report 3363). These methods associate a binary designation number of 2^n digits to each of n generating propositions (elements) E_i: where if $(E_i)_j$ represents the jth position digit of the designation number for E_i, then $(\overline{E}_i)_j = 0, 1$ when $(E_i)_j = 1, 0$; $(E_i \cap E_j)_k = (E_i)_k \odot (E_j)_k$ where $0 \odot 0 = 1 \odot 0 = 0 \odot 1 = 0$, $1 \odot 1 = 1$; $(E_i \cup E_j)_k = (E_i)_k \oplus (E_j)_k$ where $1 \oplus 1 = 1 \oplus 0 = 0 \oplus 1 = 1$, $0 \oplus 0 = 0$. Also, representing E_i by E_i^1, $\overline{E}_i$ by E_i^0, consider the 2^n possible intersections $\bigcap_{i=1}^{n} E_i^c$ where $c = 0$ or 1, for the

2^n possible dispositions of the n superscripts c in the formula; then $(\bigcap E_i^c)_k = 1$ for only one integer k, $1 \leq k \leq 2^n$, and 0 otherwise, where k differs for each of the 2^n possible intersections. In this Boolean algebra the generating propositions are logically independent. However in practical problems one often desires to introduce logical constraints or dependencies between the generators. If $f_i(E_1, E_2, \cdots)$, $i = (1, \cdots, m)$, represent the Boolean functional constrains applied, then the designation numbers of the E_i are reduced to include only those positions k for which $(\bigcap_{i=1}^{m} f_i)_k = 1$. The computational methods for a many component propositional logic is an application of this general constraint method. All the results and computational methods for the free Boolean algebra are equally valid in any constraint case. (Received August 26, 1954.)

58t. R. S. Ledley: *A many-component propositional, two-valued logic.*

The familiar propositional calculus of symbolic logic involves propositions with two components in a two-valued logical system. However, practical logical problems in science, industry, and government frequently deal with many-component propositions, where the ith component, Q_i, of a proposition Q having r_k components is defined by $\bar{Q}_i = \bigcup_{j=1}^{r_k} Q_j$ and $Q_i \bigcap Q_j = 0$ both for $j \neq i$. The author has previously presented a digitalization and systematization of the free Boolean algebra with 2^{2^n} elements (see Journal of the Operations Research Society of America vol. 2 (1954)), which can be interpreted in terms of the two-component propositional, two-valued logic generated by n propositions. This same digitalization has now been extended to include non-free Boolean algebras with 2^n elements. These algebras can be interpreted as representing many-component propositional, two-valued logics, where for a system of m propositions with $r_1, r_2, \cdots, r_m$ components each, $n = \prod_{i=1}^{m} r_i$. All results and computational methods of the free Boolean case are equally valid in this non-free case. The number of units in the binary designation number of length n for a single component of the kth proposition is n/r_k. The digitalized techniques enable these propositions to be manipulated with the same facility as ordinary algebraic type functions, the methods providing easy implementation by an electronic "logic machine." (Received August 26, 1954.)

59. R. S. Ledley: *Digitalization and systematization of some aspects of the functional calculus of symbolic logic.*

Aspects of the functional calculus are digitalized and systematized to achieve straightforward computational methods enabling solution of those practical problems arising in industry, science, and government, which need the functional as well as the propositional logical calculus. The theory is based on elementary universally valid formulas ($i, j, \cdots$ representing universal, $m, n, \cdots$ existential quantifiers) such as: $x_i y_j P(x, y) = y_j x_i P(x, y)$; $x_m y_i P(x, y) \supset y_i x_m P(x, y)$; $x_i P(x) \supset x_m P(x)$. Formulas are treated as expressions in a modified prenex normal form; the prefix represented by a_{mi}, where $a_{mi} = 1$ for x_i in the scope of y_m, $a_{mi} = 0$ otherwise. The weight of the mth row is $\sum_i a_{mi}$; that of the ith column is $\sum_m a_{mi}$. The predicates themselves are handled by the author's digitalized method for the propositional calculus. With this notation, systematic methods are formulated for: generating all implications of any number of expressions deducible by elementary formulas; testing any number of expressions for contradictions, redundancy and tautology; manipulating and analyzing functional expressions; obtaining the solution to expression equations, etc. Both the restricted and extended functional calculus are treated. This digitalization is amenable to easy mechanization by an electronic "logic machine." The notation is extended to include additional class types of quantifiers such as 'there exist at least n,' etc. where for r types, the array is $a_{i_1,\dots,i_r}$. (Received August 24, 1954.)

60t. Eric Reissner: *On transverse vibrations of thin shallow elastic shells.*

It is shown that in the range of frequencies governing *transverse* vibrations of shallow shells the differential equations of the problem may be simplified by omitting *longitudinal inertia* terms. The practical importance of this simplification is illustrated by explicitly determining the frequencies of free transverse vibrations of simply-supported shells with second-degree middle surface equation and rectangular base-plane projection of the boundary. (Received August 19, 1954.)

61t. Eric Reissner: *On axi-symmetrical vibrations of shallow spherical shells.*

The paper determines explicitly the frequencies of free, axi-symmetrical, transverse vibrations of shallow spherical shell segments with clamped or free edges. The calculations further indicate the practical advantages of being able to neglect longitudinal inertia in problems of this type (see preceding abstract). A comparison is made with an earlier study of the problem of the shell segment with clamped edge, carried out without neglecting longitudinal inertia (J. Appl. Phys. vol. 17 (1946) pp. 1038–1042. (Received August 19, 1954.)

GEOMETRY

62t. Louis Auslander and L. F. Markus: *Holonomy of flat affinely connected manifolds.*

Let M be a differentiable (C^∞) n-manifold with a differentiable affine connection Γ which is flat (torsion and curvature tensors are zero). Then M admits a covering by local coordinates such that on each intersection of coordinates the Jacobian matrix is constant and thus linear phenomena, i.e. linear differential equations with constant coefficients, are meaningful relative to this covering. Conversely, if M admits a covering by local coordinates with constant Jacobian matrices, then M admits a differentiable flat affine connection. For differentiable n-manifolds M with flat affine connection Γ there exists a homomorphism from the fundamental group $\Pi_1(M)$ onto the holonomy group $H(M; \Gamma)$. In the principal bundle $B(M)$ the set of bases joined by "horizontal" curves to a given base defines a holonomy covering space $\widehat{M}$ for M. With the raised connection $\widehat{\Gamma}$, $H(\widehat{M}; \widehat{\Gamma}) = 0$ and the covering transformation group of $\widehat{M}$ over M is $H(M; \Gamma)$. In a certain sense, $\widehat{M}$ is the (minimal) universal covering space of M with regard to holonomy. Finally, suppose $H(M; \Gamma) = 0$ and Γ is complete. Then M is a Riemannian manifold with Christoffel connection Γ and M is isometric (differentiably) with E^n modulo a discrete vector subspace. (Received September 16, 1954.)

63. C. C. Hsiung: *On the total curvature of a simple closed curve in a Riemannian manifold.*

Let C be a simple closed curve in an n-dimensional Riemannian manifold V_n ($n \geq 2$) which is twice differentiably imbedded in a Euclidean space E_m of m ($>n$) dimensions, and let κ_g be the first curvature of C relative to V_n. Furthermore, let $N_{n+1}|, \dots, N_m|$ be any $m-n$ mutually orthogonal vectors normal to V_n at a point P,

$\kappa_{\nu|1},\cdots,\kappa_{\nu|n}$ the principal curvatures of V_n associated with any normal vector $N_{\nu|}$ $(\nu=n+1,\cdots,m)$, and M the first mean curvature of V_n at P defined by $M^2=\sum_{\nu=n+1}^{m}(\sum_{\lambda=1}^{n}\kappa_{\nu|\lambda})^2$. If $\kappa_{\nu|1},\cdots,\kappa_{\nu|n}$ for each ν have the same sign at each point of C, then it is shown that (*)$\int_C|\kappa_g|ds\geq 2\pi-\int_C|M|ds$, where s is the arc length of C. For a Euclidean V_n, the inequality (*) was obtained by W. Fenchel for $n=3$, and independently by B. Segre, H. Rutishauser and H. Samuelson, and K. Borsuk for a general n. (Received September 15, 1954.)

64. Valdemars Punga: *On the determination of affine connection in metric spaces. Application to Cartan-Finsler space.*

The usual way of determining the affine connection in a metric space is to set the covariant differential of a metric tensor equal to zero (i.e. $\delta g_{\alpha\beta}=0$) and to solve the corresponding equation for $\Gamma^{\alpha}_{\beta\gamma}$. The disadvantages of this method are: (a) we should know the formula for the covariant differential of a metric tensor, (b) in more general metric spaces the metric is not always defined by the tensor $g_{\alpha\beta}$. Here is a method for the determination of an affine connection for any metric: equate the length of the vector at the initial point (or at an initial line element) to the length of the same vector when parallelly displaced to the neighbouring point (or neighbouring line element) and solve the resulting equation for the affine connection using infinitesimals up to the first order. If the space is defined so that these two lengths are not equal (as in Weyl space), then use the method of linear displacement of length (see V. Punga, *The principle of linear displacement of length and Weyl's geometry*, in this Bulletin). This method will be demonstrated for Cartan-Finsler space of line elements. The length of a vector $v^\alpha(x,x')$ in Cartan-Finsler space is given by $v=(g_{ij}(x,x')v^iv^j)^{1/2}$. The vector at the line element $(x^\alpha+dx^\alpha, x'^\alpha+dx'^\alpha)$ parallel to the vector v^α at (x^α, x'^α) is the vector $v^\alpha+Dv^\alpha=v^\alpha-\Gamma^{*\alpha}_{\beta\gamma}v^\gamma dx^\beta-C^{\alpha}_{\beta\gamma}v^\gamma\delta x'^\beta$, $\delta x'^\beta=dx'^\beta+\Gamma^{\beta}_{\sigma\rho}x'^\rho dx^\sigma$ (the proof that $v^\alpha+Dv^\alpha$ is a vector is similar to that of V. Punga, *On parallel displacement of a vector*, in this Bulletin). According to the author's method he sets up $g_{\alpha\beta}(x,x')v^\alpha v^\beta=g_{\alpha\beta}(x+dx, x'+dx')(v^\alpha+Dv^\alpha)(v^\beta+Dv^\beta)=(g_{\alpha\beta}(x,x')+(\partial g_{\alpha\beta}/\partial x_\gamma)dx^\gamma+(\partial g_{\alpha\beta}/\partial x'^\gamma)dx'^\gamma)(v^\alpha v^\beta+g_{\alpha\beta}v^\beta Dv^\alpha+g_{\alpha\beta}v^\alpha Dv^\beta)$, or $[\partial g_{\alpha\beta}/\partial x^\gamma-g_{\epsilon\beta}\Gamma^{*\epsilon}_{\gamma\alpha}-g_{\alpha\epsilon}\Gamma^{*\epsilon}_{\gamma\beta}-(\partial g_{\alpha\beta}/\partial x'^\epsilon)\Gamma^{*\epsilon}_{\gamma\rho}x'^\rho]dx^\gamma+[(\partial g_{\alpha\beta}/\partial x'^\gamma)-g_{\epsilon\beta}C^{\epsilon}_{\gamma\alpha}-g_{\alpha\epsilon}C^{\epsilon}_{\gamma\beta}]\cdot\delta x'^\gamma=0$. Since dx^γ and $\delta x'^\gamma$ are independent it follows that the expressions in brackets are zero. Solving them for $\Gamma^{*\alpha}_{\beta\gamma}$ and $C^{\alpha}_{\beta\gamma}$ one obtains Cartan's formulas (E. Cartan, *Les espaces de Finsler* II). (Received September 13, 1954.)

65*t*. Valdemars Punga: *On parallel displacement of a vector.*

The parallel displacement of a vector v^α from the point x^α to $x^\alpha+dx^\alpha$ is defined by the affine connection $\Gamma^{\alpha}_{\beta\gamma}(x)$ with the law of transformation from coordinate system (α, β, $\cdots$ Greek indices) to (a, b, $\cdots$ Roman indices): (1) $\Gamma^{a}_{bc}=(\partial x^a/\partial x^\alpha)(\partial x^\beta/\partial x^b)\cdot(\partial x^\gamma/\partial x^c)\Gamma^{\alpha}_{\beta\gamma}-(\partial x^\beta/\partial x^b)(\partial x^\gamma/\partial x^c)(\partial^2x^a/\partial x^\beta\partial x^\gamma)$, $x^a=x^a(x^\alpha)$. The vector at the point $x^\alpha+dx^\alpha$ parallel to the vector v^α at the point x^α is the vector $v^\alpha-\Gamma^{\alpha}_{\beta\gamma}v^\gamma dx^\beta$. It is not obvious that $v^\alpha-\Gamma^{\alpha}_{\beta\gamma}v^\gamma dx^\beta$ is a vector: $v^\gamma(x)$ is a vector, dx^β is a vector, but $\Gamma^{\alpha}_{\beta\gamma}(x)$ is not a tensor and hence the product $\Gamma^{\alpha}_{\beta\gamma}v^\gamma dx^\beta$ seems to be not a tensor. But the vector $v^\alpha-\Gamma^{\alpha}_{\beta\gamma}v^\gamma dx^\beta$ is located not at the point x^α, but at the point $x^\alpha+dx^\alpha$ and because of this location it is a vector; to prove this will be the object of this paper. Thus, one has to prove that (2) $(\partial x^a/\partial x^\alpha)_{x^\alpha+dx^\alpha}\cdot(v^\alpha-\Gamma^{\alpha}_{\beta\gamma}v^\gamma dx^\beta)=v^a-\Gamma^{a}_{bc}v^c dx^b$. Assuming that the relation $f(x^\alpha+dx^\alpha)=f(x^\alpha)+(\partial f/\partial x^\sigma)(x)dx^\sigma$ holds for any admissible function $f(x)$, one has similarly: (3) $(\partial x^a/\partial x^\alpha)_{(x^\alpha+dx^\alpha)}=\partial x^a/\partial x^\alpha+(\partial^2x^a/\partial x^\alpha\partial x^\sigma)dx^\sigma$. Thus the left member of (2) under substitution (3) changes to (4) $(\partial x^a/\partial x^\alpha)_{x^\alpha+dx^\alpha}\cdot(v^\alpha-\Gamma^{\alpha}_{\beta\gamma}v^\gamma dx^\beta)=(\partial x^a/\partial x^\alpha$

$+(\partial^2 x^\epsilon/\partial x^\alpha \partial x^\sigma)dx^\sigma)(v^\alpha - \Gamma^\alpha_{\beta\gamma}v^\gamma dx^\beta)$. Assuming (3) the author restricts the use of infinitesimals to those of the first order only (like dx^α) and hence is justified in (4) in dropping out any term containing infinitesimals of order higher than one (like $dx^\sigma dx^\gamma$). Hence (4) reduces to (5) $(\partial x^\epsilon/\partial x^\alpha)v^\alpha + (\partial^2 x^\epsilon/\partial x^\alpha \partial x^\sigma)v^\alpha dx^\sigma - (\partial x^\epsilon/\partial x^\alpha)\Gamma^\alpha_{\beta\gamma}v^\gamma dx^\beta$. Substituting $(\partial x^\epsilon/\partial x^\alpha)v^\alpha = v^\epsilon$, $v^\alpha = (\partial x^\alpha/\partial x^\epsilon)dx^\epsilon$, $dx^\sigma = (\partial x^\sigma/\partial x^\epsilon)dx^\epsilon$ into (5) and changing correspondingly the dummy indices, (5) reduces to (6) $v^\epsilon + [(\partial^2 x^\epsilon/\partial x^\gamma \partial x^\beta)(\partial x^\gamma/\partial x^\epsilon)\cdot(\partial x^\beta/\partial x^\lambda) - (\partial x^\epsilon/\partial x^\alpha)(\partial x^\beta/\partial x^\lambda)(\partial x^\gamma/\partial x^\epsilon)\Gamma^\alpha_{\beta\gamma}]v^\epsilon dx^\lambda$. The expression in brackets according to (1) is $\Gamma^\epsilon_{\lambda\epsilon}$ so that (6) reduces to $v^\epsilon + \Gamma^\epsilon_{\lambda\epsilon}v^\epsilon dx^\lambda$. Q.E.D. (Received September 13, 1954.)

66*t*. Valdemars Punga: *The principle of linear displacement of length and Weyl's geometry.*

Given a metric space $(g_{\alpha\beta}(x) = g_{\beta\alpha}(x))$ with a symmetric affine connection $\Gamma^\alpha_{\beta\gamma}(x) = \Gamma^\alpha_{\gamma\beta}(x)$. The vector at $x^\alpha + dx^\alpha$ parallel to the vector v^α at x^α is the vector $v^\alpha + Dv^\alpha = v^\alpha - \Gamma^\alpha_{\beta\gamma}v^\gamma dx^\beta$ (see V. Punga, *On parallel displacement of a vector*, in this Bulletin). The principle of parallel displacement of vectors (and tensors of any rank) is the leading principle in modern differential geometry. At the same time the principle of parallel displacement of the length of a vector in metric spaces, taken independently from vector displacement, is not so widely appreciated. The expression "parallel displacement of length" would be replaced by "linear displacement of length." In this paper a reasonable formula for displacement of length will be obtained and it will be shown that if two independent principles of length and vector displacement are taken simultaneously, Weyl geometry results. Assume that displacing a length v_0 from a point x_0^α to the point x^α the length v is obtained, where $v = f(x_0, x)\cdot v_0$ or, in differential form: $Dv^\alpha = v_0(\partial f/\partial x^\alpha)_{x=x_0}dx^\alpha$. Setting up $(\partial f(x, x_0)/\partial x^\alpha)_{x=x_0} = -\phi_\alpha(x_0)/2$, for any point $x = x_0$ and any vector $v = v_0$, one obtains $Dv = -(v/2)\cdot\phi_\alpha(x)dx^\alpha$. Hence $Dv^2 = 2vDv = -v^2\phi_\alpha(x)dx^\alpha$. On the other hand $Dv^2 = g_{\alpha\beta}(x+dx)(v^\alpha + Dv^\alpha)(v^\beta + Dv^\beta) - g_{\alpha\beta}(x)v^\alpha v^\beta = (g_{\alpha\beta} + dg_{\alpha\beta})(v^\alpha - \Gamma^\alpha_{\lambda\mu}v^\mu dx^\lambda)(v^\beta - \Gamma^\beta_{\sigma\rho}v^\rho dx^\sigma) = -v^2\phi_\alpha(x)dx^\alpha$. Solving this equation for $\Gamma^\alpha_{\beta\gamma}$ (up to infinitesimals of the first order), one obtains $\Gamma^\alpha_{\beta\gamma} = \left\{ {\alpha \atop \beta\gamma} \right\} + (\delta^\alpha_\beta\phi_\gamma + \delta^\alpha_\gamma\phi_\beta - g^{\alpha\nu}g_{\beta\gamma}\phi_\nu)$, which is the Weyl connection. (Received September 13, 1954.)

LOGIC AND FOUNDATIONS

67. Kurt Bing: *On arithmetical classes not closed under direct union.*

Horn (J. Symbolic Logic vol. 16 (1951) pp. 14–21) has shown that all classes of algebras characterizable by closed sentences of conditional type are closed under passage to direct union and that this result cannot be improved by allowing a larger class of characterizing sentences describable in terms of the quantifiers and propositional structure of their prenex normal forms. The problem whether all classes of algebras characterizable by closed sentences, but not by those of conditional type, fail to be closed under direct union is still unsolved. In this paper, building on a suggestion and work by Tarski (Proceedings of the International Congress of Mathematicians, Cambridge, 1950, vol. 1, pp. 705–720) and work by McKinsey (J. Symbolic Logic vol. 8 (1943) pp. 61–76), two sufficient conditions for nonclosure under direct union of arithmetical classes of algebras not characterizable by functions of conditional type, and a sufficient condition for closure, not known to be co-extensive with Horn's condition, are proved. (Received September 14, 1954.)

68*t*. A. R. Schweitzer: *A survey of differential equation theories and related subjects.*

The author first discusses means of transition from integral equations (considered in a previous paper) to differential equations and then classifies research in differential equations according to leading schools indicated by the following names. I. Gauss, Pfaff, Grassmann, Riemann, Fuchs; Weierstrass. II. Pincherle, Volterra. III. Lagrange, Cauchy, Lamé, Picard, Poincaré. IV. F. R. Moulton, G. D. Birkhoff. It is assumed that the subject of differential equations had its origin in the mathematical interpretations of natural phenomena such as astronomy and physics and that differential equations, as well as integral equations and integro-differential equations, are encompassed by functional equations. As a basis for orientation in the extensive domain of differential equations the author selects differential equations of the Fuchsian class. Subjects connecting investigations of Fuchs with researches of Picard, Poincaré, Birkhoff include the method of successive approximations (Picard), automorphic functions (Poincaré), and singular points of differential equations (Birkhoff). Subjects related in a wider sense to differential equations include transformation groups, boundary value problems, differential geometry, and topology. Reference is made to the treatises and papers of L. Schlesinger and to the collected mathematical papers of G. D. Birkhoff. (Received September 14, 1954.)

STATISTICS AND PROBABILITY

69. Miriam Lipschutz: *On the approximation to stable distributions.*

Consider a sequence of identically distributed, independent, positive, continuous random variables X_k, with distribution function $F(x)$. Let $1-F(x)=h(x)/x^{\gamma}$, $0<\gamma<2$, where $\lim_{x\to\infty} h(cx)/h(x)=1$ for any positive constant c. The sum $S_n=\sum_{k=1}^{n} X_k$ properly normalized will tend to a stable distribution $G_\gamma(x)$ of index $(\gamma, -1)$. Let $r(m)\to\infty$ as $m\to\infty$. If for $x<r(m)^{s/\gamma}$, say, the expansion $h(mx)/h(m)=1+\sum_{k=1}^{s} a_k l_k(x)/r(m)^k$ be valid, then under some mild auxiliary conditions on the functions $l_k(x)$, the following results hold for $0<\gamma<1$: (a) $\operatorname{Sup}_{-\infty<x<\infty} \left|P(S_n<b_nx)-G_\gamma(x)\right|<A/r(b_n)$. (b) If $x\to 0$ as $n\to\infty$, but $x>[((1+\epsilon)/k_\gamma) \lg r(b_n)]^{(\gamma-1)/\gamma}$, then $P(S_n<b_nx)\sim Bx^{\gamma/2(1-\gamma)} \exp -k_\gamma x^{-\gamma/(1-\gamma)}$, where B and k_γ are constants. Similar results are obtained in the case $1<\gamma<2$, for $P(S_n-n\mu<b_nx)$ where $\mu=E(X)$, when x is finite and also when $x\to-\infty$. The method used is that of characteristic functions. The difficulties previously encountered by investigators in an attempt to use this method are overcome by noting that in convergence to a stable distribution the major contribution is due to the maximum term. Hence the author has first estimated the condition c.f. of S_n for a given value of the maximum. This bounds the range of integration of the other $n-1$ terms and enables the necessary expansions to be made. The probability that the maximum term exceeds this range is shown to be negligible. (Received September 16, 1954.)

TOPOLOGY

70*t*. E. H. Brown, Jr.: *Finite computability of the homotopy groups of finite groups.*

Let G be an abelian group and n an integer. According to J. C. Moore (Ann. of Math. vol. 59 (1954) pp. 549–557) a space X is said to be of homotopy type (G, n) if and only if X is simply connected, has trivial homology except in dimension n, and $H_n(X)$ is isomorphic with G. The homotopy groups of such a space depend only on G and n and, therefore, are denoted by $\pi_i(G, n)$. The aim of this paper is to prove that $\pi_i(g, n)$ is finitely computable when G is finite, and from this to prove the finite computability of the stable homotopy groups of spheres. (Received September 24, 1954.)

71. J. R. Munkres: *A note on the Hurewicz theorem.*

Proofs of the Hurewicz theorem appeal (usually implicitly) to the homotopy addition theorem, an "obvious" proposition whose proof entails considerable complications of a technical nature (See Hu, *The homotopy addition theorem*, Ann. of Math. vol. 58, p. 108). The situation becomes somewhat simpler when one uses (as has become common in recent years) the cubic rather than the simplicial singular homology. Eilenberg's proof of the Hurewicz theorem (Ann. of Math. vol. 45, pp. 439–443) is readily adapted to this situation, and the appeals to the homotopy addition theorem may be replaced by the following special case of it, which is proved in this paper: Let u be a map of the unit $n+1$ cube, I_{n+1}, into X which maps the $n-1$ skeleton into $x \in X$. Let d be the homology boundary operator; then du may be written $\sum_1^m \epsilon_i v_i$, where $\epsilon_i = \pm 1$ and v_i is a map of $(I_n, \dot{I}_n) \rightarrow (X, x)$, so that it defines an element $[v_i]$ of $\pi_n(X, x)$. Then $[v_1]^{\epsilon_1} \circ \cdots \circ [v_m]^{\epsilon_m} = 0$, where $\circ$ is the group operation in π_n. (Received August 2, 1954.)

L. W. Cohen,
Associate Secretary

THE NOVEMBER MEETING IN UNIVERSITY, ALABAMA

The five hundred and seventh meeting of the American Mathematical Society was held at the University of Alabama in University, Alabama on Friday and Saturday, November 26–27, 1954. About 130 persons registered, including 99 members of the Society.

By invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings, Professor Nathan Jacobson of Yale University addressed the Society Friday evening on the subject *Division rings*, with Professor Alfred Brauer presiding.

Sessions for contributed papers were held Friday afternoon and Saturday morning, Professors A. S. Householder, G. B. Huff, Wallace Givens, A. D. Wallace, and F. A. Lewis presiding.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title. Where a paper has more than one author, that author whose name is followed by "(p)" presented it. Professor González was introduced by Professor F. A. Lewis, Dr. Swift by Professor J. H. Roberts, and Mr. Stallard by Professor W. M. Whyburn.

ALGEBRA AND THEORY OF NUMBERS

72. A. T. Brauer: *Limits for the characteristic roots of a matrix.* VII.

Some of the results of the former parts of this paper [I, Duke Math. J. vol. 13 (1946) pp. 387–395; II, ibid. vol. 14 (1947) pp. 21–26; III, ibid. vol. 15 (1948) pp. 871–877; IV, ibid. vol. 19 (1952) pp. 75–91; V, ibid. vol. 19 (1952) pp. 553–562; VI, ibid. vol. 22 (1954) to be published] will be improved further by obtaining bounds for the components of the characteristic vectors. Let $A=(a_{\kappa\lambda})$ be an arbitrary matrix, ω a characteristic root, $\mathfrak{X}=(x_1, x_2, \cdots, x_n)$ a characteristic vector belonging to ω, and x_r the absolute greatest component. Then ω lies in the circle C_r with center at a_{rr} and radius $P_r=\sum_{\lambda\neq r}|a_{r\lambda}|$. Let R be a closed subregion of C_r containing ω and d_κ be the minimum distance of $a_{\kappa\kappa}$ from R. Set $t_{r\kappa}=1$ if $d_\kappa=0$, and otherwise $t_{r\kappa}=\min(1, P_\kappa/d_\kappa)$ for $\kappa=1, 2, \cdots, n$; $\kappa\neq r$. It is shown that $|x_\kappa|\leq t_{r\kappa}|x_r|$, and this result is improved by iteration. It follows that ω lies in the circle $|z-a_{rr}|\leq\sum_{\lambda\neq r}|a_{r\lambda}|t_{r\lambda}$ contained in C_r. Moreover, if the inequalities for the x_κ contradict the rth of the corresponding linear equations, then ω does not lie in R. If A is non-negative, ω the greatest positive root, and x_m the smallest component, then lower bounds for x_κ/x_m are obtained. Using these results it is often easy to compute ω as exactly as wanted. (Received October 13, 1954.)

73. C. C. Buck: *The algebraic aspect of integration in space.*

By "integration in space" is meant the n-dimensional analogue of the notions of integration along a curve, integration over a surface, etc. The term "algebraic" indicates that the discussion is restricted to topics that can be studied without the use of a limit process. An algebraic definition for space integrals has been improvised from a set of postulates for the notion of integral which were stated by Lebesgue in 1904. (Received October 13, 1954.)

74t. Leonard Carlitz: *On a theorem of Stickelberger.*

Making use of a slight extension of the theorem $(D/p)=(-1)^{n-r}$, where D is the discriminant of $f(x)$ and $f(x)\equiv g_1(x)\cdots g_r(x)$ (mod p), where the $g_i(x)$ are distinct irreducible polynomials, the following result is obtained. Let $D(a_1,\cdots,a_n)$ denote the discriminant of $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$. Let N denote the total number of solutions of $D(\xi_1,\cdots,\xi_n)=\eta^2$, ξ_i, $\eta\in GF(q)$, q odd, N' the number of solutions with $\eta\neq 0$. Then $N=q^{n-1}$, $N'=q^n-q^{n-1}$, $n>1$. (Received October 4, 1954.)

75t. Leonard Carlitz: *Some arithmetic properties of elliptic functions.*

Making use of the familiar Fourier series for sn x, ns x, etc., it is shown in this paper that one can prove rapidly such results as Kummer's congruences for the coefficients in the power series expansions (see Duke Math. J. vo.l 20 (1953) pp. 1–12). In the case of ns $x=\sum \beta_m x^m/m!$, an explicit relation connecting β_m with the Bernoulli number B_m leads to many properties of β_m, including some new ones. Moreover, congruences (of a restricted kind) are obtained for the coefficients of the Weierstrass Al-functions. (Received October 4, 1954.)

76t. Leonard Carlitz and J. H. Hodges: *Representations by hermitian forms in a finite field.*

Let $q=p^n$, $p>2$, and suppose that $\theta\in GF(q^2)$, $\theta^2=\nu\in GF(q)$, but $\theta\notin GF(q)$. Then $\alpha=a+b\theta\in GF(q^2)$ if $a, b\in GF(q)$. By the conjugate of α is meant $\bar{\alpha}=a-b\theta$. If $A=(\alpha_{ij})$ is a square matrix, $\alpha_{ij}\in GF(q^2)$, let $A^*=\bar{A}'=(\bar{\alpha}_{ij})'$, where the prime denotes transpose. Then A is hermitian if $A^*=A$. In this paper the authors determine the number $N_t(A, B)$ of $m\times t$ matrices U such that $U^*AU=B$, where A and B are hermitian, A is nonsingular of order m, B is of order t and rank $r\leqq m$, and all matrices have elements in $GF(q^2)$. Incidentally, the number $N(m, r)$ of hermitian matrices of order m and rank r is found. Also a certain exponential sum $H(B, s)$, which for $B=0$ reduces to $N(t, s)$, is considered. Using a relation between $N_t(A,B)$ and $H(B, s)$, they are able to explicitly evaluate the latter. They include an application of the sum $H(B, s)$ to the solution of a problem in partitions of hermitian matrices. It is worth noting that all of the results obtained hold as well if A and B are skew-hermitian ($A^*=-A$, $B^*=-B$), for then the matrices θA, θB are hermitian and may be used in place of A and B above. (Received October 4, 1954.)

77. P. F. Conrad: *Extensions of ordered groups.*

For definitions see Proc. Amer. Math. Soc. vol. 5 (1954) pp. 323–328. The factor group C'/C'' of convex subgroups of an o-group G where C' covers C'' is called a component of G. If the components of G are d-closed, then there exists an a-closed a-extension of G with each component o-isomorphic to the group R of real numbers w.r.t. addition. A condition on the components of G that is necessary and sufficient for G to be d-closed is derived. Each a-closed o-group of rank 2 is d-closed and has both components o-isomorphic to R. Methods for construction of o-groups are given. (Received October 18, 1954.)

78. D. O. Ellis: *Note on reflexive relations.*

One studies the algebra of reflexive relations in a given set under the customary partial ordering and the generalized relational product recently employed in universal algebra (A. W. Goldie, *The Jordan-Hölder theorem for general abstract algebras*, Proc. London Math. Soc. vol. 52 (1950) pp. 107–131). One also shows that any such product of two reflexive relations gives rise to a partial ordering in a subset of the original set and that every such partial ordering may be obtained in this way. The methods are those of elementary set-theory and lattice-theory. (Received September 27, 1954.)

79. Seymour Ginsburg (p) and J. R. Isbell: *Certain real functions on Boolean algebras.*

Let B be a complete Boolean algebra. f is a quasi-simple function on B if f is a real-valued function, the domain of each value being a principal ideal of B. Multiplication and addition of two quasi-simple functions is defined. Let $S(B)$ ($S^*(B)$) be the set of all (bounded) quasi-simple functions, endowed with the uniform convergence topology ($S^*(B)$ is normed in the usual manner). The ideal structure of the algebra $S(B)$ is intimately connected with the ideal structure of B. There is a one-one correspondence between the ideals of $S(B)$ and the ideals of B which preserves inclusion, principal ideals, and prime ideals. Another sample result: an ideal in $S(B)$ is closed if and only if the corresponding ideal in B is closed under countable unions. The ideal structure of $S^*(B)$ is less intimately bound to the ideal structure of B. The linear space conjugate to $S^*(B)$ is obtained. A necessary and then a sufficient condition for $S^*(B)$ to be complete, together with some examples, are given. (Received September 22, 1954.)

80t. Seymour Ginsburg and J. R. Isbell: *On a theorem of Sikorski.*

The pertinent results of R. Sikorski (Fund. Math. (1952)) are to the effect that if an $\aleph$-complete Boolean algebra B is isomorphic to a C-additive field of sets, F, $C<\aleph$, then B is isomorphic to some $\aleph$-additive field of sets. Here it is shown under somewhat weaker conditions that any such F is already $\aleph$-additive. The point is contained in two observations: (1) a field representation of B is $\aleph$-additive if and only if the two-valued measure on B determined by each point in an element of the field is $\aleph$-additive; (2) the results of S. Ulam (Fund. Math. (1930)) on two-valued measures on complete atomic Boolean algebras apply for all Boolean algebras. A different proof of Ulam's results is given. (Received September 22, 1954.)

81. E. H. Hadlock: *On the equivalence of ternary quadratic forms.*

A necessary condition for a ternary quadratic form f to be equivalent to a form f' with no term involving the product of distinct variables is that there exist a certain arithmetical invariant associated with f and represented primitively by f. A necessary and sufficient condition for the equivalence of f and f' is that there exist values of the variables for which the ternary quadratic form U will be a common divisor of the three linear forms Y_i for these same values of the variables. Moreover the coefficients of U and of Y_i are quadratic functions of X_{i3}, $i=1, 2, 3$, and the coefficients of X_{i3} are the coefficients of f. Finally X_{i3} are three linear functions of c_{j3}, $j=1, 2, 3$, associated with f, and c_{j3} are the elements of the third column of the transformation which takes f into f'. (Received October 13, 1954.)

82. A. S. Householder: *Norms of vectors and matrices.*

In the terminology of Faddeeva (*Computational methods of linear algebra* (in Russian), Moscow and Leningrad, 1950), a matrix norm $\|A\|$ is said to be consistent with a vector norm $\|x\|$ in case $\|Ax\| \leq \|A\| \cdot \|x\|$ for every A and x, and to be subordinate in case for every A there exists an $x \neq 0$ with the equality holding. Given an arbitrary vector g of positive elements, one can define a vector norm $\|x\|_g$ (and another $\|x\|_{g'}$) and a subordinate norm $\|A\|_g$ (or $\|A\|_{g'}$). These have the property that, given a particular matrix A, if $|A|$ is the matrix of absolute values, and if $|A|g = g_1$, then $\|A\|_{g'} \leq \|A\|_g$. Such a property is desirable for sharpening convergence proofs and estimates of computational error. (Received August 27, 1954.)

83. C. W. Huff: *Existence of noncommuting matrices satisfying the logarithmic equation.*

John von Neumann has shown (Math. Zeit. vol. 30 (1929) p. 11) that when A and B commute and the norms of $A-I$, $B-I$, and $AB-I$ are all less than one, then A and B satisfy the logarithmic equation $\log AB = \log A + \log B$, where the norm of A is $(AA^*)^{1/2}$. However, this paper shows that there are noncommuting matrices of any order greater than one satisfying the logarithmic equation by means of the following theorem, which is a consequence of previous results (Rend. Circ. Mat. Palermo (2) vol. 2 (1953) pp. 1–5): Let $f(w) = (e^w - 1)/w$ for $w \neq 0$ and $f(0) = 1$. If $A = \begin{pmatrix} 0 & a \\ s & 0 \end{pmatrix}$ and $B = \begin{pmatrix} s & b \\ 0 & s \end{pmatrix}$ where $f(-s) = 1$ and $bs \neq 0$, then the characteristic roots of $\exp A - I$, $\exp B - I$ and $\exp (A+B) - I$ are all less than one in absolute value. Furthermore, $\log (\exp A)(\exp B) = \log (\exp A) + \log (\exp B)$ and $(\exp A)(\exp B) \neq (\exp B)(\exp A)$. (Received October 14, 1954.)

84. W. V. Parker: *A note on a theorem of Roth.*

In a recent paper, Roth (Proc. Amer. Math. Soc. vol. 5, pp. 1–3) proved the following theorem. If A and B are $n \times n$ matrices with elements in the field F, whose characteristic polynomials are $a_0(x^2) - xa_1(x^2)$ and $b_0(x^2) - xb_1(x^2)$ respectively, where $a_0(x)$, $a_1(x)$, $b_0(x)$, and $b_1(x)$ are elements in the polynomial domain, $F[x]$, of F; and if the rank of $A - B$ does not exceed unity, then the characteristic polynomial of AB is $(-1)^n[a_0(x)b_0(x) - xa_1(x)b_1(x)]$. In this note the theorem is extended to give the characteristic polynomials of several other matrices which are polynomials in the matrices A and B. (Received October 13, 1954.)

85. P. B. Patterson: *Almost regular forms*

Almost characteristics of the first, second, and third kinds are defined. Also defined are almost regular forms and almost regular forms of the first, second, and third classes. It is shown that there exist infinitely many genera containing at least two classes of positive forms possessing an almost characteristic of the second or third kind where the invariants associated with the forms may or may not contain a common odd prime factor to an odd power. The method used is similar to that employed by B. W. Jones and E. H. Hadlock (Proc. Amer. Math. Soc. vol. 4 (1953) pp. 539–543) to show primitive ternary indefinite quadratic genera of more than one class. (Received October 11, 1954.)

86. B. M. Seelbinder: *Bounds for positive solutions of pairs of linear Diophantine equations.*

Let P be a $2 \times k$ matrix with positive integral elements a_i ($i = 1, 2, \cdots, k$) in the first row and positive integral elements b_i in the second row such that the elements of each row are relatively prime and such that the set of all minors $D_{ij} = a_i b_j - a_j b_i$ of order two has no common divisor. Assume that the columns are so arranged that

$b_1/a_1 \leq b_2/a_2 \leq \cdots \leq b_k/a_k$. Set $t_i = (a_i, b_i)$; let $L_1 = D_{1k}(D_{12}+D_{13}+\cdots+D_{1,k-1})/t_1t_k$ and $L_2 = D_{1k}(D_{2k}+D_{3k}+\cdots+D_{k-1,k})/t_1t_k$. It is proved that for every lattice point (m, n) in the interior of that angular region which is bounded by the straight lines $a_1n - b_1m = L_1$ and $b_km - a_kn = L_2$ and which lies completely in the first quadrant, the pair of linear Diophantine equations $a_1x_1 + a_2x_2 + \cdots + a_kx_k = m$ and $b_1x_1 + b_2x_2 + \cdots + b_kx_k = n$ always has a solution in positive integers $x_1, x_2, \cdots, x_k$. (Received September 22, 1954.)

87*t*. A. D. Wallace: *The Green substructures of a compact connected mob.*

A mob is a Hausdorff space together with a continuous associative multiplication. Let S be a compact connected mob. If $a \in S$ let $L_a = \{x \mid x \cup Sx = a \cup Sa\}$, see J. A. Green, Ann. of Math. (1950). Theorem. Let K be the minimal ideal of S, let $a \in S \backslash K$, and let C be the component of $S \backslash L_a$ containing K. Then $C^* \backslash C = L_a$. This result is related to a theorem due to Professor W. M. Faucett but the two theorems seem to be logically independent. If R_a is defined dually and if $H_a = R_a \cap L_a$ then an analogous result holds for H_a. Hence we have the corollary: If e is an idempotent not in K and if C is the component of $S \backslash H(e)$ containing K, then $H(e) = C^* \backslash C$. It should be observed that the conditions $\mathfrak{M}$ used by Green in his fundamental Theorem 8 do not hold for compact mobs. However, Green's proof mutatis mutandis is valid for a compact mob. Special cases of Green's Theorem 8 have been noted by Professor Faucett and Professor R. J. Koch in the compact case. (Received July 26, 1954.)

88*t*. A. D. Wallace: *The local structure of clans.*

A clan is a compact connected Hausdorff space together with a continuous associative multiplication with unit u. Let S be one such and let $H(u)$ be the maximal subgroup of S containing u. If $x \in S$ let $T(x)$ be the intersection of the sets $(S \backslash A)^*$ where A runs through all subcontinua of S not containing x. For each $x \in S$ the set $T(x)$ is a continuum (F. Burton Jones). Theorem: If $x \in H(u)$ and if C is the component of $H(u)$ containing x then $T(x) \subset C$. Corollary: If $H(u)$ is totally disconnected, then S is semilocally connected at each $x \in H(u)$. It may happen that $H(u)$ is precisely the set of points where S is semilocally connected. If $H(u)$ is connected then S may be semilocally connected at no point. (Received July 26, 1954.)

89. R. A. Willoughby: *Almost singular systems of linear homogeneous equations.* Preliminary report.

Given $A(\mu) \cdot x = 0$ where $A(\mu)$ is an $n \times n$ ($n = 30$ app.) matrix whose elements are real-valued functions of a real variable μ, the proposed problem is to find a computationally effective procedure for finding (approximately) (1) the least positive μ (if one such exists) for which $A(\mu)$ is singular and (2) the family of solution vectors x. It is assumed that one is given non-negative real numbers μ_1, μ_2 such that the desired μ is the only μ in the range $\mu_1 \leq \mu \leq \mu_2$ for which $A(\mu)$ is singular. A plan of attack being pursued is to form $B(\mu) = A^T(\mu) \cdot A(\mu)$ and form an (approximately) equivalent triangular system using the method outlined in Goldstine and von Neumann's *Numerical inverting of matrices of high order*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1021–1099. The desire is to choose μ so that at least one of the triangular diagonal elements is (almost) zero ($A(\mu)$ is so scaled that all the elements involved in the triangular system are less than one numerically). The resulting triangular system is then used to obtain the solution vectors. (Received October 4, 1954.)

ANALYSIS

90. L. P. Burton: *Conditions which preclude the existence of maximum, minimum, and minimax solutions of an ordinary differential system.*

The system $y'_{i+1}=f_{i+1}(x, y_i)$ $(i=1, \cdots, n)$, where $y_{n+1}(x)$ and $f_{n+1}(x, y_n)$ are, by definition, $y_1(x)$ and $f_1(x, y_n)$ respectively, is considered. Let R be a region in $(n+1)$-dimensional space and R: $(x_0, y_{10}, \cdots, y_{n0})$ be a point in R. By assuming continuity and certain monotone properties of the f_i in the subsets of R for which $x>x_0$ and $x<x_0$, it is shown that there cannot exist two distinct solutions $y_1(x), \cdots, y_n(x)$ and $z_1(x), \cdots, z_n(x)$, one of which has a limit point at P, the other of which approaches P as a limit, which are such that for any k, $1\leq k\leq n$, $y_k(x)\neq z_k(x)$ over an interval I: $x_0<x<x_0+a$, $a^2>0$. More precisely, by assuming continuity of the f_i in R it follows that in the case of nonunique solutions through P such solutions cannot be maximum, minimum, or minimax solutions. An extension is thus provided to the theory of critical solutions as originated by E. Kamke [Acta Math. vol. 58 (1932) pp. 57–85] and extended jointly by W. M. Whyburn and the author [Proc. Amer. Math. Soc. vol. 3 (1952) pp. 794–803]. It is further noted that the above general theorem gives a uniqueness theorem in case $n=1$ and, in case $n=2$, is closely related to the classical oscillation theory for the second-order differential system. (Received October 13, 1954.)

91*t*. V. F. Cowling: *Taylor methods of summation.*

In this paper the author obtains some new results regarding Taylor methods of summability and generalizes others previously obtained by Cowling and Piranian (Michigan Mathematical Journal vol. 1 (1952) pp. 73–78). Some typical results follow. Let $\{a_n\}$ be a sequence of complex numbers and suppose $\limsup |a_n|^{1/n}=1/R<\infty$ and let α be a real number $(0\leq\alpha<1/3,\ \alpha<R)$. If the series $\sum_0^\infty a_n n!/s(s+1)\cdots(s+n)$ is absolutely summable V_α at s_0, it is absolutely summable V_α throughout the half-plane $\mathrm{Re}(s)>\mathrm{Re}(s_0)$. Similar results are obtained regarding ordinary summability of these and Newton series. The Euler and Taylor summability of series of the form $\sum_0^\infty a_n e^{-sn^{\alpha}}$, $0<\alpha<1$, is also studied. (Received October 13, 1954.)

92. M. K. Fort, Jr.: *Imbedding homeomorphisms in flows.* Preliminary report.

Let J be an interval of real numbers, and let f be a continuously differentiable, order preserving homeomorphism of J onto J. Conditions are found under which there exists a flow (one-parameter group of homeomorphisms) F_t, $-\infty<t<\infty$, for which $F_1=f$ and for which each F_t is continuously differentiable. It is shown that if f has no fixed points except perhaps end points of J, and f' is positive-valued and nondecreasing, then: (i) if J is an open interval, many such flows exist; (ii) if J is a half open interval, a unique such flow exists; (iii) if J is a closed interval, there may not exist such a flow. (Received October 11, 1954.)

93. Tomlinson Fort: *Solving linear difference equations by means of the Dirichlet series transform.*

The solution of linear differential equations by means of the Laplace transform is now a commonplace. This same transform can be used to solve linear difference equations with constant coefficients although the ideas are more complicated. A power series transformation has also been used for this same purpose. In the present paper it is shown how a Dirichlet series transform is a simpler and more natural instrument for solving difference equations. A table of transforms is made for the more common functions and illustrative equations are solved. (Received October 4, 1954.)

94. Harry Gonshor: *On uniformly closed rings generated by certain operators on a Hilbert space.*

Let A be an operator which can be decomposed by direct integral theory onto two-dimensional operators. Then with suitable identifications, it has been proved that for all such A, A can be regarded as the identity function on a certain definite space $Z \cup Q$ (the same for all such A), where Z is the set of all complex numbers. To obtain the structure of the uniformly closed ring generated by A and A^*, it is convenient to define the notion of the spectrum which generalizes the usual definition for normal operators. Then the ring is isomorphic and isometric to a subring of the ring of all functions which are continuous and second order matrix valued on Q, and continuous and complex-valued on Z. For a large class of such operators the subring may be taken to be the whole ring. (Received October 25, 1954.)

95. M. O. González: *Bilateral Laplace transformation of distributions.*

If q is a distribution, its bilateral Laplace transform $Q(s)$ (s complex) is defined and the fundamental rules of operational calculus are proved. Some transforms are given and also applications to the derivation of certain results involving the δ functional and to the solution of several extended integral equations of the form $f(t) * T = S$, where $f(t)$ is a given function and S a given distribution. (Received October 13, 1954.)

96. J. W. Jewett: *Differentiable approximations to light interior transformations of plane domains.* Preliminary report.

Let D be a domain in the plane, let n be a positive integer, and let a continuous mapping f of $\bar{D}$ into the plane be light and strongly interior on D. Then there exists a sequence of continuous functions $f_k: \bar{D} \to E^2$ converging uniformly to f on $\bar{D}$ such that each f_k is light and strongly interior on D, agrees with f on the boundary of D, and is n times continuously differentiable on D. (Received October 18, 1954.)

97. W. B. Jurkat: *The consistency of Nörlund and Hausdorff methods.*

Let N, H denote the regular and real summability methods of Nörlund and Hausdorff respectively. Together with A. Peyerimhoff the following result is proved: If a series is N-summable to s' and H-summable to s'', then it is also summable in the generalized Abel sense to the common sum $s = s' = s''$. In particular, the methods of Nörlund and Hausdorff are consistent as it was conjectured by E. Ullrich. A corresponding result holds for functional methods. (Research supported by U.S.A.F., Office of Scientific Research, Contract AF 18(600)-691.) (Received November 9, 1954.)

98. C. W. McArthur: *Some applications of a theorem on uniform Cauchy points.*

It is well known that the permutation group $P(N)$ of the natural numbers N is of the second category relative to the topology induced on it by the Fréchet metric. Two theorems are proved which have as corollaries the following results. Suppose $(X, +, d)$ is a complete commutative semigroup with a pseudo metric d. Then a series $\sum_{i=1}^{\infty} x(i)$ in X is unconditionally convergent if and only if $[t \in P(N): \{\sum_{i=1}^{n} x(t(i))\}$ converges$]$ is second category in $P(N)$. If X is a weakly complete Banach space, $\sum_{i=1}^{\infty} x(i)$ is unconditionally convergent if and only if $[t \in P(N): \{\sum_{i=1}^{n} x(t(i))\}$ is bounded$]$ is second category. Suppose $s = \{s(i)\}$ is a sequence of real numbers. Let $E_{+\infty}(s) = [t \in P(N): \limsup_n \sum_{i=1}^{n} s(t(i)) = +\infty]$ and $E_{-\infty}(s) = [t \in P(N): \liminf_n \sum_{i=1}^{n} s(t(i)) = -\infty]$. Then $E_{+\infty}(s) \cap E_{-\infty}(s)$ is a residual everywhere dense G_δ subset of $P(N)$ if and only if $E_{+\infty}(s)$ is non-null and $E_{-\infty}(s)$ is non-null. (Received August 13, 1954.)

99t. J. S. MacNerney: *Continuous products in linear spaces.*

Let B denote the linear normed complete space of continuous linear transformations from a LNC space to itself. A *quasi-harmonic operator* is a function M from the ordered number pairs s, t to B (for each t), of bounded variation in s on each interval, such that (for each s) each of the limits $M(s, t-)$ and $M(s, t+)$ exists and such that $M(a, b)M(b, c) = M(a, c)$ and $M(a, a) = [M(a-, a) + M(a, a-)]/2 = [M(a, a+) + M(a+, a)]/2 = 1$ for each ordered number triple a, b, c. Extending the notion of a continuous product previously studied by the author (*Stieltjes integrals in linear spaces*, to appear in Ann. of Math.), a one-to-one correspondence, $M(s, t) = {}_s\prod^t \{1 + dF\}$, is established between the class of quasi-harmonic operators M and the class of functions F from the real numbers to B, of bounded variation on each interval, such that $F(0) = 0$ and $[F(s) - F(s-)]^2 = [F(s+) - F(s)]^2 = 0$ for each number s. Application is made to continuous continued fractions and to certain discontinuous and nonlinear integral equations. (Received October 13, 1954.)

100. E. P. Miles, Jr. (p) and Ernest Williams: *A basic set of homogeneous harmonic polynomials in k variables.*

For any set of non-negative integers (b_j) such that $b_1 \leq 1$ and $\sum_{j=1}^{k} b_j = n$, let $MW^n_{b_1 b_2 \cdots b_k}(x_1, x_2, \cdots, x_k) = \sum (-1)^{[a_1/2]}(n![a_1/2]!/\prod_{j=1}^{k} a_j! \prod_{j=2}^{k} ((b_j - a_j)/2)!) \cdot \prod_{j=1}^{k} x_j^{a_j}$ where the summation extends over all (a_j) such that (1) $a_j \equiv b_j \bmod 2$, $j = 1, 2, \cdots, k$, (2) $\sum_{j=1}^{k} a_j = n$, (3) $a_j \leq b_j$, $j = 2, 3, \cdots, k$. The polynomials MW form a basic set of homogeneous harmonic polynomials in k variables. The authors modify them to form solutions of the generalized wave equation $\sum_{j=2}^{k} \partial^2 u/\partial x_j^2 = \partial^2 u/\partial x_1^2$ and the generalized Laplace equation $\sum_{j=1}^{k} \partial^2 u/\partial x_j^2 = 0$. (Received October 4, 1954.)

101. E. P. Miles, Jr. and Ernest Williams (p): *A basic set of homogeneous harmonic polynomials in k variables.* II.

It is shown that elements $MW^n_{b_1 \cdots b_k}(x_1 x_2 \cdots x_k)$ of the author's basic set of homogeneous harmonic polynomials of degree n in k variables satisfy the following differential recursion formulas: (a) $(\partial/\partial x_1)\ MW^n_{1 b_2 \cdots b_k} = n MW^{n-1}_{0 b_2 \cdots b_k}$, $1 + \sum_{j=2}^{k} b_j = n$; (b) $(\partial/\partial x_j)\ MW^n_{b_1 \cdots b_j \cdots b_k} = n\ MW^{n-1}_{b_1 \cdots b_j - 1 \cdots b_k}$, $j = 2, 3, \cdots, k$, $b_1 = 0, 1$, $\sum_{j=1}^{k} b_j = n$; (c) $(\partial/\partial x_1)\ MW^n_{0 b_2 \cdots b_k} = -n \sum_{j=2}^{k} MW^{n-1}_{1 b_2 \cdots b_j - 2 \cdots b_k}$, $\sum_{j=2}^{k} b_j = n$; where $MW^n_{c_1 c_2 \cdots c_j \cdots c_k} = 0$, if $c_j < 0$. For $k = 3$ the author's basic set gives a single formulation for the four types given by Protter (Portugaliae Mathematica vol. 10 (1951)); the differential recursion formulas correspond to those of Protter. (Received October 13, 1954.)

102. E. J. Pellicciaro: *Concerning critical sets of type* 0.

In a paper presented to the Society [Bull. Amer. Math. Soc. Abstract 59-1-61] the author proposed a classification of critical sets. The present paper concerns itself with critical sets of type 0 and proves the following theorem: Let $M=M(H, p)$ be a critical set of $f(x)$ of type 0. If q is a boundary point of M belonging to M and U is a neighborhood of q, then U contains a collection T of critical points of $f(x)$ such that T intersects an infinite number of critical sets of $f(x)$ and if t belongs to T, then (i) $f(t)=f(p)$, (ii) for some neighborhood u of t, $f(x)\leqq f(p)$ or $f(x)\geqq f(p)$ throughout u, and (iii) t is a boundary point of some related complementary domain of $M(H, t)$. (Received October 13, 1954.)

103. W. C. Royster: *A coefficient problem for some classes of univalent Laurent series.*

Let $f(z)=\sum_{n=-\infty}^{\infty} a_n z^n$, $0<\rho<|z|<1$, $a_0=0$, be regular and univalent for $\rho<|z|<1$. Nehari and Schwarz (Proc. Amer. Math. Soc. vol. 5 (1954)) showed that if $f(z)$ belongs to the class of typically real Laurent series, the $|a_n|$ satisfy a sharp inequality and that if $f(z)$ belongs to the class of starlike functions represented by univalent Laurent series, the $|a_n|$ satisfy an inequality which was not sharp. In this paper the author improves the bound on $|a_n|$ for the starlikeness case; however, the bound again is not sharp. Sharp bounds are obtained for the class of functions which are starlike and satisfy the additional restriction that there is a real number μ such that Im $\{f(\rho e^{i(\theta+\mu)})\}$ and Im $\{f(e^{i(\theta+\mu)})\}$ change sign on the same ray. (Received October 13, 1954.)

104. F. W. Stallard: *Differential systems with interface conditions.* Preliminary report.

This paper considers systems of n first order ordinary linear differential equations with two point boundary conditions and interface conditions at other points of the interval of the independent variable. Solutions, which ordinarily have discontinuities of the first kind at the interfaces, are studied. It is shown that there exist a discontinuous matrix, D, satisfying the interface conditions and $D'(x)=0$, and an absolutely continuous matrix $W(x)$, satisfying an associated system, such that $D(x)W(x)$ is a solution of the given system and satisfies the boundary and interface conditions. The notion of a Green's matrix for such problems is introduced and its properties explored. (Received October 13, 1954.)

105. George Swift: *Irregular Borel measures on topological spaces.*

Let X be a topological space, and let $\mathcal{B}$, $\mathcal{C}$, $\mathcal{U}$, and $\mathcal{W}$ be the classes of all Borel, compact, open, and open Borel sets of X, respectively. Recall that a Borel measure μ is outer (inner) irregular at a set $B\in\mathcal{B}$ when $\mu(B)<\inf\{\mu(\mathcal{W}): B\subset\mathcal{W}\in\mathcal{W}\}$ $(>\sup\{\mu(C): B\supset C\in\mathcal{C}\})$. If X is σ-compact and μ is totally finite, then μ is outer irregular at $B\in\mathcal{B}$ if and only if μ is inner irregular at B'. Counterexamples exist for non-σ-compact X. Theorems concerning unions and differences of Borel sets at which μ is irregular are exhibited; in particular, if $\{A_i\}_{i=1}^{\infty}\subset\mathcal{B}$ are pairwise separable by sets of $\mathcal{W}$, $A=\bigcup_{i=1}^{\infty}A_i$, and $\mu(A)<\infty$, then μ is outer (inner) irregular at A if and only if there exists an i_0 $(i_0=1, 2, \cdots)$ such that μ is outer (inner) irregular at A_{i_0}. This cannot be relaxed to pairwise disjoint $\{A_i\}_{i=1}^{\infty}$. Sums and differences of irregular measures are found to be irregular in certain cases; for example, if $\{\mu_i\}_{i=1}^{\infty}$ is a sequence of Borel measures on X, $\mu=\sum_{i=1}^{\infty}\mu_i$, and $\mu(X)<\infty$, then μ is a Borel measure on X which is outer (inner) irregular at a set $B\in\mathcal{B}$ if and only if there exists an i_0 $(i_0=1, 2, \cdots)$ such that μ_{i_0} is outer (inner) irregular at B. Let X be Hausdorff, let Z be the set of all $x\in X$ such that $\mu(\{x\})>0$, and let $\mu(Z)<\infty$, then the set function

ψ defined by the relation $\psi(A)=\mu(A)-\mu(A\cap Z')$ for every $A\in\mathfrak{B}$ is a Borel measure on X such that (1) ψ is outer (inner) irregular at $B\in\mathfrak{B}$ if and only if μ is outer (inner) irregular at B, and (2) $\psi(\{x\})=0$ for every $x\in X$. (Received October 13, 1954.)

106. W. M. Zaring: *Multiply monotone complex sequences.*

Denote by $\{c_n\}$ a null sequence of complex numbers. Then trivially $\Delta^{\alpha}c_n=\sum_{k=0}^{\infty}C_{k-\alpha-1,k}c_{n+k}$ converges for $\alpha\geqq 0$. If $|\arg\Delta^{\alpha}c_n|\leqq\phi<\pi/2$, then we say that $\{c_n\}$ is (α,ϕ)-monotone. Using this definition of complex monotonicity some results of K. Knopp, *Mehrfach monotone Zahlenfolgen*, Math. Zeit. vol. 22 (1925) pp. 75–85, and L. Fejer, *Trigonometrische Reihen und Potenzreihen mit mehrfach monotone Koeffizientenfolge*, Trans. Amer. Math. Soc. vol. 39 (1936) pp. 18–59, are extended to complex sequences. (Received October 13, 1954.)

Applied Mathematics

107. R. G. Blake: *Finite differences in laminated orthotropic plane strain.*

In order to obtain an approximate solution to the problem of plane strain in laminated orthotropic material, the differential equation is replaced by a set of finite equations. Stencils are then obtained for use in solving these equations by the relaxation method. (Received October 13, 1954.)

108t. Henry Gerhardt: *Stresses in a catenoid membrane shell and the hyperbolic metric.* Preliminary report.

A. Foppl in his *Festigkeitslehre* derives the differential equation of the stress function for a circular membrane. This paper attempts to extend his equation to the more general case of the catenoid membrane shell. A plane diagram of the stresses is given, using the hyperbolic rather than the euclidean metric. (Received October 11, 1954.)

109. Nathaniel Macon: *A continued fraction for e^x.*

Using the continued fraction of Gauss, one can expand e^x as a continued fraction $b_0+a_1/[b_1+a_2/(b_2+\cdots)]$, where $b_0=1$, $a_1=x$, $a_{2n}=-x/[2(2n-1)]$, $a_{2n+1}=x/[2(2n+1)]$, and $b_n=1$ $(n=1,2,3,\cdots)$. It is shown that this expansion can be contracted to yield one in which $b_0=1$, $b_1=1-x/2$, $a_1=x$, $b_n=2n-1$, and $a_n=x^2/4$ $(n>1)$. This latter expansion converges quite rapidly and is well suited for digital computation. For example, if $-1\leqq x\leqq 1$, one has $|e^x-A_6/B_6|<10^{-10}$. (Received October 14, 1954.)

110. C. L. Seebeck, Jr. (p) and H. Hoelzer: *Curve fitting using differential equations.*

The problem treated is: Given a set of points (x_j, y_j), to find a differential equation plus initial conditions whose integral curve will approximate the given data. Linear homogeneous differential equations are used in a manner analogous to polynomials in the usual least square theory. The original set of y_j are damped by multiplication by e^{-px}. It is assumed that a curve $y=f(x)$ passes through this set of points and that its Laplace transform converges with its real part equal 0. If iw is its imaginary part, an identity in w is established by applying the same Laplace transform to a given type of differential equation. Using this identity, the constants in the differential equation are determined in 3 different ways: 1. Using computed points on the Laplace transform of a set of chords joining the given damped points; 2. Using approximate initial conditions and moments of area of $f(x)$; 3. Using only the moments of area. When linear homogenous differential equations are used, the techniques employed involve only function evaluation and solution of linear equations. Illustrations are given. (Received October 13, 1954.)

GEOMETRY

111. J. C. Morelock: *A note on the complete system of curves $|A|$ cut out on F_4.*

In this paper it is shown by simple algebraic processes that the complete system of curves $|A|$ cut out on F_4 by all surfaces of order P (where P is a fixed prime) will in each case have certain characteristics, in fact: Theorem: *A necessary and sufficient condition that the complete system of curves $|A|$ (cut out of F_n by all surfaces of order P) shall have its genus equal to its dimension is that $n=4$.* (Received October 13, 1954.)

112. J. C. Morelock and N. C. Perry (p): *A classification of algebraic surfaces invariant under cyclic collineations.*

In algebraic geometry it is of interest to examine homogeneous polynomial surfaces of prime degree p which transform into themselves under the collineation T defined by $(x', x_2', x_3', x_4') = (x_1, Ex_2, E^2x_3, E^3x_4)$ where $E^p = 1$. A surface $F(x_1, x_2, x_3, x_4) = 0$ is invariant if each of its terms $x_1^a x_2^b x_3^c x_4^d$ is transformed by T into the expression $E^s x_1^a x_2^b x_3^c x_4^d$ (where s is of the same residue class (modulus p) for all terms). This property of a term is called s-invariance. The present paper proves the following results: Theorem: Necessary and sufficient conditions that $x_1^a x_2^b x_3^c x_4^d$ be s-invariant are that a, b, c, and d are respectively members of the congruence classes $\overline{n}$, $\overline{r-2n-s}$, $\overline{n-r+s}$ $\overline{r}$ (determined by the parameters n and r); and that n and r be restricted by certain lengthy inequalities which restrict the range of n and r as functions of s and p. (Received October 13, 1954.)

113. T. L. Saaty: *The number of vertices of a polyhedron.*

Relationship between convex polyhedra and a set of linear inequalities with a linear function to optimize is indicated. The number of steps involved in using iterative procedures for solving this type of problem, such as the simplex process, depend on the number of vertices of the convex set formed by the inequalities. It is desired to find upper bounds to the number of vertices as a function of the number of inequalities. Such an upper bound is determined, using Euler's formula and other auxiliary relations. It is found to be a polynomial of degree $(n-2)$ in the number of inequalities where n is the number of the dimension dealt with. A theorem is given showing that an upper bound as a linear function of the number of inequalities F_{n-1} dominated by the fundamental upper bound $F_{n-1}!/(F_{n-1}-n)!n!$ does not exist for $n \geq 4$. Generalizing this theorem using polynomials in F_{n-1} would presumably show that the results obtained here cannot be further refined. (Received October 11, 1954.)

114. H. C. Wang: *A note on invariant affine connections.*

Let G be a transitive group of differentiable homeomorphisms of a differentiable manifold M, and H the isotropic subgroup at a point p of M. Nomizu [Amer. J. Math. vol. 76 (1954) pp. 33–65] recently discussed and proved the existence of invariant affine connections II under G when G/H is a reductive homogeneous space. In this note, the author gives a both necessary and sufficient condition for the existence of such a Π. In fact, let $\mathfrak{L}$ denote the tangent space of M at p, and τ the natural linear representation of H over $\mathfrak{L}$. Then there exists an affine connection over M invariant under G if and only if τ is faithful. Moreover, if there exists one, there must exist a symmetric one. The proof consists in merely geometrical arguments about the bundle of affine frames over M. (Received October 13, 1954.)

TOPOLOGY

115. R. W. Bagley: *A maximal set in the lattice of topologies.*

Let Λ be the lattice of T_1 topologies on an infinite set. The ordering in Λ is defined as follows: $\alpha \leq \beta$ whenever each set open under β is also open under α. Let Λ_0 be the subset of Λ (with the same ordering) consisting of all elements which are lattice products of hyperplanes in addition to the last element of Λ. Λ_0 is a full set algebra and is maximal with respect to containing unique complements and finite lattice products. (Received October 8, 1954.)

116. T. R. Brahana: *The local Betti numbers in topological product spaces.*

The local Betti numbers $p^r(x, y)$, $r=0, 1, \cdots$ (see Wilder, *Topology of manifolds*), of points (x, y) in the topological product $X \times Y$ of two locally compact topological spaces X and Y can be determined by knowledge of the local Betti numbers $p^s(x)$ and $p^t(y)$ of the projections of the points on the factor spaces, by the formula $p^r(x, y) = \sum_{s+t=r} p^s(x)p^t(y)$, which is established in this paper. The proof depends on the Čech homological equivalence of the following form:, $H^r(X \times Y, X \times B \cup A \times Y) \approx \sum_{s+t=r} H_s(X, A) \otimes H_t(Y, B)$, where (X, A) and (Y, B) are compact pairs. Application of the formula above is made to prove that the topological product of generalized manifolds (subject to mild restrictions) is again a generalized manifold; the more general statement holds: the bundle space of a fibre bundle whose base and fibre are in the class of generalized manifolds referred to is again a generalized manifold. Application of the theorem is made to the problem of dimension of product spaces. (Received September 29, 1954.)

117. C. E. Capel: *Space of subsets and cartesian products.*

For a space X, let $S(X)$ denote the space of all non-void, closed subsets of X with the Frink topology. (If U and V are open subsets of X then the collection of all closed sets contained in U and meeting V is a subbasis element of this topology.) Let $\{X_a\}$ be a collection of spaces with $P\{X_a\}$ and $P\{S(X_a)\}$ denoting the cartesian products; then $P\{S(X_a)\}$ can be considered as a subset of $S(P\{X_a\})$. It is shown that if the projections p_a: $P\{X_a\} \rightarrow X_a$ are closed functions, then for any space Y and any continuous function f: $P\{S(X_a)\} \rightarrow Y$ there is a continuous extension f^*: $S(P\{X_a\}) \rightarrow Y$. Hence $P\{S(X_a)\}$ is a retract of $S(P\{X_a\})$. (Received October 15, 1954.)

118. W. M. Faucett and R. J. Koch (p): *Complements of maximal ideals in compact semigroups.*

Let S denote a compact Hausdorff topological semigroup. It is known that S contains maximal proper ideals, necessarily open. Let J be a maximal proper ideal of S; it is shown that S/J (Rees quotient) is either the trivial semigroup of two elements or else is completely simple [Rees, Proc. Cambridge Philos. Soc. vol. 36 (1940) pp. 387–400]. It follows that the complement of J is the disjoint union of compact isomorphic groups and of sets whose square lies in J. The equivalence between regular and simple elements [J. A. Green, Ann. of Math. vol. 54 (1951) pp. 163–172] in a compact semigroup is demonstrated. The maximal subgroup containing an idempotent e is characterized as the set of bi-ideal [Good and Hughes, Bull. Amer. Math. Soc. Abstract 58-6-575] generators of eSe. Maximal elements are defined and studied. A forthcoming result of R. J. Koch and A. D. Wallace is generalized as follows: if $S^2 = S$, then $S = S\tilde{E}S$ where $\tilde{E}$ is a set of idempotents chosen one from the complement of each maximal proper ideal. (Received October 7, 1954.)

119. W. L. Gordon: *Separating sets in Euclidean spaces.*

The following theorem, upon application to compact subsets X of Euclidean $n+2$ space, provides sufficient intrinsic conditions for X to cut R^{n+2}, thus yielding a generalization and variation of a classical theorem of Janiszewski. Let X be paracompact with A and B closed subsets satisfying $X = A \cup B$ and the homomorphism $i^*: H^n(A) \rightarrow H^n(A \cap B)$ (the nth Alexander-Kolmogoroff cohomology groups, coefficients arbitrary) not onto. If either (a) $H^n(B) = 0$ or (b) there is a continuous mapping of A onto B whose restriction to $A \cap B$ is deformable in B to the identity map of $A \cap B$, then $H^{n+1}(X)$ is not zero. (Received October 13, 1954.)

120. J. S. Griffin, Jr.: *Intrinsic systems of covering paths.*

Let $\mathcal{B} = \{B, X, \pi, F, \mathcal{V}, \phi, G\}$ be a fibre bundle and let $\mathcal{P}$ be a family of paths in X. Families $\mathcal{J}$ subject to the following condition are studied: for each $x \in X$ and $p \in \mathcal{P}$ such that $\pi(x) = p(t)$, there is a unique $q \in \mathcal{J}$ such that $q(t) = x$ and $\pi q = p$. Since $\mathcal{J}$ is a function space, there are natural topological restrictions; in particular, in the presence of suitable hypotheses $\mathcal{J}$ may be made to yield covering homotopies. In this case, $\mathcal{J}$ gives a family of paths in the space B^F and thus it is natural to consider the associated principal bundle, or more generally any associated bundle. Finally such an $\mathcal{J}$ may be shown to be equivalent to a family of paths in G. On the one hand this notion includes the families of (unique) covering paths for F totally disconnected, and the families of (unique in a sense) covering paths for G totally disconnected. On the other, if $\mathcal{B}$ is the tangent bundle for a differentiable manifold and $\mathcal{P}$ is the family of all differentiable paths, then an affine connection for X is equivalent to a family $\mathcal{J}$. (Received November 29, 1954.)

121. O. G. Harrold, H. C. Griffith, and E. E. Posey (p): *The imbedding in three-space of cells which are locally polyhedral except at boundary points.*

In this paper the authors give a new proof of their theorem on a characterization of tamely imbedded curves in three-space (Bull. Amer. Math. Soc. Abstract 60-4-542. To appear in Trans. Amer. Math. Soc.). The proof of the present paper is direct in the sense that a simple closed curve in three-space that has property P and which bounds a topological 2-cell that is locally polyhedral modulo the boundary is approximated "uniformly" by a sequence of unknotted polygonal simple closed curves. The essential construction is that of a sequence of piecewise linear homeomorphisms on three-space whose limit is a homeomorphism. (Received October 13, 1954.)

122. S. T. Hu: *On differential forms in differentiable spaces.*

As an obvious generalization of differentiable manifolds of class C^k, the author defined in 1949 the notion of the differentiable spaces of class C^k (See Ann. of Math. vol. 50, p. 266). A subset M of a euclidean space E^n is said to be a differentiable space of class C^k if there exist an open set U of E^n containing M and a differentiable mapping $r: U \rightarrow M$ of class C^k such that $r(x)=x$ for each x in M. r is called a retraction of class C^k. In the present paper, the differential forms in a differentiable space are defined in a natural way and the theorems of de Rham are established for this generalization. However, the main interest of the work is not the generalization but the simplification of the classical case by getting rid of the clumsy systems of local coordinates. (Received October 13, 1954.)

123. I. S. Krule: *On ordered topological spaces*. Preliminary report.

Let X be a Hausdorff space, and let L be a nonvoid closed transitive subset of $X \times X$. If $A \subset X$, let $L(A)=\{x \in X \mid (x, a) \in L \text{ for some } a \in A\}$. L is *monotone* if $L(x)$ is connected for each $x \in X$. L is *continuous* if $L(A^*) \subset L(A)^*$ for each $A \subset X$. If X is a continuum and L is continuous, then for each $x \in X$, $L(x) \neq \emptyset$, and X contains reflexive elements. Using definitions and techniques analogous to those used by Wallace and Koch in the study of topological semigroups, generalizations and order analogues of results in topological semigroups are obtained quite easily. For example, if X is a metric indecomposable continuum and L is reflexive, monotone, and continuous, then every element in X is L-minimal (cf. Wallace, Math. J. of Okayama Univ. vol. 3 (1953) p. 1). Examples are obtained which show that the analogy considered between topological semigroups and ordered spaces is incomplete. (Received October 13, 1954.)

124t. R. L. Plunkett: *A note on convex metrics with unique segments*. Preliminary report.

It is pointed out by R. H. Bing [*A convex metric with unique segments*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 167–174] that every continuous curve which has a convex metric with unique segments is contractible and locally contractible and examples are given of spaces which fail to have a convex metric with unique segments because of failure to be contractible, in one case, and locally contractible, in another. In this note, it is proved that each continuous curve which has a convex metric with unique segments is an absolute retract for separable metric spaces. Thus, the example by Borsuk [*Sur un espace compact localement contractile qui est n'est pas un rétracte absolu de voisinage*, Fund. Math. vol. 35 (1948) pp. 175–180] of a compact space which is contractible and locally contractible but which is not an absolute retract affords an example of a space which does not have a convex metric because it fails to be an absolute retract, although it is both contractible and locally contractible. (Received October 13, 1954.)

125. W. L. Strother: *A new characterization of Peanian spaces*.

Hahn and Mazurkewicz showed that a compact metric space X is a Peanian space if and only if X is a locally connected continuum. This paper shows that a metric space X is a Peanian space (continuous image of the unit interval) if and only if X is homeomorphic to a multi-retract of a Tychonoff cube. (A is said to be a multi-retract of X if there exists a continuous multi-valued function F on X to A such that $F(a)=a$ for all a in A.) (Received October 15, 1954.)

126t. P. M. Swingle: *Locally n-indecomposable and related concepts*.

Let R be a region, $x \in R$; W has a local essential sum at x w.r.t. R of the class (C) of subconnexes of W if there exists a region R' in R, $x \in R'$, such that $W \cdot R' \subset \bigcup(C)$

but no $C\cdot R'$ is contained in the closure of the sum of the other $C\cdot R''$s. A connexe W is locally n-indecomposable at x if, for every region R, $x\in R$, W has a local essential sum at x w.r.t. R of a class of n, but not more than n, subconnexes of W. If n is an integer >1, there does not exist a connexe W which is locally n-indecomposable at each of its points; also in a completely separable space there does not exist a connexe W which is locally finitely-indecomposable at each of its points, but then for $x\in$ connexe W, W at x either is locally (aleph-zero)-indecomposable, or locally finitely-indecomposable, or there exists an integer $n_x>0$ such that W is locally n_x-indecomposable there. If (1) the space also is compact, (2) W is locally n-indecomposable at $x\in W$, but (3) not locally (aleph-zero)-indecomposable at any point of the closure of W, then W is locally n-irreducible at x; also, if (1) and (3), W is locally n-indecomposable at x if and only if, for every region R, $x\in R$, W has a local essential sum at x w.r.t. R of a class (I) of n indecomposable subconnexes of W. (Received March 11, 1954.)

127. A. D. Wallace: *Struct ideals on compact spaces.*

A struct on a space X is a nonvoid closed and transitive subset of $X\times X$. If $A\subset X$ let $L(A)$ be all x such that $(x, a)\in L$ for some $a\in A$, where L is a struct on X. The set A is an L-ideal if $A\neq\square$ and if $L(A)\subset A$. We let $L_0(A)$ be the union of L-ideals contained in A. Lemma. If X is compact and if U is open, then $L_0(U)$ is open. Theorem. Let X be compact, let A be an L-ideal, and let B be a closed set with B not contained in A. There exists an L-ideal maximal among all L-ideals that contain A and do not contain B and each such is open. If $a\in X$ let $L_a\neq\{x \mid a\cup L(a)=x\cup L(x)\}$. One defines L-maximal and L-minimal elements. The complements of sets L_a with a L-maximal and $L_a\neq X$ are maximal L-ideals and conversely. If X is compact and if K is the set of L-minimal elements, then K is the union of all minimal L-ideals. If X is compact and if L satisfies also $L(A^*)\subset L(A)^*$ for all A, then K is closed. (Received July 26, 1954.)

128. R. F. Williams: *A note on unstable homeomorphisms.*

An example of an unstable homeomorphism of a compact metric continuum (the "two-solenoid") onto itself is given. This answers a question raised by W. R. Utz (Proc. Amer. Math. Soc. vol. 1 (1950) pp. 769–774) who defined the term *unstable homeomorphism* in the same paper: a homeomorphism f of a metric space X onto X is said to be *unstable* provided that there exists a fixed positive number δ, such that if x and y are distinct points of X, then there exists an integer n, such that $\rho[f^n(x), f^n(y)]>\delta$. (This research was sponsored by The National Research Foundation under contract NSF-G358.) (Received June 28, 1954.)

129. C. T. Yang: *Mappings from spheres to euclidean spaces.* II.

As a continuation of recent works of the author (Bull. Amer. Math. Soc. Abstracts 59-6-720 and 60-4-547) it is proved that a mapping of a $(3m)$-sphere into the euclidean m-space maps the end points of some 3 mutually orthogonal diameters into a single point. (Received October 14, 1954.)

J. H. ROBERTS,
Associate Secretary

THE NOVEMBER MEETING IN IOWA CITY

The five hundred eighth meeting of the American Mathematical Society was held in conjunction with a meeting of the Central Section of the Institute of Mathematical Statistics at the State University of Iowa on Friday, November 26, 1954. There were about 90 registrations, including 60 members of the Society.

Professor R. P. Boas presided at an address delivered by Professor Vaclav Hlavaty of Indiana University. The address was entitled *Unified field theory* and was given by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings. There was a session for contributed papers in the afternoon presided over by Professor H. T. Muhly.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title.

ALGEBRA AND THEORY OF NUMBERS

130. H. W. Becker: *A case history of the impotence of abstract algebra in Diophantine analysis.*

A Pythagorean tetrahedron has $t, u, v, y, x, z = (\alpha^2+\beta^2)(\gamma^2+\delta^2) \pm 4\alpha\beta\gamma\delta$, $2(\beta\gamma+\alpha\delta)(\alpha\gamma\pm\beta\delta)$, $(\alpha^2-\beta^2)(\gamma^2-\delta^2)$, $4[\alpha\beta\gamma\delta(\alpha^2+\beta^2)(\gamma^2+\delta^2)]^{1/2}$. In Dickson's *History* II, pp. 644–647, z is satisfied in large integers by $\alpha, \beta, \gamma, \delta = \boxed{\square}$, or else by (1) $\alpha\beta(\alpha^2+\beta^2) = \gamma\delta(\gamma^2+\delta^2)$. The 2 known solutions of (1), Euler[illegible] (5) and Gerardin[illegible], have duals on reversing alternate signs which satisfy (2) $abcd(a^2-b^2)(c^2-d^2) = \square = y^2/16$. In addition to these 2, "Je possede 3 formules de degres respectifs: 7, 13, 19."—M. Rignaux, L'Int. des Math. vol. 26 (1919) p. 5, which 3 do not seem to have come to light. In the Petrus transform of a, b, c, d (see Euler[illegible], p. 661 ibid.): $\alpha, \beta, \gamma, \delta = k, l, j, i = a^2-b^2, 2ab, c^2-d^2, 2cd$. The only known simple parametric solution for z is $\alpha, \beta, \gamma, \delta = (r^2-s^2)^2, 4rs(r^2+s^2), r, s$, Euler's[illegible] (ii) PT (ibid., p. 667–668), dual of Rolle's PT (ibid. p. 172, Fermat[illegible]; p. 504, O'Riordan[illegible], last line). But the simplest solutions are mavericks (as yet unformulated) found by tabulating $\alpha\beta(\alpha^2+\beta^2)/\square$. Of the 10 known solutions in integers $\leqq 35$: 8, 1, 5, 1; 9, 8, 5, 2; 13, 9, 5, 1; 13, 9, 8, 1; 15, 8, 3, 1; 19, 17, 19, 9; 20, 9, 13, 4; 29, 2, 5, 2; 29, 2, 9, 8; 35, 12, 21, 20, only the last is not a maverick, an algebraic coverage or potency of only 10%. If $ab(a^2\mp b^2)/\square = \alpha\beta(\alpha^2\pm\beta^2)/2\square$, then $\alpha, \beta, 2ab, a^2\mp b^2$ is a solution of z or y, by tabular algebra. Thus from 9, 4 and 5, 1 resp. (upper signs) one gets $\alpha, \beta, \gamma, \delta = 5, 1, 72, 65$. (Received October 13, 1954.)

131*t*. Ellen Correl and Melvin Henriksen: *A note on rings of bounded continuous functions with values in a division ring.*

Let $C^*(X, A)$ denote the ring of all bounded, continuous A-valued functions on a topological space X that is completely regular with respect to a topological division ring A (for definitions and background, see Goldhaber and Wolk, Duke Math. J. vol. 21 (1954) pp. 565–569). *Stone's theorem* is said to hold for A if $C^*(X, A)/M = A$ for every maximal ideal M of $C^*(X, A)$. *Theorem*: *Stone's theorem holds for locally compact* *division rings*. Because of known structure theorems for locally compact division rings (Otobe, Jap. J. Math. vol. 19 (1945) pp. 189–202), this theorem is new only when A is totally disconnected (in which case X is zero-dimensional). Goldhaber and Wolk have shown (loc. cit.) that if A is of type V (in the sense of Kaplansky), then Stone's theorem holds for A only if A is locally sequentially compact. They also ask if all the maximal ideals of $C^*(X, A)$ are of a certain specified form (too complicated to be stated here). The answer to this question is in the negative, even if both X and A are the real field. (The second author was supported in part by the National Science Foundation, contract no. NSF-G1129.) (Received October 13, 1954.)

132. W. G. Leavitt: *Modules over rings of words.*

In this paper are considered finitely based modules over certain "word" rings of the type introduced by Malcev. Over the word ring itself (all sums with integral coefficients of words in a finite set of symbols) any finitely based module, while containing infinite independent sets, nevertheless has invariant basis number. This result also holds for Malcev's ring. Thus imbeddability in a division ring, while sufficient for invariance of basis number, is not necessary. It is shown, finally, that for any integer $n>1$ an integral domain is constructable over which a finitely based module has invariant basis number if and only if it has a basis of length $<n$. The basic ring used is a word ring R in the symbols of n by $(n+1)$ and $(n+1)$ by n matrices A and B. The set H of all members of R having form xyz, with $x, z \in R$ and y an element of either $AB-I_{n+1}$ or $BA-I_n$, is shown to be a two-sided ideal, and the factor ring $K=R/H$ is found to be an integral domain of the desired type. (Received September 24, 1954.)

133. Bernard Vinograde: *A property of the product of division algebras.*

Let D_1 and D_2 be division algebras of finite dimension over field K, and let J be any ideal of $D=D_1\times D_2$ (over K). If Z_1 and Z_2 are the centers of D_1 and D_2 respectively, then the center of D/J is isomorphic (with respect to K, Z_1, or Z_2) to Z/J_z, where $Z=Z_1\times Z_2$ and $J_z=J\cap Z$. This enables one to prove structure theorems on algebras which are generated by every residue system modulo the radical. In particular, D has the latter property if and only if its center has the same property. (Received October 14, 1954.)

ANALYSIS

134. P. C. Rosenbloom: *An operational calculus.*

Let L be a linear operator on a linear vector space such that analytic functions of L can be defined with the usual properties. Then identities between ordinary analytic functions yield the corresponding identities between functions of L. These can often be interpreted as relations between solutions of different partial differential equations. Particular cases lead to the Riemann-Liouville fractional integrals, the Riesz potentials for elliptic equations, the work of Weinstein and his associates on the Euler-Poisson-Darboux equation, and the John identities for spherical means. The formal calculations can be justified in part by Dunford's theory of the spectral resolution of operators. (Received November 12, 1954.)

135. M. E. Shanks: *Remarks on derivations and differentiability.*

Let $C^0(X)$ be the algebra of all continuous real functions on the space X, and if

X is a differentiable manifold of class C^k, let $C^k(X)$ be the functions with at least k continuous derivatives. For class C^∞ manifolds tangent vector fields are usually defined as derivations on $C^\infty(X)$ to $C^\infty(X)$. It is shown that the same definition applies in the case C^k, in that if D is any derivation of $C^k(X)$ into $C^r(X)$; $r<k$, then $Df = \lambda^i \partial f/\partial x^i$ in the local coordinates with $\lambda^i \in C^r$. Something like converses to the above are obtained when $X = R =$ the real numbers. For example, if $C(R) \subset C^0(R)$ is an algebra over the reals which contains the polynomials and for which there is a derivation D mapping $C(R)$ onto $C^0(R)$ such that $Dx = 1$ and Df vanishes locally if and only if f is locally constant, then $C(R) = C^1(R)$ and $Df = f'$. Also, if $C \subset C^0(R)$ is a vector space closed under composition and D is an operator from C onto $C^0(R)$ satisfying the chain rule with local vanishing as above, then C is an algebra, $C = C^1(R)$, and $Df = f'$. Examples are given of algebras of nondifferentiable functions on which derivations onto $C^0(R)$ exist. Also examples of derivations on algebras of functions over spaces which are not manifolds. (Received October 14, 1954.)

136t. Leonard Tornheim: *Approximation by families of functions.*

An n-parameter family of functions F on $[a, b]$ is a set of continuous functions for which exactly one member passes through each set of n points on $[a, b]$ with distinct abscissas. It is a linear family if it is closed under the linear operations. For a given continuous function g on $[a, b]$, let $M_k(f) = \int_a^b (f-g)^{2k} dx$. Then f_k in F is a best k-approximant to g if $M_k(f_k) = \text{g.l.b.}_{f \in F} M_k(f)$. (1) A best k-approximant always exists. (2) It is unique if F is linear but need not be in general. (3) Also it need not cross g. (4) $\text{Lim}_{k\to\infty} f_k$ exists and equals f_∞, the unique best approximant for which $\int_a^b |f-g| dx$ is a minimum. Properties (2) and (4) can be proved using the method of Pólya, C. R. Acad. Sci. Paris vol. 157 (1913) pp. 840–843. (Received October 13, 1954.)

APPLIED MATHEMATICS

137t. L. E. Payne: *Inequalities for certain eigenvalues of a membrane.* II.

Let D be a closed convex domain with boundary C in the xy-plane and let $u(x, y)$ be a solution of the differential equation $\Delta u + ku = 0$ in D (where k is a positive constant) and either (1) $u = 0$ on C or (2) $\partial u/\partial n = 0$ on C. We denote the eigenvalues in case (1) by $\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots$, and in case (2) by $0 = \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \cdots$. In a previous abstract (Bull. Amer. Math. Soc. Abstract 60-6-734) the author showed that $\mu_3 \leqq \lambda_1 - 1/2(\rho k)_{\max}$. This inequality is now improved to read (3) $\mu_3 \leqq \lambda_1 - 1/(\rho k)_{\max}$, and in addition the general inequality (4) $\mu_n \leqq \lambda_{n-2}$, which is valid for all $n \geqq 3$, is obtained. The research of this author was supported by the United States Air Force through the Office of Scientific Research. (Received September 27, 1954.)

GEOMETRY

138t. P. C. Hammer: *Planar curves of equivalent breadth.*

Earlier, the writer achieved an inversion of central symmetrization in the plane using geometrical methods (Bull. Amer. Math. Soc. Abstract 59-2-235). In this paper he uses analytical methods to achieve the same result and extends the results to include nonconvex curves of breadth equivalent to that of a given curve. In general, for any closed convex curve, the problem of determining all convex curves of equivalent breadth is reduced to that of determining all continuous increasing functions. (Received October 13, 1954.)

LOGIC AND FOUNDATIONS

139. J. W. Addison: *Analogies in the Borel, Lusin, and Kleene hierarchies.* I.

Mostowski (Fund. Math. vol. 34 (1947) pp. 81–112) utilized an analogy with the projective hierarchy on the real line R to develop the Kleene hierarchy on the set N of natural numbers. An imperfection in this analogy was later found (Kleene, Neder. Akad. Wetensch. vol. 53 (1950) pp. 800–802). An analogy without this imperfection is shown to exist between the *Borel* hierarchy on R and the Kleene hierarchy on N. Its cause is traced through the Borel hierarchy on N^N (a G_δ-subset of R) and the Kleene hierarchy on N^N. The finite Kleene hierarchy on N^N is just the finite Borel hierarchy on N^N with enumerable unions and intersections required to be effectively enumerable. Many parallel theorems, including separation theorems, about the Borel and Kleene hierarchies on N^N are demonstrated by essentially the same proofs. Systematically changing a few symbols these proofs apply also to the Kleene hierarchy on N. Another less direct way of perfecting Mostowski's analogy is also exhibited. A purely recursive-function-theoretic result is proved by direct application of the Baire category theorem. Identifying $x(\in N)$ with the constant function x the nth effective class of Baire-de la Vallée Poussin relative to N (as a subspace of N^N) is just the class of sets of Post-Kleene degree x for $n-1<x\leqq n$. The projective hierarchy corresponds to the Kleene function-quantifier hierarchy. (Recieved September 24, 1954.)

STATISTICS AND PROBABILITY

140*t*. H. W. Becker: *Weighted factorial tables, an idea of Andre.*

Let $\sum {}_xP_{n,m}=P_n=n!$, $\sum m\cdot {}_xP_{n,m}=P_n'$. Andre classified permutations by alternations, ${}_{\mathrm{I}}P_{n,m}$, Netto, *Combinatorik* (1927) pp. 105–112. Then ${}_{\mathrm{I}}P_n'=n!(2n-1)/3$. The number of permutations of n letters with given letter m fixed is ${}_{\mathrm{II}}P_{n,m}=(n-1)!$ For table, bibliographies, etc. of the number with m inversions, ${}_{\mathrm{III}}P_{n,m}$, see Pixa, Math. Mag. vol. 21 (1948) p. 259, ibid. vol. 22 (1949) pp. 213–214, and Riordan, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 429–432. Then ${}_{\mathrm{II}}P_n'={}_{\mathrm{III}}P_n'=(n+1)!/2$; ${}_{\mathrm{II}}P_{n,m}$ and ${}_{\mathrm{III}}P_{n,m}$ are unequal, but isobaric. Call ${}_{\mathrm{IV}}P_{n,m}$ the unsigned Stirling numbers of the first kind, Steffensen, *Interpolation* (1927) p. 57, which enumerate cycles of substitutions, Touchard, Acta Math. vol. 70 (1939) p. 243–297. Then ${}_{\mathrm{IV}}P_n'={}_{\mathrm{IV}}P_{n+1,2}=P(P+Q)^{n-1}$ where $Q_0=1$, otherwise $Q_m=(m-1)!$ The number of permutations of n letters, m underanged, is ${}_{\mathrm{V}}P_{n,m}=(n,m)(P-1)^{n-m}$. Then ${}_{\mathrm{V}}P_n'=n!$, $n\geqq 1$; here m is isobaric to unity, and $V=V(1,0)$. Let $V(r,s)$ denote weighting by $m+s$ of the r-place menages numbers enumerated for $r=2$ and 3 by Riordan and Kaplansky, Scripta Math. vol. 12 (1946) p. 124, and ibid. vol. 20 (1954) p. 23. Then ${}_{\mathrm{V}(r,s)}P_n'=(r+s)\cdot n!$, $n\geqq r$, as far as known. Interpretation of $m\cdot {}_xP_{n,m}$ is that each of the m letters is starred in turn. If any combination is starred simultaneously, the wt. functions is 2^m, the sum ${}_xP_n''$. Then ${}_{\mathrm{II}}P_n''=2(2^n-1)(n-1)!$; ${}_{\mathrm{III}}P_n''=2K_n=(K+1)^n$ where $e^{tK}=1/(2-e^t)$, Bull. Amer. Math. Soc. vol. 56 (1950) p. 61; ${}_{\mathrm{IV}}P_n''=(n+1)!$; ${}_{\mathrm{V}}P_n''=(P+1)^n$. (Received October 13, 1954.)

TOPOLOGY

141*t*. A. L. Blakers: *A new Whitehead product for homotopy groups.*

Let $(X;A,B)$ be a triad. A product is defined pairing $\pi_p(X;A,B)$, $p\geqq 3$, and $\pi_q(A\cap B)$, $q\geqq 1$, to $\pi_{p+q-1}(X;A,B)$. The definition is geometric and similar to those of previous generalizations of the Whitehead product. (See, for example, Ann. of Math. vol. 58 (1953) pp. 295–324.) The new product is bilinear when $p>3$, $q>1$. It is determined by the operators from $\pi_1(A\cap B)$ when $q=1$, and is shown to be related to the previous generalizations under the inclusion and boundary homomorphisms. (Received October 8, 1954.)

142*t*. M. E. Shanks: *An example where the covering homotopy theorem fails.*

Let B be the Sierpinski universal curve, a 1-dimensional Peano space, and let G be the group of all homeomorphisms of B with the compact-open topology. R. D. Anderson has shown that B is homogeneous (Bull. Amer. Math. Soc. Abstract 59-2-249). Then, if H is the subgroup leaving a point invariant, B is homeomorphic to the coset space G/H. It is shown that no arc in B may be "lifted up" into G. Thus the covering homotopy theorem fails for maps of a point and G is not a fiber space with fiber H, even though the base space is about as "nice" as possible. It follows from the proof that no locally compact subgroup of G can act transitively on B. (Received October 14, 1954.)

J. W. T. Youngs,
Associate Secretary

THE NOVEMBER MEETING IN LOS ANGELES

The five hundred ninth meeting of the American Mathematical Society was held at the University of California, Los Angeles, on Saturday, November 27, 1954. Attendance at the meeting was approximately 95, including 80 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Edmund Pinney, of the University of California, Berkeley, delivered an address entitled *Nonlinear differential equations*. Professor Pinney was introduced by Professor H. F. Bohnenblust.

The sessions for contributed papers were presided over by Professor Robert Steinberg and Dr. G. E. Forsythe.

Abstracts of papers presented at the meeting follow. Abstracts whose titles are followed by "*t*" were presented by title. In the case of joint authorship, the name of that author who presented the paper is followed by "(p)." Mr. Rosenblum was introduced by Professor F. Wolf and Mr. Moore by Dr. G. W. Evans II.

ALGEBRA AND THEORY OF NUMBERS

143. Harvey Cohn: *Approach to Markoff's minimal forms through modular functions.* II.

In an earlier preliminary report (Bull. Amer. Math. Soc. Abstract 59-4-443), the author outlined a formal procedure which can now be referred entirely to the fundamental domain of a modular function. Consider a fundamental domain D of genus 1 bounded by four geodesic arcs in the upper half plane with vertices on the boundary and subject to the following conditions: vertices are rational, transformations of the group are integral unimodular. Then Markoff's forms are associated with the fixed points of substitutions connecting the opposite sides. The chain of generating operations of the forms is nothing else but the (modular) change of basis operations. The domain D does not correspond to any congruence subgroup (in fact the linear substitutions associated with D include "all" denominators). Chains of analogous forms can be generated, however, corresponding to the fundamental domains of genus 1 for congruence subgroups such as modulo 11, 17, etc. (Received October 8, 1954.)

144. Alfred Horn: *A characterization of unions of linearly independent sets.*

Let X be a vector space of finite or infinite dimension over any division ring. Let S be a subset of X and let k be a positive integer. If for any finite subset T of S we have $|T| \leq k \cdot \text{rank } T$, where $|T|$ is the cardinal of T, then S can be divided into k linearly independent sets. This confirms a conjecture of K. F. Roth and R. Rado. If k is an infinite cardinal and $|T| \leq k \cdot \text{rank } T$ for any subset T of S, then in general S cannot be divided into k linearly independent sets. (Received October 18, 1954.)

145t. Joachim Lambek: *Groups and herds*. Preliminary report.

A *herd* H is a set with a ternary operation f such that $fabb=a=fbba$ for all a, b in H; it is called *associative* if $ffabcde=fabfcde$ for all a, b, c, d, e in H. The congruence relations in any herd commute; this is known to imply the Jordan-Hölder-Schreier theorem for herds with a selected element. Every associative herd H, some element of which has been designated as 1, may be regarded as a group G with $a\cdot b=fa1b$, $a^{-1}=f1a1$; conversely H is obtained from G by defining $fabc=a\cdot b^{-1}\cdot c$; this correspondence is one-to-one. If any other element of H is chosen to serve as identity, a group isomorphic with G results. The subherds of H are the cosets of the subgroups of G; the group of automorphisms of H is the holomorph of G. The relation between associative herds and groups is analogous to that between affine spaces and vector spaces. (Received October 14, 1954.)

146. T. S. Motzkin: *Bounds relating to Hilbert's Nullstellensatz.*

Using a new proof of that theorem it is shown by simultaneous induction that if the polynomials $f_1, \cdots, f_r$ in $K[x]$ (or: in $K[x, y]$), of degree $\leqq d$, are $\neq 0, \cdots, 0$ for every $\xi=(\xi_1, \cdots, \xi_n)$, $\xi_1, \cdots, \xi_n$ in the algebraic closure of K (or: of $K(y)$), then $\sum f_i g_i=a$ for some $g_1, \cdots, g_r$ in $K[x]$ (or: $K[x, y]$) of degree $\leqq d_n \leqq (d_{n-1}+d)\delta_{n-1}-d$, $d_0=0$ (or: of degree $\leqq \delta_n - d$) with $a=1$ (or: for some a in $K[y]$ of degree $\leqq \delta_n \leqq C_{n+d_n+d,n+1}-C_{n+d_n,n+1}$); here K is any given field, $x=(x_1, \cdots, x_n)$ and $y=(y_1, \cdots, y_m)$ are indeterminates, $K[\ ]$ and $K(\ )$ denote ring and field extension of K. All g_i but $r_0=C_{n+d,d}$ may be chosen as 0; the intersectional (or Helly) dimension of the family of hypersurfaces of degree d in affine or projective n-space over an infinite field is r_0-1. (Received October 13, 1954.)

ANALYSIS

147t. Stefan Bergman: *Stream function of a doublet in a subsonic flow.*

Using the results of [1] Trans. Amer. Math. Soc. vol. 62 (1947) p. 452 ff., the author constructs the stream function of a doublet for a subsonic flow. The modified stream function (see (2.13b) of [1]) $\psi^*=(\psi/\mathcal{H})$ satisfies in the pseudologarithmic plane the equation (1): $\psi^*_{Z\bar{Z}}+F(Z+\bar{Z})\psi^*=0$, see (2.15b) of [1]. Here $Z=\lambda+i\theta$ where λ is a function of speed and θ the angle which the velocity vector forms with the positive x-axis (in the physical plane). By the transformation $\zeta=\zeta(Z)=(Z-Z_0)^{1/2}+\zeta_0$ the Riemann surface R with the branch points at Z_0 is mapped into the ζ-plane. (1) goes over into (2): $\Psi^*_{\zeta\bar{\zeta}}+|dZ/d\zeta|^2F[Z(\zeta)+\bar{Z}(\bar{\zeta})]\Psi^*=0$. Let $\Psi^{*(L,1)}(\zeta, \bar{\zeta}, \zeta_0, \bar{\zeta}_0)$ be the fundamental solution of (2) (see (5.3a) of [1]) with the affix at $\zeta_0=\alpha_0+i\beta_0$. Differentiating $\Psi^{*(L,1)}$ with respect to α_0 one obtains a singularity $\Psi^{*(1,1)}$ of the first order at the point ζ_0. If further one replaces $(\zeta-\zeta_0)$ by $(Z-Z_0)^{1/2}$ and multiplies by H, one obtains a solution $\psi^{(1,1)}$ of (1) which becomes infinite at the branch point Z_0. It represents the stream function of a doublet. (Received November 27, 1954.)

148t. Stefan Bergman: *A formula for stream functions of subsonic flows around profiles of a certain type.*

Let p be the integral operator transforming analytic functions of a complex variable into solutions of the equation (2) of the previous abstract. The real and imaginary parts of $p[\zeta^n(Z)]$, $n=0, 1, 2, \cdots$, where $\zeta(Z)=(Z-Z_0)^{1/2}+\zeta_0$, represent a set of particular solutions of equation (1) of the previous abstract. They are defined on the Riemann surface of the function $(Z-Z_0)^{1/2}$. Orthonormalizing these functions with respect to a domain B bounded by segments of $\operatorname{Re} Z=\text{const.}$ and segments of $\operatorname{Im} Z=\text{const.}$, one determines the stream function ψ_0 so that $\psi=\psi^{(1,1)}+\psi_0$ becomes constant along the boundary of B. Here $\psi^{(1,1)}$ is the stream function representing a doublet (see the previous abstract). Then ψ represents the stream function of a flow around a body, bounded (in the physical plane) by segments of straight lines and by segments of free boundaries. (See also Proceedings of First U.S. Nat. Congress of Appl. Mechanics (1951/52) pp. 705 ff.) (Received November 27, 1954.)

149. F. H. Brownell: *Extended asymptotic eigenvalue distributions for plane domains with corners.*

Let D be a bounded, open, connected set in the plane with boundary B, and let λ_p, $0<\lambda_p\leq\lambda_{p+1}$, be the eigenvalues of the membrane problem, $-\nabla^2u=\lambda u$ on D, $u=0$ on B. It is assumed that B is the disjoint union of $n+1$ Jordan curves (so D has n holes) and that except at a finite number of corners, where the angle of the tangent vector to B proceeding in the positive sense has a jump increase of α_j, $-\pi<\alpha_j<\pi$, the curvature $c(s)$ of B exists satisfying on the closed arcs between corners $|c(s)-c(s')|\leq|s-s'|^{\eta}M$, $0<\eta\leq 1$, s being arc length on B. Under these conditions over $\omega\geq 1$, $\sum_{p=1}^{\infty} 1/\lambda_p(\lambda_p+\omega^2)=(\mu_2(D)/4\pi)(\ln(\omega^2)/\omega^2)+C/\omega^2+(l(B)/8)1/\omega^3)-[(1-n)/6+\sum_{\alpha_j} Q(\alpha_j)](1/\omega^4)+O(1/\omega^{4+\eta})$, which extends some results of Pleijel [Arkiv Mat. vol. 2 (1953) no. 26] applying to simply connected D $(n=0)$ with infinitely differentiable boundary. The function $Q(\alpha)$ is continuous over $|\alpha|<\pi$, differentiable at $\alpha=0$ with $Q'(0)=0=Q(0)$, and is given explicitly as the sum of a series of computable integrals with convergence ratio $\leq|\alpha|/\pi$. If B is polygonal, $O(1/\omega^{4+\eta})$ can be replaced by $O(e^{-\omega\rho})$, $\rho=r_0/16$, r_0 the minimum distance between corners. The above estimates yield [Brownell, Pacific Journal of Mathematics (1954)] the asymptotic distribution of λ_p as over $y\geq 1$, $\sum_{\lambda_p\leq y} 1=N(y)=(\mu_2(D)/4\pi)y-(l(B)/4\pi)y^{1/2}+[(1-n)/6+\sum_{\alpha_j} Q(\alpha_j)]+R(y)$, $R(y)=\tilde{O}(y^{-1/2}\ln y)$, or $=\tilde{O}(y^{-r})$ for every $r>0$ if B is polygonal. These $\tilde{O}$ estimates on the remainder $R(y)$ have the sense that certain Gaussian averages of $R(y)$, which drop out oscillating parts, are of the indicated order. Analogous results for euclidean k-space, $k>2$, are readily obtained. (Received September 28, 1954.)

150. G. W. Evans, II, R. L. Brousseau, and Ralph Keirstead (p): *Instability considerations for various difference equations derived from the diffusion equation.*

Several methods of iterating the implicit difference equation $\alpha(U_{j+1}^{k+1}-2U_j^{k+1}+U_{j-1}^{k+1})/(\Delta x)^2=(U_j^{k+1}-U_j^k)/\Delta t$, where $U_j^k=U(j\Delta x, k\Delta t)$, for a solution of the differential equation $\alpha U_{xx}=U_t$ are described. The methods are applicable to one or more space dimensions and to cylindrical as well as Cartesian coordinates. A discussion of convergence of the iteration schemes and a comparison of solutions obtained by these methods with the analytic solution of a simple diffusion problem is included. (Received October 4, 1954.)

151. J. M. G. Fell: *A generalization of a theorem of Rellich.*

The author proves an infinite-dimensional generalization, applicable to second quantization of Rellich's Theorem (Gottingen Nachrichten, Math.-Phys. Klasse, 1946, pp. 107–115) on the uniqueness of operators satisfying the usual quantum-mechanical commutation relations. For each x in a Hilbert space H, let A_x be a (not necessarily bounded) linear operator on a second Hilbert space K, satisfying the algebraic laws: $A_{ax+by}=\bar{a}A_x+\bar{b}A_y$, $[A_x, A_y]=0$, $[A_x, A_y^*]=(y, x)$. Under convergence restrictions similar to those of Rellich, and requiring irreducibility, the author shows that to each orthonormal basis $\{x_r\}$ of H, there corresponds an orthonormal basis $\{F(n_1, n_2, \cdots)\}$ of K, where $n_1, n_2, \cdots$ ranges over all sequences of non-negative integers with $\sum n_i<\infty$; such that, for $y=x_r$, one has $A_y[F(n_1, \cdots)]=(n_r)^{1/2} \cdot F(n_1, \cdots, n_r-1, \cdots)$ and $A_y^*[F(n_1, \cdots)]=(n_r+1)^{1/2}F(n_1, \cdots, n_r+1, \cdots)$. Similar results are obtained with antisymmetric commutation rules. (Received October 13, 1954.)

152. C. J. A. Halberg, Jr. (p) and A. E. Taylor: *On the spectra of linked operators*. Preliminary report.

Let X, Y be complex linear spaces, Z a non-null complex linear space contained in both X and Y. Let X be a Banach space X_1, Y a Banach space Y_2 under the norms n_1, n_2 respectively. Let Z be a Banach space Z_N under the norm $N(z)=\max[n_1(z), n_2(z)]$. Let T_1, T_2 be bounded distributive operators on X_1, Y_2 respectively, such that $T_1z=T_2z\in Z$ when $z\in Z$. Operators satisfying these conditions will be said to be "linked." If, in addition, it is assumed that Z is dense in X_1, T_1 and T_2 will be said to be "linked densely with respect to X_1." Let T be the operator defined on Z_N by $Tz=T_1z$ and let $\rho(S)$ indicate the resolvent set of an operator S. Then the main result of this report is that, if T_1 and T_2 are linked densely relative to X_1 and $\rho(T_1)\cap\rho(T_2)\subset\rho(T)$, then any component in the spectrum of T_1 has a nonvoid intersection with the spectrum of T_2. The proof involves the integration of operator-valued functions around suitable contours in the complex plane. Special cases involving the sequence spaces l_p are considered. (Received October 13, 1954.)

153. C. H. Meng: *On ϵ-unitary operators*.

A linear bounded operator T on a Hilbert space H is called an ϵ-unitary operator if $\|T^*T-I\|\leqq\epsilon$. The adjoint of an ϵ-unitary operator can easily be shown to be an ϵ-unitary operator. For $\epsilon<1$, a unitary operator U can be constructed such that $\|T-U\|\leqq\epsilon$. The relationship between the spectrum of T and that of U is studied. In particular, a criterion that $e^{i\phi}$ lies in the spectrum of U is that there exists a sequence of unit vectors $\{x_n\}$ in H such that $\|(T-e^{i\phi}R)x_n\|\to 0$ as $n\to\infty$. (Received October 14, 1954.)

154. R. E. Moore: *Numerical approximation of multiple integrals.*

A general procedure is developed for "minimizing" the number of evaluations of the integrand (hence the time) necessary to obtain a numerical approximation of prescribed accuracy to a given multiple integral. The procedure is designed specifically for use in connection with digital computing machines. (Received October 8, 1954.)

155. Marvin Rosenblum: *Perturbation of the continuous spectrum and unitary equivalence*. Preliminary report.

Let H be a Hilbert space and let A and B be possibly unbounded self-adjoint operators in H such that $B-A=\sum_{j=1}^{\infty}\lambda_j(\cdot, \phi_j)\phi_j$, where the ϕ_j are orthonormal and

$\sum_{j=1}^{\infty} |\lambda_j| < \infty$. Suppose that A and B have resolutions of the identity E_x and F_x respectively, and for all f in H and $j=1, 2, \cdots$, (E_xf, ϕ_j) and (F_xf, ϕ_j) are absolutely continuous functions of x such that $d(E_xf, \phi_j)/dx$ and $d(F_xf, \phi_j)/dx$ belong to L_p for some fixed $p>1$. Theorem: Under the above conditions, it follows that B is unitarily equivalent to A. The theorem is proved by exhibiting a densely defined isometric operator U_0 that satisfies $BU_0=U_0A$ and extending U_0 to a unitary operator. The case $p=1$ is now under study. (Received October 13, 1954.)

156. Marvin Shinbrot: *A boundary value problem with a small parameter.*

The following boundary value problem is considered: (1) $\epsilon\ddot{x}+a(t)\dot{x}+b(t)x=0$, $x(0)=\alpha$, $x(1)=\beta$, where $\epsilon>0$ is a small parameter and $a(t)$ and $b(t)$ are of class C^1. Two situations are permitted: $a(t)\geqq\delta>0$ or else $a(t_0)=0$, $0\leqq t_0<1$ while $a(t)>0$, $t>t_0$, $a(t)<0$, $t<t_0$ (if $t_0=0$, this last condition is omitted, of course). If $t_0>0$, let $u_1(t)$ and $u_2(t)$ denote solutions of the "reduced equation" $a(t)\dot{u}+b(t)u=0$ having the initial values $u_1(0)=1$, $u_2(1)=1$; if $t_0=0$, let $u_2(t)$ be defined as before, but take $u_1(t)\equiv 0$. It is then shown that (1) possesses a unique solution provided that $u_1(t)$ is bounded as $t\to t_0-0$, $u_2(t)$ is bounded as $t\to t_0+0$, and ϵ is small enough. Furthermore, there is an $\eta=\eta(\epsilon)=o(1)$ as $\epsilon\to 0$ such that $x(t)\to\alpha u_1(t)$, $\dot{x}(t)\to\alpha\dot{u}_1(t)$ on $0\leqq t\leqq t_0-\eta$, $x(t)\to\beta u_2(t)$, $\dot{x}(t)\to\beta\dot{u}_2(t)$ on $t_0+\eta\leqq t\leqq 1$ and these modes of approach are uniform. It is noted that the proof admits generalization also to the case $a(0)=a(t_0)=0$, $0<t_0<1$. (Received September 7, 1954.)

157*t*. D. A. Storvick: *Extension of a theorem of Seidel to a class of meromorphic functions.*

Let $f(z)$ be meromorphic with bounded characteristic in $|z|<1$, and let the values which $f(z)$ assumes in $|z|<1$ lie in a domain G whose boundary Γ has positive capacity. Furthermore, let the radial limit values of $f(z)$ belong to Γ for almost all $e^{i\theta}$ on $|z|=1$. Such functions have been studied by O. Lehto [Ann. Acad. Sci. Fennicae, A. I., no. 160 (1953) pp. 1–15] and M. Tsuji [J. Math. Soc. Japan vol. 4 (1952) pp. 91–95]. It is shown that every point of Γ which is arcwise accessible from G is a radical limit value of such a function $f(z)$; this result extends a theorem of W. Seidel [Trans. Amer. Math. Soc. vol. 36 (1934) p. 208] for bounded analytic functions. (Received October 7, 1954.)

158*t*. František Wolf: *On scalar operators in Banach space.* Preliminary report.

Let U be a bounded operator in a reflexive Banach space, such that it satisfies, with all its polynomials and for all x, the relation $\|p(U)x\|^2\leqq M\|x\|\cdot\|p^2(U)x\|$ always with the same constant M. Then U is a scalar operator in the sense of Dunford (cf. Pacific Journal of Mathematics vol. 4 (1954) p. 332), i.e. there exists a spectral measure E, such that $f(U)=\int f(\theta)dE(\theta)$, and for every Borel set σ of the complex plane $\|E(\sigma)\|\leqq M$. By applying the above inequality successively to $p^2(U)$, $p^4(U)$, etc. and using the spectral mapping theorem one obtains $\|p(U)x\|\leqq M\|x\| \sup_{\theta\in\sigma(U)} p(\theta)$. From here, by a well-known theorem, we deduce the scalarity of U. The above condition was stated for the first time by F. Y. Atkinson (Monatshefte für Mathematik vol. 53 (1949) p. 278) and used in the case of a completely continuous U to deduce in a different way what amounts to the scalarity of U. The above can be extended to unbounded operators if, for x, we use a convenient everywhere dense set of x in the domain of U. (Received October 13, 1954.)

Applied Mathematics

159*t*. Marvin Shinbrot: *Singular perturbations with a turning point.* Preliminary report.

Asymptotic expansions (as $\epsilon \to +0$) have been sought for the solutions of the differential equation $\epsilon\ddot{x}+a(t)\dot{x}+b(t)x=0$ in an interval $[0, T]$ in which $a(t)$, $b(t)$ are of class C^1. The function $a(t)$ is allowed to have a zero at a point $t=t_0$, $0 \leqq t_0 \leqq T$ (the "turning point"). Let $u_1(t)$ and $u_2(t)$ denote solutions of the "reduced equation" $a(t)\dot{u}+b(t)u=0$, satisfying the following conditions: if $t_0=0$, $u_1(t)=0$, $u_2(1)=1$; if $t_0=T$, $u_1(0)=1$, $u_2(t)=0$; if $0<t_0<T$, $u_1(0)=1$, $u_2(1)=1$. Then, the desired expansions have been found provided (i) $u_1(t)$ is bounded as $t \to t_0-0$ and $u_2(t)$ is bounded as $t \to t_0+0$; (ii) $|a(t)| \geqq c|t-t_0|^\gamma$, $c=$constant, $0 \leqq \gamma \leqq 1$. The generalization to the case where $a(t)$ has finitely many zeros in $[0, T]$ is trivial. Work is proceeding on more general situations; in particular, an attempt is being made to eliminate the condition $\gamma \leqq 1$. (Received September 7, 1954.)

Geometry

160*t*. T. K. Pan: *Centers of curvatures of a vector field.*

Let v be a vector field on a surface in an ordinary space. Along a curve C on S, v has several curvatures. In this paper, definitions of centers of various curvatures of v along C are given to include those of a curve on a surface as special cases. Relations among these centers are investigated. Most theorems on centers of curvatures of a curve on a surface are generalized; for example: (1) With respect to C at P which is neither the asymptotic line of v at P nor the indicatrix of v at P, the three centers of curvatures for P of v—center of absolute curvature, center of angular spread, and center of normal curvature—are collinear. (2) The centers of normal curvature for P of v along any two curves making equal angles with the line of curvature of v at P are coincident; the locus of centers of angular spread for P of a pencil of vector fields along C is a circle. (3) The locus of orthocenter of associate curvature of the orthogonal vector field of v with respect to a line of curvature of v is the edge of regression on the developable surface formed by straight lines on the orthogonal trajectories of v along the line of curvature of v. (Received October 11, 1954.)

161. F. A. Valentine: *Three point arcwise convexity.*

A convex arc is, by definition, an arc which is contained in the boundary of a plane convex set. A convex curve, as distinguished from a convex arc, is a closed connected portion of the boundary of a plane convex set. Definition 1. A set S is said to satisfy the three point arcwise convexity property if each triple of points $x \in S$, $y \in S$, $z \in S$ is contained in a convex arc belonging to S. *Theorem* 1. *Let S be a closed set in the plane which has at least three points. Then S has the three points arcwise convexity property if and only if it satisfies at least one of the following three conditions*: (1) *It is a closed convex set.* (2) *It is a convex curve.* (3) *It is a closed convex set except for one bounded convex hole, that is, it is obtained by deleting from a closed convex set a bounded open convex subset.* (Received October 13, 1954.)

LOGIC AND FOUNDATIONS

162*t*. R. M. Robinson: *Primitive recursive functions*. II.

This supplements *Primitive recursive functions*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 925–942, especially §7, Theorem 3. Let J, K, L be the Cantor pairing functions, defined by $2J(u, v)=(u+v)^2+3u+v$, $KJ(u, v)=u$, $LJ(u, v)=v$. It is now shown that all primitive recursive functions of one variable can be obtained from S and K (or from S and L) by repeated use of the formulas $Fx=Ax+Bx$, $Fx=BAx$, and $Fx=B^x0$ to define new functions. Also, the scheme $Fx=Ax+Bx$ may be replaced by $Fx=J(Ax, Bx)$. The latter form proves useful in studying the arithmetical representation of recursively enumerable set. (Received October 4, 1954.)

STATISTICS AND PROBABILITY

163. Donald Davidson and Patrick Suppes (p): *A finitistic axiomatization of subjective probability and utility*.

This paper is concerned with a simultaneous axiomatization of subjective probability and utility, which is based on the following six primitive notions: (1) a finite set K, ordinarily a set of things valued; (2) a binary relation P of preference whose field is K; (3) a finite set X of outcomes of a random mechanism; (4) a family $\mathcal{J}$ of subsets of X (the elements of $\mathcal{J}$ are the chance events for which a subjective probability is determined); (5) a distinguished chance event E^* whose subjective probability is determined as 1/2; (6) a quinary relation M such that for x, y, u and v in K and E in $\mathcal{J}$, $x,y\ M(E)\ u,v$ if and only if the individual in question is indifferent between (a) the prospect of x if E occurs and y if E does not occur and (b) the prospect of u if E occurs and v if E does not occur. The main result is that for every system $\langle K, P, X, \mathcal{J}, E^*, M\rangle$ satisfying the axioms there exists a unique subjective probability function s and a utility function ϕ unique up to a linear transformation such that (for x, y, u and v in K and E and F in $\mathcal{J}$): $x\,P\,y$ if and only if $\phi(x)\geqq\phi(y)$; $s(E)+s(\tilde{E})=1$; $s(\tilde{E})\geqq 0$; $s(X)=1$; if $E\subseteq F$ then $s(E)\leqq s(F)$; $x,y\ M(E)\ u,v$ if and only if $s(E)\phi(x)+s(\tilde{E})\phi(y)=s(E)\phi(u)+s(\tilde{E})\phi(v)$. For fundamental reasons unrestricted additivity of probability fails in this finitistic set-up; in fact $\mathcal{J}$ is closed under complementation but not under union. (Received October 14, 1954.)

164*t*. T. E. Harris: *Recurrent Markov processes*. Preliminary report.

Let X be a Borel-measurable subset of the real line and let m be a countably additive measure on the Borel subsets of X; $m(X)$ may be infinite, but m is assumed sigma-finite. Let $P(x, E)=\text{Prob}\ (x_{n+1}\in E \mid x_n=x)$ be the transition function of a Markov process defined on X. The process is called recurrent if $m(B)>0$ implies that for all starting points x_0 the probability is 1 that $x_n\in B$ infinitely often. *Theorem*. If x_n is recurrent, there exists a sigma-finite measure π satisfying $\pi(E)$ $\int_X\pi(dx)P(x, E)$ for every Borel set E in X. If $\pi(X)$ is finite it can be normalized to give a stationary absolute probability distribution. The significance of the case $\pi(X)=\infty$ has been discussed by Harris and Robbins, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 860–864. (Received October 13, 1954.)

TOPOLOGY

165*t*. C. E. Burgess: *Continua and certain types of homogeneity*.

Definitions of *nearly-homogeneous* point sets and *n-homogeneous* point sets were given in the author's paper *Some theorems on n-homogeneous continua*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 136–143. Let M be a nearly-homogeneous bounded plane continuum which separates the plane and has only a finite number of complementary domains. It is shown that (1) every indecomposable proper subcontinuum of M is a continuum of condensation of M and (2) M is a simple closed curve if it either is aposyndetic or is arcwise connected or contains a simple closed curve. (In papers cited in the author's paper mentioned above, F. B. Jones and Herman Cohen have presented similar results for homogeneous continua.) Let K be a compact metric continuum which is n-homogeneous, where $n>1$. It is shown that K is a simple closed curve if either some two of its points are not separated by any of its subcontinua or it is locally connected and separated by some set consisting of n points. (Received October 13, 1954.)

J. W. GREEN,
Associate Secretary

BOOK REVIEWS

Fonctions analytiques. By G. Valiron. Paris, Presses Universitaires de France, 1954. 4+236 pp. 1.500 fr.

This book is intended for the student who has had an introductory course in functions of a complex variable, and the first chapter gives a brief summary of the basic notions of function theory. The second chapter treats the classical theory of univalent functions using the area principle of Gronwall. The author gives the theorem $|a_2| \leq 2$, the one-quarter theorem and its consequences, and the Littlewood bound for the coefficients of a univalent function.

Boundary values of bounded analytic functions are discussed in the third chapter, culminating in Fatou's theorem and the Riesz theorem on the determination of a function by the boundary values on a set of positive measure. In the next chapter the author discusses the behavior of conformal mappings on the boundary and gives criteria that they be conformal there.

Chapter five takes up "functions of Fatou," i.e., functions defined in the unit circle which are bounded by 1 and have absolute value 1 on the unit circumference. In the following chapter we are led to the study of functions which map the upper half-plane into itself and the iteration of such functions. Here the author introduces the notion of a normal family.

Chapter seven discusses the multiplicative functions of Poincaré and notions of their order of growth. Chapter eight deals with the question of prolongation of a function. The final chapter gives the methods and results of Wiman and Valiron on the growth of entire functions.

Altogether this is a very useful little book which has collected together a number of diverse topics in function theory in a form which is easy to read and quite suitable for the student who wishes to familiarize himself with some of the classical tools available in function theory.

H. L. ROYDEN

Uniformisierung. By R. Nevanlinna. Berlin, Göttingen, Heidelberg, Springer, 1953. 391 pp.

There has long been a lack of a reference for students wanting to learn about Riemann surfaces, and Nevanlinna's new book, which might equally have been called ***An introduction to Riemann surfaces***, goes far toward filling this need. The first eight chapters of this book, which the reviewer found very readable, are a complete and modern approach to the classical theory of Riemann surfaces.

The book begins with a chapter which gives the algebraic origins of the notion of a Riemann surface and discusses the problem of uniformizing an algebraic curve. The second chapter adopts the abstract viewpoint and is concerned with the development of the topological tools necessary for Riemann surface theory. The first section begins with point set theory and continues through the definitions of two-dimensional manifold and Riemann surface. A detailed discussion of the Prüfer example is given, but, rather than follow Radó's direct but tedious proof of the separability of a Riemann surface, Nevanlinna prefers to deduce this at a later stage as a consequence of the general uniformization theorem. The second section of this chapter develops the homology theory of surfaces using the singular theory. Remaining sections cover the fundamental group, covering surfaces, cover transformations, and triangulation.

The third chapter develops the exterior differential calculus for Riemann surfaces, discusses analytic functions and differentials and their properties. A concluding section develops the various integral and residue theorems. The integration theory is all developed using singular differentiable simplexes and chains.

Chapter four deals with the fundamental existence theorems for harmonic and analytic functions on Riemann surfaces. The author uses the "alternating method" of Schwarz, feeling that it is superior because it is constructive.

Closed Riemann surfaces are taken up in the next chapter, and the Abelian differentials of first, second, and third kinds. The interchange of argument and parameter is discussed, and the reviewer regrets that Nevanlinna does not follow this up with a proof of the Riemann-Roch theorem, which is alluded to but not stated.

The Riemann mapping theorem for arbitrary simply-connected Riemann surfaces is proved in the sixth chapter. After a chapter on groups of linear transformations, the general Riemann mapping theorem is used to uniformize a Riemann surface by mapping its universal covering surface onto the plane or unit-circle. Having done this, the author takes up in some detail a variety of topics, including the Poincaré hyperbolic metric, the elliptic modular function, and conformal self-mappings of a Riemann surface.

The ninth chapter derives the various mapping functions for planar (schlicht-artige) surfaces. They are characterized by extremal properties, and their use for uniformization is discussed.

While the preceding nine chapters have given a relatively complete coverage of the classical theory of Riemann surfaces, the last chapter turns to the subject of open Riemann surfaces. Since this is an extremely active subject at the present time, it is impossible to give any sort of definitive survey. However Nevanlinna does manage to introduce the reader to the principal problems in the field and to give an idea of some of the methods available.

This book arose out of the author's lectures at the University of Helsinki and is admirably suited for anyone who wishes to learn about Riemann surfaces. The author always has the present work in open Riemann surfaces in mind, so that the reader will find that he is prepared for the literature in this field.

H. L. ROYDEN

Lezioni sulle equazioni a derivate parziali. By F. G. Tricomi. Editrice Gheroni Torino, 1954. 4+484 pp.

This book furnishes an excellent introduction to the rapidly expanding theory of partial differential equations, written in the author's usual lucid and interesting style.

The work is divided into five parts. The first part, consisting of one hundred and four pages, presents a rapid but thorough summary of classical analytic tools required in the remainder of the book. The theory of integral equations, the gamma function, the hypergeometric function, the Legendre and Bessel functions are all treated. This part is well worth reading on its own.

The second part, consisting of seventy-five pages, is devoted to a discussion of the theory of characteristics for equations of the first and second order. It includes a section devoted to the Hamilton-Jacobi theory and its connection with the calculus of variations.

The third part, one hundred pages, is devoted to equations of hyperbolic type. Various classical approaches, such as those of Laplace and Riemann, are presented, and there is large section on the movement of a compressible fluid.

The fourth part, ninety-five pages, treats the equations of elliptic type. The classical techniques are given, together with a discussion of more modern methods based upon difference equations, and numerical methods such as the "relaxation" method of Southwell. A section on incompressible fluids is included.

The fifth and concluding part is devoted to equations of parabolic type and equations of mixed-type. The greater part of this section is concerned with equations of mixed type, a topic investigated in great detail by Tricomi in 1923, and which in recent years has become of great importance in the study of transonic flow. The applications to this theory are given in several sections.

Two very useful features are the ample references to both quite recent and classical papers and the eighty or so representative problems gathered at the ends of the various sections.

The book is particularly to be recommended to anyone intending to work in the mathematical theory of hydrodynamics or aerodynamics.

RICHARD BELLMAN

Infinite abelian groups. By I. Kaplansky. (University of Michigan Publications in Mathematics, no. 2.) Ann Arbor, University of Michigan Press, 1954. 5+91 pp. $2.00.

The theory of finite and infinite abelian groups is comparatively rich in structure theorems and these form the central theme of Kaplansky's excellent monograph. A satisfactory characterization is available, and presented here, for the groups in the following classes of abelian groups: finite groups [Frobenius-Stickelberger], torsion groups with bounded order, finitely generated groups, countable torsion groups [Ulm], groups with division, countably generated torsionfree modules over complete discrete valuation rings [Kaplansky]. Direct sums of cyclic groups are naturally prominent in this discussion, since the groups in the classes mentioned are either direct sums of cyclic groups or direct sums of cyclic groups and groups with division or else contain direct sums of cyclic groups as essential building blocks. Thus one finds here the theorems of Prüfer and Kulikoff, characterizing certain direct sums of cyclic groups, and their application proving that every subgroup of a direct sum of cyclic groups is itself a direct sum of cyclic groups.

Various structures may be derived from an abelian group. The ring of endomorphisms is known to reflect particularly faithfully the properties of the original group; and the author proves for a large class of groups that they are completely determined by their endomorphism rings—this is one of many instances where results in this work go beyond what had been known before. Similarly the theory of characteristic subgroups has been developed considerably further than had been done by the author's predecessors.

The first eleven chapters lead up to Ulm's theorem which is proved with admirable simplicity. This is no mean task, considering the difficulty of Ulm's original proof and of Zippin's simplified version thereof. Operators make their first appearance in the 12th chapter, quite rightly in our opinion, since in a large part of the theory of abelian groups the operators do not contribute any more to our discussion than the—purely ring theoretical—question: which properties of the integers have been really needed for our arguments? Quite naturally the answer to this question does not throw too much light on the theory of abelian groups. Still the operators have their uses: firstly a special selection of the ring of operators may produce a class of abelian groups that is amenable to treatment—a trivial example is provided by the modules over a field; and secondly they open the way to interesting applications and the author has not missed his opportunities here.

This book is written with admirable simplicity and lucidity; and it is a pleasure to be led by the author through this field, formerly so inaccessible, now so easy of access. Here it should be noted that the interesting and deep results presented in this book had been scattered through the mathematical periodicals of the world if they had been published at all. But let the reader not be misled by the apparent slimness of this monograph: There are a hundred theorems given without proof [under the misleading name of exercise] and if their proofs had been supplied [instead of the "hints" given in difficult situations] the size of the book would have been doubled at the least.

A substantial bibliography with a useful "guide to the literature"—that might be considered as a rather concentrated report on the state of the theory of abelian groups—deserves our particular attention.

We have explained why we believe that the student of the theory of abelian groups will be grateful to the author for this work. But there are various reasons why this book will be a great help to those of us who attempt to educate mathematicians and to introduce young people to the present-day ways of mathematical thinking. For instance, one likes to impress on a student's mind the central importance of structure theorems. But there do not exist many of them; and without examples of structure theorems it is impossible to explain their significance and in particular what constitutes a satisfactory structure theorem. Now these theorems form, as we mentioned before, the central theme of the book under review. Next the transfinite tools, so helpful in various branches of mathematics, are used here extensively so that the reader will learn how to avail himself of these methods. As a matter of fact this was one of the author's aims in writing this book; and in this as in his other aims he has succeeded brilliantly.

REINHOLD BAER

Handbook of elliptic integrals for engineers and physicists. By P. F. Byrd and M. D. Friedman. (Die Grundlehren der mathematischen

Wissenschaften, vol. 67.) Berlin, Springer, 1954. 13+355 pp. 36 DM.; bound, 39.60 DM.

This work is essentially a large table of elliptic integrals and, as such, is a valuable addition to the literature of the subject.

After an introduction containing a rich collection of formulas on the Jacobian elliptic functions and integrals (about 40 pages), the greater part of the book consists of 148 pages of formulas giving the reduction of more than a thousand elliptic integrals, with algebraic or elementary transcendental integrands, to (far less numerous) integrals whose integrands are rational combinations of Jacobian elliptic functions. A further section (about 30 pages) gives the expressions of the latter integrals in terms of elliptic functions and of the three classical kinds of elliptic integrals, in Legendre's notation. The last part of the book (about 70 pages) deals first with methods for evaluating the integrals of the third kind (the weak point of many works on the subject), which are generally expressed in terms of ϑ-functions. There are also a table of integrals and derivatives with respect to the modulus, an appendix on Weierstrassian functions and integrals (in the rest of the book only the notations of Legendre and Jacobi are used), and some supplements to the principal table of elliptic integrals. The book closes with about 30 pages of numerical tables, the most extensive of which gives the values of the Legendrian elliptic integrals of the first and second kinds.

In the background of the work there seems to be a certain, not unfounded, skepticism about the ability of the average applied mathematician to reduce, himself, a given elliptic integral to one of the canonical types. I myself have often had a similar feeling, considering the poor help which many "practical" books on the subject offer to an unskilled reader. With this handbook at hand, the difficulty is largely eliminated, since in most cases one will find his integral, or a similar one, in the tables.

Despite three pages of errata and some misspellings in the bibliography, the book seems to be reliable; the typography is excellent.

F. G. TRICOMI

BRIEF MENTION

The geometry of René Descartes. Trans. from the French and Latin by D. E. Smith and M. L. Latham. With a facsimile of the first edition, 1637. New York, Dover, 1954. 14+244 pp. $2.95 cloth, $1.50 paper.

This is a photographic reproduction of the 1925 edition published by Open Court, Chicago (and not mentioned in this edition). The French text and the translation appear on facing pages.

Selected papers on noise and stochastic processes. Ed. by N. Wax. New York, Dover, 1954. 6+337 pp. $3.50 cloth, $2.00 paper.

Reprints of papers by S. Chandrasekhar, G. E. Uhlenbeck and L. S. Ornstein, Ming Chen Wang and G. E. Uhlenbeck, S. O. Rice, M. Kac, and J. L. Doob.

Modern physics for the engineer. Ed. by L. N. Ridenour. New York, McGraw-Hill, 1954. 20+499 pp. $7.50.

This volume contains two chapters of possible interest to mathematicians: Communication theory and transmission of information, by J. B. Wiesner; Computing machines and the processing of information, by L. N. Ridenour.

Table of binomial coefficients. Prepared for the Mathematical Tables Committees of The British Association and The Royal Society under the editorship of J. C. P. Miller. Cambridge University Press, 1954. 8+162 pp. $5.50.

Exact values of ${}_nC_r$ are given for $r \leqq n/2 \leqq 100$, as well as for some small values of r and larger values of n; for $r=2, 3$ the tables extend to $n=5000$.

RESEARCH PROBLEMS

1. Richard Bellman: *Dynamic programming.*

Solve

$$f(x, y) = \operatorname{Max}\begin{bmatrix} p_1[r_1x + f((1 - r_1)x, y)], \\ q_1[s_1y + f(x, (1 - s_1)y)], \\ p_2[r_2x + s_2y + f((1 - r_2)x, (1 - s_2)y)] \end{bmatrix}, \qquad x, y \geq 0,$$

where $0 < p_1, p_2, q_1, r_1, r_2, s_1, s_2 < 1$; see Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 1077–1082; vol. 38 (1952) pp. 716–719. (Received September 23, 1954.)

2. Richard Bellman: *Probability theory.*

Let $\{Z_k\}$ be a sequence of random matrices having a common probability distribution, say p, of being A and $(1-p)$ of being B, where A and B are two matrices having all positive elements. Let $X_N = \prod_{k=1}^{N} Z_k$, and $x_{ij}(N)$ be the ijth element in X_N. Determine the limiting distribution of $\log x_{ij}(N)$, suitably normalized. See Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) and Rand Paper 398, to appear in a forthcoming issue of Duke Math. J. (Received September 23. 1954.)

3. Richard Bellman: *Analysis.*

Let $\prod_{k,l=0}^{\infty} (1-x^k y^l t)^{-1} = \sum_{n=0}^{\infty} q_n(x, y)t^n$. Obtain a formula for $q_n(x, y)$. (Received September 23, 1954.)

4. Richard Bellman: *Number theory.*

Let $f(x)$ be an irreducible polynomial with integer coefficients and the property that $f(x) > x$ for $x \geqq a$. Prove that the sequence $\{x_n\}$ defined by the recurrence relation $x_{n+1} = f(x_n)$, $x_0 = a$, an integer, cannot represent primes for all large n. See Bull. Amer. Math. Soc. vol. 53 (1947) pp. 778–779. (Received October 2, 1954.)

5. Richard Bellman: *Number theory.*

Let x be a rational number greater than one, and let $[y]$ denote, as customary, the greatest integer contained in y. Prove that $[x^n]$ cannot be prime for all large n. (Received October 2, 1954.)

6. Richard Bellman: *Number theory.*

Prove that $\sum_{n=1}^{N} d(n^2+2) \sim cN \log N$ as $N \to \infty$. See Duke Math. J. vol. 17 (1950) pp. 159–168. (Received October 2, 1954.)

7. A. D. Wallace: *Manifolds with multiplication.*

Let M be a compact connected manifold without boundary and provided with a continuous associative multiplication such that $MM = M$. Does the following hold: either (i) M is a group or (ii) $xy = y$ for each x, y or $xy = x$ for each x, y? It is known that (i) holds if we replace "$MM = M$" by "there is a two-sided unit"; see Summa Brasil. Math. vol. 3 (1953) pp. 43–55. (Received October 4, 1954.)

8. A. D. Wallace: *Fixed points for topological lattices.*

A topological lattice is a Hausdorff space X together with two maps (continuous functions) $\wedge: X\times X\to X$ and $\vee: X\times X\to X$ satisfying the usual conditions. Let X be a compact connected topological lattice. (1) Does X have the fixed point property? (2) If X is also metric, is X an absolute retract in the sense of Borsuk? It is clear that (2) implies (1) when X is metric. Easy examples show that (2) fails if X is not metric. These conjectures are supported by numerous special cases (unpublished) as well as by the fact that X has to be trivial in the sense of cohomology for any coefficient group; see Summa Brasil. Math. vol. 3 (1953) pp. 43–55. (Received October 4, 1954.)

9. A. D. Wallace: *Two problems on topological semi-groups.*

Let S be a clan. That is, S is a compact connected Hausdorff space together with a continuous associative multiplication with two-sided unit. (1) If S is finite-dimensional and a homogeneous space, is S a group? This is known to hold if also S is a manifold (Summa Brasil. Math. vol. 3 (1953) pp. 43–55) or if S is indecomposable (Math. J. Okayama Univ. vol. 3 (1953) pp. 1–3). (2) Let K be the smallest nonvoid subset of S such that $SK\subset K\supset KS$. If S is an AR (ANR), is K an AR (ANR)? It is easy to affirm this if K meets the center of S (using an unpublished result of R. J. Koch) since K is then a retract of S. In this case K is a group and hence is a zero for S if S is an AR. (Received October 4, 1954.)

10. A. D. Wallace: *Differentiability of continuous multiplications.*

Let S be a clan, that is, let S be a compact connected Hausdorff space together with a continuous associative multiplication with two-sided unit. Let S be topologically contained in R^n, let u be the unit of S, let $H(u)$ be the maximal sub*group* of S containing u (Anais Acad. Brasil. Ci. vol. 25 (1953) pp. 335–336), and let F be the boundary of S relative to R^n. It is known (Math. J. Okayama Univ. vol. 3 (1953) pp. 23–28) that $H(u)\subset F$. If $H(u)=F$, it follows from an unpublished result of R. H. Bing, together with well-known results on topological groups, that $H(u)$ is a Lie group. Assume that this is so. (1) Can the multiplication be assumed differentiable on all of S? (2) If T is the quotient space of S mod $H(u)$, is T a differentiable manifold with boundary? (Received October 4, 1954.)

11. A. D. Wallace: *An addition theorem for cohomology.*

Let X be a compact Hausdorff space, let $X=X_1\cup X_2$ with X_1 and X_2 closed, and let H^n be the n-dimensional cohomology group over a fixed abelian group. Suppose that $h\in H^n(X_1\cap X_2)$ is such that $i^*h=0$ if $i: A\to X_1\cap X_2$ is the inclusion map and A is any closed proper subset of $X_1\cap X_2$. Assume also that h is not extendable to $H^n(X_1)$ but is extendable to $H^n(B)$ if B is a closed proper subset of X_1 which includes $X_1\cap X_2$. If, finally, a similar condition holds for X_2, is it true that $H^{n+1}(X_1\cap X_2)\neq 0$? An analogue of this result is known to hold for homology and is easy to prove. (Received October 4, 1954.)

12. Dieter Gaier: *A problem on entire functions.*

Let $f(z)$ be an entire function with $|f(z)|\leq Ae^{B|z|}$, and assume that $\lim f(z)=0$ for $z\to\infty$ along a path C. Is it true that then also $\lim f'(z)=0$ for $z\to\infty$ along C? The case that C is a straight line was considered in Math. Zeit. vol. 58 (1953) p. 454. (Received October 18, 1954.)

1954

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THE STRUCTURE OF TOPOLOGICAL SEMIGROUPS

A. D. WALLACE

The title of this address might incline one to the notion that here is to be found a small number of large theorems. To the contrary, I shall talk about a large number of small theorems. Actually, there does not exist at this time any corpus of information to which the title "structure of topological semigroups" is in any fashion applicable. Whether such a body of theorems will ever exist is a matter for the future and is likely to depend on the use to which it might be put as well as to the tastes of mathematicians who are not yet such.

When the investigation of topological groups began there was at hand a theory of abstract groups and much of a fundamental character in Lie groups was available. Beyond this there existed a great body of geometry even if some of it was in a nebulous state insofar as the then held standards of rigor were concerned.

With topological semigroups the situation is quite contrariwise. Here we are faced with a lack of satisfactory algebraic results. I do not think that there are so many as twenty-five papers each exceeding ten pages which are concerned exclusively with the algebraic aspects.

We are more fortunate than were the pioneers who forayed the frontiers of topological groups in that we have at our disposal a greater wealth of topology. Much that they could not—or at least did not—use is at hand for our use. Furthermore we can rely, at least if for no more than analogy, on their results. The state of both algebraic and set-theoretic topology is a somewhat happier one now than then. Still we are likely to be troubled for awhile for lack of something like Haar measure without which we shall be at a loss for representation theorems. At present there seems to be no line of attack on the representation problem and it is probable that we shall need to rely to a greater extent on geometry and topology than was the case with groups.

1. **Introduction.** My interest in this field began many years ago when, as a graduate student, I first learned of the beautiful theorem of E. Cartan,

If an n-sphere is a topological group then $n=0$, 1 or 3.

An address delivered before the Annual Meeting of the Society in Baltimore, Maryland, on December 28, 1953, by invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings; received by the editors July 2, 1954.

CALC
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No thought is required of a graduate student to turn this into an all-embracing question—

What spaces admit what algebraic structures?

But my training and a few simple examples indicated that more progress was likely to be made if the question were reformulated;

What compact connected Hausdorff spaces admit a continuous associative multiplication with unit?

The question is still a large one. We shall concentrate on other, and shorter, themes and leave the original as a basso ostinato, an insistent reminder not to stray too far, in our development, from the announced problem.

A ***mob*** is a map (=continuous function) $m: S\times S\to S$ such that

(i) S is a Hausdorff space and

(ii) m is associative. If we write $xy=m(x, y)$ then (ii) becomes the more familiar $(xy)z=x(yz)$.

A word about terminology is in order. One commonly uses such phrases as "G is a compact topological group" when, of course, G is actually a set. Tradition will be followed here and we will use "S is a compact mob" in place of a longer and more correct form.

A ***clan*** is a compact connected mob with a (two-sided) unit. A schematic theorem of the sort that interests me is as follows:

THEOREM. H_1: *S is a clan*
H_2: *Topological*
C: *Algebraic.*

Let us say that a space is indecomposable if it is *not* the union of two closed connected proper subsets. We commonly think of indecomposable spaces as being monstrous things created by set-theoretic topologists for some evil (but purely mathematical) purpose.

THEOREM. H_1: *S is a clan*
H_2: *S is indecomposable*
C: *S is a group.*

A ***manifold*** is a locally Euclidean space. Surely manifolds and indecomposable spaces are antipodal points in the sphere of topology. But we modulate from the tonic minor of the pathological (by way of algebraic topology) to the dominant major of the commonplace to obtain this result,

THEOREM. H_1: *S is a clan*
H_2: *S is a manifold*
C: *S is a group.*

Of course not all theorems of interest fall into the scheme we indicated. Let R^n be n-dimensional Euclidean space. Denote the (set-theoretic) boundary of the set A by $F(A)$. Now if u is the unit of the clan S let $H(u)$ be the set of all elements of S with (two-sided) inverses relative to u. As we shall see later $H(u)$ is a compact group.

THEOREM. H_1: *S is a clan*
H_2: *S is topologically contained in R^n, $n \geq 2$.*
C: $H(u) \subset F(S)$.

With this introduction we turn to the pedestrian task of indicating in more detail what has been accomplished thus far in mobs. We shall make a definite effort to show that there are few domains of topology into which mob-theory has not penetrated. We shall not mention the work of Eckmann, Hopf, Leray, Borel and others discussed by Samelson in his invited address [25]. The applications to analysis were discussed by Hille in his Colloquium Lectures [13].

A word about notation and terminology. In general, topological meanings will have precedence of algebraic ones. We use A^* for the topological closure of A so that "A is closed" means $A = A^*$ and *not* $A^2 \subset A$. We denote the null set by $\square$, 0 and $\square$ being homeomorphic. The symbol $A \backslash B$ is used rather than $A - B$ for the complement of B in A. "Compact" will be used for "bicompact." A space is *connected* if it is *not* the union of two nonvoid disjoint closed sets.

2. **Maximal subgroups.** It is natural to examine first those substructures of the mob S (*and in all that follows S will be a mob*) with which we are most familiar. A *subgroup* of S is a nonvoid set A satisfying $xA = A = Ax$ for all $x \in A$. With the aid of the Hausdorff Maximality Principle (or Zorn's Lemma) one readily proves [32]:

THEOREM 2.1. *Each subgroup of S is contained in a maximal such and no two of these intersect.*

It may happen that S contains no subgroups. But when S is compact there is at least one. Now a subgroup is a self-contained object, and, in order to bring into play all of S, we say that a set $L \subset S$ is a *left ideal* if $L \neq \square$ and if $SL \subset L$. Here, $AB = \{xy \mid x \in A \text{ and } y \in B\}$ or $AB = m(A \times B)$. Similarly one defines *right ideal* and (two-sided) *ideal*. We shall denote by K the (unique if it exists) *minimal ideal* of S and by E all those $e \in S$ such that $e^2 = e$. Recall that if S has a (two-sided) unit it is designated by u. If e is an idempotent (i.e., $e \in E$) let $H(e)$ be the maximal subgroup of S containing e.

THEOREM 2.2. *If S has a minimal left and a minimal right ideal then*

S has a minimal ideal K and

$$K = \bigcup \{H(e) \mid e \in E \cap K\}.$$

Any pair $H(e_1)$, $H(e_2)$ of subgroups with e_1, $e_2 \in E \cap K$ are topologically isomorphic.

This result goes back to Suschkewitsch [27], Rees [24], and is due essentially to Clifford [3] with these hypotheses. It is useful to note that $H(e) = eSe$ if and only if $e \in K$. Recall that a space X has the *fixed point property* if each map f: $X \to X$ provides an $a \in X$ such that $f(a) = a$. When the hypotheses of Theorem 2.2 subsist and if S has the fixed point property, then $K \subset E$. It is interesting to know what properties of S are inherited by K. Thus if S is an absolute retract (absolute neighborhood retract) in the sense of Borsuk, is K also one such? This is true if, for example, S is abelian or indeed if S contains a (nonvoid) normal subset because then $K = eSe$ and the function $x \to exe$ retracts S onto K, see Clifford-Miller [5] and Koch [15]. We shall see later that, if S is a clan, then the cohomology structure of K is exactly that of S.

The maximal subgroups of S seem rather mysterious beings. Here is a result reformulated from Green [12].

THEOREM 2.3. *Let, for $x \in S$,*

$$T(x) = xS \cap Sx \cap \{y \mid x \in yS \cap Sy\}.$$

Then $T(e) = H(e)$ if $e \in E$ and $T(x)$ is a maximal subgroup if and only if $T(x)$ is a submob.

THEOREM 2.4. *Let S be compact and let*

$$H = \bigcup \{H(e) \mid e \in E\}.$$

Then H is closed. If $x \in H$ let $y(x)$ be the unit of the unique maximal subgroup containing x and let $\theta(x)$ be the inverse of x in this group. Thus

$$xy(x) = x = y(x)x,$$

$$x\theta(x) = y(x) = \theta(x)x,$$

$$y(y(x)) = y(x), \qquad \theta(\theta(x)) = x.$$

Then y: $H \to E$ is a retraction and θ: $H \to H$ is an involutorial homeomorphism.

This result (see [32]) shows that H has a "continuous unit" and a "continuous inverse." It is a corollary that when S is a compact algebraic group it is also a topological group. It does not yet seem to be known if "locally compact" can replace "compact" in the last sentence. A useful algebraic analysis of H can be found in Clifford [2] under the condition that $HH \subset H$.

There is a sort of dual to Theorem 2.4 (see [1; 9; 18 and 34]). Let

$$P(S) = \{x \mid xS = S\}, \qquad Q(S) = \{x \mid Sx = S\}.$$

THEOREM 2.5. *Let S be compact and let $P(S) \neq \square$. Then*

$$P(S) = \bigcup \{H(e) \mid e \in E \cap P(S)\}$$

and each pair of the groups $H(e_1)$, $H(e_2)$ are topologically isomorphic. Indeed, if $e_0 \in E \cap P(S)$, there is a topological isomorphism

$$P(S) = H(e_0) \times (E \cap P(S)).$$

Moreover $E \cap P(S)$ is the set of left units of S. Finally $P(S) \neq \square \neq Q(S)$ if and only if S has a unit and then $P(S) = H(u) = Q(S)$.

In a sense Theorem 2.5 is "best possible." For if G is a compact group with unit e and if X is a compact space with multiplication $xy = y$ for all $x, y \in X$, then $S = G \times X$ is a compact mob, using coordinate-wise multiplication. We have $E = E(S) = \{e\} \times X$, $P(S) = S$, and each maximal subgroup of S is $G \times \{x\}$, $x \in X$.

It is a corollary to Theorem 2.5 (Gelbaum-Kalisch-Olmsted, Iwasawa, Peck) that a compact mob with two-sided cancellation is a topological group.

The results in this section do not exhaust our knowledge of the structure of S related to its maximal subgroups. We cite without further comment the papers of Faucett [8], Numakura [21], Koch [15; 16] and Koch-Wallace [17]. The maximal subgroups will continue to appear as we proceed.

3. Submobs.

We define, for $x \in S$,

$$\Gamma_n(x) = \{x^n, x^{n+1}, \cdots\}^*,$$
$$\Gamma(x) = \Gamma_1(x),$$
$$K(x) = \bigcap \{\Gamma_n(x) \mid n \geqq 1\}.$$

Because of its elegance, simplicity, and useful character we state first a result which can be materially strengthened (Koch [15], Numakura [21], Peck [23]).

THEOREM 3.1. *If $\Gamma(x)$ is compact, then $K(x)$ is the minimal ideal of $\Gamma(x)$ as well as the unique maximal subgroup of $\Gamma(x)$. If $O(x) = \{x, x^2, \cdots\}$ then*

$$\Gamma(x) = K(x) \cup \{O(x)\backslash K(x)\}$$

and $O(x)\backslash K(x)$ *is a discrete open subset of* $\Gamma(x)$.

The first person to examine the structure of $\Gamma(x)$ seems to have been Koch [15] from whose results we quote the following.

THEOREM 3.2. (a) *If* $\Gamma(x)$ *is locally compact and is either zero-dimensional with unit or an algebraic group, then* $\Gamma(x)$ *is a compact topological group.*

(b) *If* $\Gamma(x)$ *is compact, then any one of the following is sufficient to imply that* $\Gamma(x)$ *is a group:*

(i) $\Gamma(x)$ *has a unit.*
(ii) $\Gamma(x)$ *is connected.*
(iii) $\Gamma(x)=\Gamma(a)$ *for some* $a\neq x$.
(iv) $\{x\}$ *is not open in* $\Gamma(x)$.

Koch [16] has been able to generalize Theorem 3.1 in an interesting fashion but lack of space does not allow inclusion of this result.

Of capital importance in group-theory is the device of "translating" to the identity. This is not possible in mobs and an ingenious substitute for this is due independently to Koch [15] and Numakura [21].

THEOREM 3.3. *If* $\Gamma(x)$ *is compact and if* A *is a compact set such that* $xA\subset A$, *then*

$$eA = \bigcap\{x^nA \mid n \geqq 1\}$$

where e *is the unit of* $K(x)$.

We can "dualize" this result as follows (see [34]).

THEOREM 3.4. *Let* $\Gamma(x)$ *and* A *be compact. If* $A\subset xA$, *then* $yA=A$ *for each* $y\in\Gamma(x)$ *and each such* y *acts as a homeomorphism on* A.

Both of these results have two-sided analogues (see [34]) and an interesting interpretation connecting them with a result due to G. T. Whyburn [37]. They also have extensions to the case of a mob acting on a space, [17] and [34].

It is a corollary to Theorem 3.1 that if S is compact then $E\neq\square$. This result has been used by Wendel [36] to show Haar measure exists on a compact group.

THEOREM 3.5. *Let* S *be compact, let* J *be a nonvoid family of nonvoid sets of* S *and let* $R=\bigcup\{T \mid T\in J\}$. *If*

(i) $T_1, T_2\in J$ *implies* $T_3\subset T_1\cap T_2$ *for some* $T_3\in J$ *and if*
(ii) $t\in R$ *and* $T\in J$ *imply* $T_1\subset tT$ *and* $T_2\subset Tt$ *for some* $T_1, T_2\in J$,

then

$$S_0 = \bigcap\{T^* \mid T \in J\}$$

is a group and the minimal ideal of the smallest closed submob of S containing R.

This result, a manifest extension of (3.1), is a generalization of a theorem of Peck's [23]. It has several reformulations which the reader may find for himself or in [34].

If S is a mob and if G is an algebraic subgroup of S when is G a topological group, i.e., under what conditions on G or S can we assert that inversion is continuous in G? The first important result of this kind stems from Montgomery and asserts that if G is separable metric and complete then G is topological [19]. Other versions have been given by Ellis [7], N. J. Rothman (unpublished), and Moriya [20]. The latter's theorems can be extended as we shall see in (3.6) and (3.7).

Recall that a space is rim compact if each point has a neighborhood basis of open sets with compact boundaries. A rim compact Hausdorff space is completely regular. This follows from the fact that if X is one such and if A is a closed set with a compact boundary, then A has a neighborhood basis of open sets with compact boundaries. We recall that a space is locally connected if each point has a neighborhood basis of connected sets.

Despite several published assertions that "a countably compact mob with two-sided cancellation is a topological group" there remains some doubt as to the validity of this claim. The techniques of McShane [41] and Pettis ([42], [43] and [44]) may be useful in this connection as well as in the problem just raised concerning the continuity of inversion. The results of Rothman were obtained, in part, in this framework.

THEOREM 3.6. *If G is a maximal subgroup of S and if S is locally connected and rim compact, then G is topological.*

THEOREM 3.7. *If S is a locally compact algebraic group and if the unit-component of S is compact, then S is topological.*

We can extend (3.7) in the manner of (3.6) but we select this special case so that we may indicate a proof. Since the unit-component C of S is compact there is a compact open set $V \supset C$. Let $T = \{x \mid xV \subset V\}$. It is easy to see that T is an open and closed submob ([28] and [31]) and is compact because $u \in V$. Now T has two-sided cancellation because S has this property so that (remarks following (2.5)) T is an open topological subgroup. Thus inversion is continuous at u and, since $x \to x^{-1}$ is an antihomomorphism, its continuity at u implies its continuity on S.

The methods in this proof are of general application. The reader will note that V might have been selected inside any open set about C and so as a corollary we have the familiar result that any totally disconnected locally compact group contains arbitrarily small open compact subgroups. The fact that S was an algebraic group did not appear until the last two sentences so we can say that a locally compact totally disconnected mob with unit has arbitrarily small open compact submobs containing the unit. We can easily generalize this to the case where the component of S containing an idempotent is compact.

The familiar process of dividing a group by its unit-component has a mob-theoretic version. Let S be a locally compact mob with *each* component compact. If T is the component *space* of S we construct in an obvious way a continuous multiplication in T and a (continuous) homomorphism f of S onto T. Of course T is locally compact and totally disconnected. The map f is closed but need *not be open.*

The use of sets like $\{x \mid xA \cap B \neq \square\}$ is typical. This device seems indicated as a replacement for the use of BA^{-1} with which the above set would be identical were S a group. It is interesting to carry through (as far as possible) the result from topological groups that, if A is compact and if B is closed, then AB is closed.

4. **Ideals.** One easily sees that any left ideal meets any right ideal. Hence the collection of all ideals of S has the finite intersection property and it follows that if S is *compact* then S has a minimal closed ideal. Now it is almost obvious that, in this case, any ideal contains a closed ideal. Thus S has a minimal ideal if it is compact. Further S has minimal left and right ideals, Numakura [21]. One way of characterizing minimal ideals is as follows (Koch [15]):

THEOREM 4.1. *Let $e \in E$. These are equivalent:*

(i) *Se is a minimal left ideal.*

(ii) *SeS is the minimal ideal.*

(iii) *eSe is a maximal subgroup.*

Numakura [22] initiated the investigation of nil-elements in a mob with zero and some extensions and completions have been given by Koch [16]. One can say that $x \in S$ (S with zero) is nil if $0 \in \Gamma(x)$. It is possible to extend Theorem 4.1 in various ways using the notion of nil-elements. Despite the interest and importance of this material we are impelled by its lack of definitive status to go no further in this direction. A purely algebraic situation has been considered by Clifford, Rich, and Schwarz (see bibliography in [4]).

Maximal ideals were first investigated by Koch and Wallace [17]. They are of great importance both structurally and as a tool. If $A \subset S$ let $L(A)$ be the union of all left ideals of S contained in A. Of course $L(A)$ may be null. The primary weapon here is the fact that if A is closed then $L(A)$ is closed and if A is open and if S is compact then $L(A)$ is open. One has (see [11] and [17])

THEOREM 4.2. *If S is compact and if S properly contains an ideal, then S has a maximal proper ideal J. If S has a unit then $J = S\backslash H(u)$.*

There are many variations on this theme and many applications of the concepts involved and we shall illustrate with some typical theorems.

THEOREM 4.3. *If S is compact, if $S^2 = S$, and if E has at most one element, then S is a group.*

We note that if S is connected then J of Theorem 4.2 is dense in S and hence we have ([17] and [31])

THEOREM 4.4. *If S is a clan, then $S\backslash A$ is connected for each $A \subset H(u)$.*

Let us agree that a ***continuum*** is a compact connected Hausdorff space. A notable result of R. L. Moore's (see [31] for a generalization) is that a nondegenerate continuum has at least two non-cutpoints. Hence a compact connected group has no cutpoint. Theorem 4.4 extends this asserting that, if S is a clan, then no point of $H(u)$ cuts S.

In order to show a typical way of using ideals we shall prove that if a clan is indecomposable then it is a group (see §1, [17], and [33]). We suppose that $S \neq H(u)$ and it follows that $K \cap H(u) = \square$. Let U be an open set about u with $U^* \cap K = \square$ and let J be the union of all ideals of S contained in $S\backslash U^*$. Thus J is open and connected [17] and J^* is a continuum not all of S. There are two cases. If $S\backslash J^*$ is connected, then $S = J^* \cup (S\backslash J^*)^*$ so that S is the union of two of its proper subcontinua. If $S\backslash J^*$ is not connected then $S\backslash J^* = A \cup B$ where A and B disjoint, open, and nonvoid. By a familiar result we know that $A \cup J^*$ and $B \cup J^*$ are closed connected subsets of S so that again S is decomposable.

In the study of continua one is led to consider *C-sets*. A C-set is such a subset C of S that if A is a subcontinuum meeting C then $C \subset A$ or $A \subset C$. The composants of an indecomposable continuum are C-sets. From [45] we have the

THEOREM 4.5. *Let S be a clan and let C be a C-set.*

(i) *If $C\cap K\neq\square$ then $C\subset K$. If $C=K$ then K is a group.*

(ii) *If $C\cap H(u)\neq\square$ then $C\subset H(u)$. If $e\in E$ and if $H(e)$ is nondegenerate then $C\cap H(e)\neq\square$ implies $C\subset H(e)$.*

With the aid of some results which involve a type of set related to the C-sets one can prove, for example, the following.

THEOREM 4.6. *Let S be a clan and let $H(u)$ be totally disconnected. Then S is semi locally connected* [37] *at each point of $H(u)$ and may possibly not enjoy this property at no point not in $H(u)$.*

It may be seen by an example that, if $H(u)$ is connected, then S may be semi locally connected at no point.

A sequence of results, combining the structure of continua with clans, has been obtained by W. M. Faucett [8]. Many of these involve the use of maximal ideals. Here are some typical ones.

THEOREM 4.7. *Let S be a continuum, let $q\in S\backslash K$, and let Q be the component of $S\backslash\{q\}$ containing K. Then qS, Sq, and SqS are all contained in $Q\cup\{q\}$.*

Now qS is an algebraic object and $Q\cup\{q\}$ is a topological one so that the inclusion $qS\subset Q\cup\{q\}$ links these two types of things.

It follows that if $q^2=q$ and if L is the union of all left ideals contained in $S\backslash\{q\}$, then $(S\backslash Q)q\subset L^*\backslash L$. If also Q is a left ideal then $(S\backslash Q)q=\{q\}$.

THEOREM 4.8. *Let S be a locally connected metric continuum and let J be a maximal proper ideal of S. If $S\backslash J$ contains at least two points, then some arc in S has only its end points in $S\backslash J$.*

Finally Faucett has proved

THEOREM 4.9. *Let S be a clan which is irreducibly connected between two elements of E. Then S is abelian if and only if it has a zero element.*

It ought to be noted that we have presented simplified versions of Faucett's results in the interest of clarity.

We close this section with an example in which we apply some of the above results. Let S be the subset of R^2 consisting of those points (x, y) with $y=\sin(x^{-1})$, $x\geqq 1$, together with the segment from $(0, 1)$ to $(0, -1)$. Note that C is a C-set of S. We show that S cannot support the structure of a mob with unit. Suppose this is false. By (4.4) we know that $u\in C$ or u is the end point of S. If $u\in C$ then $C\subset H(u)$ by (4.5) (ii). If D is the component of $H(u)$ containing C, then D is a group and hence homogeneous. But clearly no closed connected subset of S containing C can be homogeneous. Hence u is the end point of S. We may write $S\backslash C=\bigcup\{A_i|i=1, 2, \cdots\}$ where $A_1\subset A_2 \subset A_3 \cdots$ and A_i is an arc with u as one end point. For each i we know that A_iC is a locally connected continuum containing C and thus $A_iC=C$. Thus $C=(S\backslash C)C$ and $C^*=(S\backslash C)^*C=SC$. Similarly $C=CS$. Thus C is an ideal and hence $K\subset C$. By (4.5) (i) $C\subset K$ and K is a group. But C is not homogeneous.

5. **Algebraic generalities.** The definitions which we now give are to be used in the next section. They have, it may seem, no a priori intuitive raison d'etre.

Let X be a space and let G be some discrete additive abelian group. If $n\geqq 0$ let ϕ and ψ be any two functions on X^{n+1} into G. We define an addition by

$$(\phi+\psi)(x_0, x_1, \cdots, x_n) = \phi(x_0, x_1, \cdots, x_n)+\psi(x_0, x_1, \cdots, x_n).$$

We let $0(x_0, x_1, \cdots, x_n)=0\in G$ and $(-\phi)(x_0, x_1, \cdots, x_n) = -\phi(x_0, x_1, \cdots, x_n)$. With these operations the set $C^n(X, G)$ of *all* functions on X^{n+1} to G becomes a group. A function $\phi\in C^n(X, G)$ is *locally zero* if, for each $x\in X$, there is an open set U about x such that $\phi(x_0, x_1, \cdots, x_n)=0$ if $x_0, x_1, \cdots, x_n\in U$. If $\phi\in C^n(X, G)$ we define a function $\phi'\in C^{n+1}(X, G)$ by

$$\phi'(x_0, x_1, \cdots, x_{n+1}) = \sum_{i=0}^{n+1}(-1)^i\phi(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$$

where $(-1)^ng=g$ if n is even and $(-1)^ng=-g$ if n is odd, g being any element of G. The set of all $\phi\in C^n(X, G)$ such that ϕ' is locally zero is denoted by $Z^n(X, G)$. A simple argument shows that $Z^n(X, G)$ is a subgroup of $C^n(X, G)$. We define next a subgroup $B^n(X, G)$ of $Z^n(X, G)$ by taking $B^0(X, G)=0$ and by stipulating that $\phi\in B^n(X, G)$ $(n>0)$ if $\phi-\psi'$ is locally zero for some $\psi\in C^{n-1}(X, G)$. Here we need the fact that $(\psi')'=0$ and the fact that, since $\phi-\psi'$ is locally zero, $(\phi-\psi')'$ is also locally zero. Finally we let

$$H^n(X, G)=Z^n(X, G)/B^n(X, G).$$

The group $H^n(X, G)$ is the n-dimensional Alexander-Kolmogoroff cohomology group of X with coefficients in G. Modulo some easily verified assertions the reader has had a brief and painless (if unmotivated) construction of a cohomology group. We refer to Spanier [26], who gave the first exposition of this theory, for details. Further results will be found in [26; 28; 29 and 30].

We shall now sketch some salient items of the AK theory to be used in the next section. We sometimes conceal the coefficient group G so that $H^n(X, G)$ becomes $H^n(X)$. If $f: X\rightarrow Y$ is a map then one defines a homomorphism $f^*: H^n(Y)\rightarrow H^n(X)$ roughly as follows: If $\phi\in Z^n(Y)$ let $(f^* \phi)(x_0, x_1, \cdots, x_n)=\phi(f(x_0), f(x_1), \cdots, f(x_n))$ and pass to the quotient group. In particular if $A\subset X$ and if $i: A\subset X$ is defined by $i(x)=x$, then $i^*: H^n(X)\rightarrow H^n(A)$ is called the *natural homomorphism*. While the elements of $H^n(X)$ are not functions (though those of $Z^n(X)$ are functions) it is useful to employ a function-like notation. If $h\in H^n(X)$ and if $A\subset X$ we let $h|A$ be the image of h under the natural homomorphism.

If X is compact Hausdorff and if $h\in H^n(X)$ is *not* zero, then there is at least one closed set $A\subset X$ such that $h|A\neq 0$ but $h|A_0=0$ if $A_0=A_0^*\subset A$ and $A_0\neq A$. We call A a *floor* for h (see [28] and [29]). If $A=A^*\subset X$ and if $h\in H^n(A)$ is such that for no $h_0\in H^n(X)$ do we have $h_0|A=h$, then for at least one closed set $R\subset X$ we know that:

(i) If $h_0\in H^n(R\cup A)$ then $h_0|A\neq h$, and

(ii) if $R_0=R_0^*\subset R$ and if $R_0\neq R$ then for some $h_0\in H^n(R_0\cup A)$ we have $h_0|A=h$. We call R a *roof* for h (see [28] and [29]). We may say that $h\in H^n(A)$ is extendable to $H^n(X)$ (or to X) if h is the image of some element of $H^n(X)$ under the natural homomorphism $H^n(X)\rightarrow H^n(A)$.

6. Cohomology in clans.

Earlier we raised the question as to what properties of S were inherited by K, the minimal ideal of S. If S is a clan then the natural homomorphism of $H^n(S)$ into $H^n(K)$ is an isomorphism onto for any n. This is a consequence of the more general

THEOREM 6.1. *If S is a continuum with left unit and if L is a closed left ideal of S such that $t_0S\subset L$ for some $t_0\in S$, then the natural homomorphism of $H^n(S)$ into $H^n(L)$ is an isomorphism onto* [35].

Of course in the above we have left out "for each n and each coefficient group."

We obtain from this the

COROLLARY. *If S is a clan and if $e\in E$, then*

$$H^n(S) \approx H^n(eS) \approx H^n(eSe).$$

Now suppose that $n>0$, that $h\in H^n(S)$, and that S is a floor for h. If $e\in E\cap K$ and if $eSe\neq S$ then $h|eSe=0$, contrary to the isomorphism of the corollary. So $eSe=S$ and S is a group. Now a manifold M has the property that for some $n>0$ and some coefficient group G there is an $h\in H^n(M, G)$ with M a floor for h. Thus if a clan is locally Euclidean it is a group, as we stated earlier.

There are some curious applications of (6.1). Let us say that a topological lattice is a pair of maps, $\vee: X\times X\rightarrow X$ and $\wedge: X\times X\rightarrow X$, satisfying the usual conditions. Let us suppose that X is compact Hausdorff. We readily verify that the intersection of the family of sets $\{a\vee X \mid a\in X\}$ is a single element, a unit for X. Dually X has a zero. If we let C be any component of X then C is also a topological lattice and hence C is a clan and so $H^n(C)=0$ for $n>0$. It follows easily that $H^n(X)=0$ for $n>0$. Thus all compact topological lattices are cohomologically trivial. With the aid of some earlier results we can also prove the

ALPHABET THEOREM. *If the letters of the alphabet are written in block capitals, then only those letters homeomorphic with I can be topological lattices.*

With the aid of (6.1) one can exhibit finite-dimensional homogeneous continua that are not clans. It is an open question whether or not a finite-dimensional clan which is also a homogeneous space is a group.

Following Haskell Cohen we define the *codimension* of the compact Hausdorff space X (relative to G) by $\mathrm{cd}(X, G)\leqq n$ if $H^n(X, G)\rightarrow H^n(A, G)$ is *onto* for all closed $A\subset X$. Cohen [6] has justified this definition by proving the expected theorems and relations.

It is perhaps reasonable to think that if N is a "large" subgroup of S then the structure of N ought to approximate that of S. For example, let S be a clan and let V be a nonvoid open set contained in $H(u)$. It may be seen that the translates of V by elements of $H(u)$ must fill $H(u)$. Hence $H(u)$ is both open and closed and thus $H(u)=S$ because S is connected. Thus if $H(u)$ contains a non-null open set then the approximation of $H(u)$ to S leaves nothing to be desired! The next result leads to a proof that, under suitable conditions, the approximation is good but need not be devastatingly so. *In Theorem 6.2 and its corollaries it is assumed that S is a continuum and that* $\mathrm{cd}(S, G)\leqq n$ (see [35] and *Topological invariance of ideals in mobs*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 866–868).

THEOREM 6.2. *Let A be closed in S, let $p, q\in S$, and define $f: A\rightarrow qA$ by $f(x)=qx$. If $h\in H^n(qA, G)$ and if $h\mid(pA\cap qA)=0$, then $f^*(h)=0$. If also $q^2=q$ and $qA\subset A$ then $h=0$.*

COROLLARY 1. *Let N be a closed set in S with $H^n(N, G)=0$ and with $N\subset t_1N$ for some $t_1\in S$. Then K, the minimal ideal of S, is also a minimal right ideal and K contains every floor of every nonzero $h\in H^n(N, G)$ and each such floor is a left ideal for S. If also $N\subset t_2N$ for some $t_2\in S$, then K is a group and is the unique floor for each nonzero $h\in H^n(N, G)$.*

COROLLARY 2. *Let also S have a unit and let N be a closed subgroup of S with $H^n(N, G)=0$. Then $K=N$, K is a homomorphic retract of S (Clifford-Miller* [5]) *and $H^p(S, G_0)\approx H^p(N, G_0)$ for any $p\geqq 0$ and any G_0.*

ZUSATZ ZUM ZUSATZ. *Let S and T be clans and let $f\colon S\rightarrow T$ be a homeomorphism onto. If* $\operatorname{cd}(S, G)\leqq n$ *and if $H^n(S, G)\neq 0$, then f maps the minimal ideal of S onto the minimal ideal of T.*

It is easy to see by an example that f need not be homomorphic. In the space $Z\times R$ of one complex and one real coordinate (using coordinate-wise multiplication) let C be the set of all (z, t) with $|z|=1$ and $t=0$ and let W be the set of all (z, t) with $Z=\exp(2\pi ia)$, $t=\exp(-a)$, $a\geqq 0$. One easily shows that $C\cup W$ is a clan. Now with the same *space* and *any* multiplication suppose that $S=C\cup W$ is a clan. We shall show that C is the minimal ideal of S and that u is the end point of S. We apply Corollary 1 and its "left-right dual" and infer from $H^1(C, G)\neq 0$ (any $G\neq 0$), that $C=K$. If $u\in K$ then $S=H(u)$. But clearly S cannot be a group. By Theorem 4.4 we see that u is the end point of S.

An immediate result of Corollary 2 is that *if S is a clan, if* $\operatorname{cd}(S, G)\leqq n$, *and if $H^n(H(u), G)\neq 0$, then S is a group*. A more rapid and direct proof of this is as follows. In the exact sequence of the triple $(S, H(u)\cup K, K)$ we have $H^n(S, K)=0$ by (6.1) and $H^{n+1}(S, H(u)\cup K)=0$ since $\operatorname{cd} S\leqq n$. Hence $H^n(H(u)\cup K, K)=0$. If S is not a group then $K\cap H(u)=\square$ by (2.5). By excision $H^n(H(u)\cup K, K)\approx H^n(H(u))$.

An inspection of examples lends credence to the idea that, if S is a clan, then in some intuitive sense $H(u)$ is contained in the "boundary" of S. In certain cases we can make this exact [34].

THEOREM 6.3. *If S is a clan topologically contained in R^n $(n\geqq 2)$, then $H(u)\subset F(S)$, the boundary of S relative to R^n.*

THEOREM 6.4. *If S is a clan topologically contained in R^n $(n\geqq 2)$ and if I is a closed ideal containing $F(S)$, then $I=S$.*

In the remainder of this section let S be a clan.

Let us say that $A\subset S$ is *covariant* if A is closed and if $H(u)AH(u)\subset A$. It is not hard to see that, if $f(A)\subset A$ for each homeomorphism f of S onto S, then A is covariant if $A=A^*$. If A is covariant then $SAS\cap H(u)=A\cap H(u)$ and hence $A\cap H(u)\neq\square$ implies $H(u)\subset A$. On the other hand if I is an ideal containing A then $A\subset SAS\subset I$. Thus, when A is covariant the following are equivalent: "$H(u)\subset A$," "$H(u)\cap A\neq\square$" and "No proper ideal of S contains A." So (6.3) and

(6.4) are the same, because of Brouwer's theorem on regional invariance, and either follows if we can show that $H(u) \cap F(S) \neq \square$. We shall now show how it is possible to make an extension. For this purpose we need some lemmas.

LEMMA 1 [28]. *Let* X *be fully normal and let* A *be closed. If* R *is a roof for* $h \in H^n(A)$, $n > 0$, *then* R *is a roof for* $h | R \cap A$, $(R \backslash A)^* = R$, *and* $R \backslash A$ *is connected.*

For a fixed $n > 0$ and a fixed coefficient group G we shall say that $x \in F^n(X)$ provided that if U is an open set about x then there is an open set V about x with $V \subset U$ such that the natural homomorphism of $H^n(X)$ into $H^n(X \backslash V)$ is *onto*. If X is a closed set in R^{n+1}, then $F(X) = F^n(X)$ where $F(X)$ is the boundary of X in R^{n+1}, see [28]. Further we say that X is of type F^n (of course relative to the obscured coefficient group) if for each closed $A \subset X$ we have $F(A) \subset F^n(A)$, where F means boundary in X.

LEMMA 2 [28]. *Let* X *be fully normal, let* A *be closed, and let* R *be a roof for some element of* $H^n(A)$, $n > 0$. *If* $F(R) \subset F^n(R)$, *then* $R \backslash A$ *is open and* $F(R \backslash A) = R \cap A$. *Hence if* $R \cup A \neq X$, *then* A *cuts* X *into the sets* $X \backslash (R \cup A)$ *and* $R \backslash A$ *and the latter is connected.*

A slightly different form of this result was found by J. W. Keesee [Proc. Amer. Math. Soc. vol. 5 (1954) p. 193].

LEMMA 3. *Let* X *be a compact Hausdorff space of type* F^n *and let* A *be a closed set with null interior which does not cut* X. *If some* $h \in H^n(A)$ *is not extendable to* $H^n(X)$, *then* X *is a roof for* h *and* $F^n(X) \subset A$.

PROOF. Let R be a roof for h. If $R \cup A \neq X$ then A cuts X, by Lemma 2. Since this is impossible, $R \cup A = X$. By Lemma 1, $R = X$ since $A^0 = \square$. If $x \in F^n(X)$ and if $x \in X \backslash A$, then, for some open set V about x not meeting A, the natural homomorphism $H^n(X) \to H^n(X \backslash V)$ is onto. Since X is a roof for h we know that h is extendable to $H^n(X \backslash V)$ and hence to $H^n(X)$, a contradiction.

THEOREM 6.5. *Let* S *be of type* F^n *and let some* $h \in H^n(H(u))$ *be not extendable to* $H^n(S)$, *e.g.,* $H^n(S) = 0$ *and* $H^n(H(u)) \neq 0$. *Then* $F^n(S) \subset H(u)$ *and if* $F^n(S) \neq \square$ *then* $F^n(S) = H(u)$.

From Lemma 3 we see that $F^n(S) \subset H(u)$. Thus, if $F^n(S) \neq \square$, then $H(u) \subset F^n(S)$ since $F^n(S)$ is clearly covariant.

An examination of set-theoretic analogs of such theorems as (6.5) will tend to throw some light on this situation. If S is irreducible (as a continuum) about the closed set A then $A \subset SAS$ implies $SAS = S$

because SAS is a continuum. Hence, in virtue of the earlier remarks, $H(u)\subset A$ if A is also covariant. Now S is irreducible (as a continuum) about the closure, A, of the set of its non-cutpoints and since A is covariant, then $H(u)\subset A$. The set A is a sort of "boundary" for S.

On the basis of (6.5) and an unpublished result of R. H. Bing it can be shown that if $S\subset R^n$, $n\geqq 2$, and if $H(u)=F(S)$ then $H(u)$ is a Lie group and $H(u)$ irreducibly cuts R^n into the two connected sets $S\backslash H(u)$ and $R^n\backslash S$.

7. **Conclusion.** It is likely that we have said those things which we ought not to have said and we have left unsaid those things which we ought to have said. In the latter category lie the algebraic results of Clifford, Dubreil, Green, Miller, Rees and Schwarz and Suschkewitsch. The papers of the Russian mathematicians are not available to me. Many results have been announced by French students of the subject. The reader may consult Mathematical Reviews.

I am greatly obliged to Professor D. D. Miller for his kindness in communicating the results of some theses prepared under his direction. The notion of a maximal subgroup and a maximal ideal appear in the elegant 1950 thesis of Helen B. Grimble.

The work of R. E. Allan, on quotient mobs, which will form a part of his dissertation, is not yet in a sufficiently complete state for exposition.

Aside from those to whose results formal recognition has been given, I record here with gratitude both the indirect and immediate aid of various members of the Tulane Topology Seminar and members of the faculty.

Finally, it is my great pleasure to acknowledge the support of this work, insofar as it involves the Tulane group, by the Office of Naval Research.

REFERENCES

1. A. H. Clifford, *A system arising from a weakened set of group postulates*, Ann. of Math. vol. 34 (1933) pp. 865–871.

2. ———, *Semigroups admitting relative inverses*, Ann. of Math. vol. 42 (1941) pp. 1037–1049.

3. ———, *Semigroups containing minimal ideals*, Amer. J. Math. vol. 70 (1948) pp. 521–526.

4. ———, *Semigroups without nilpotent ideals*, Amer. J. Math. vol. 71 (1949) pp. 834–844.

5. A. H. Clifford and D. D. Miller, *Semigroups having zeroid elements*, Amer. J. Math. vol. 70 (1948) pp. 117–125.

6. Haskell Cohen, *A cohomological definition of dimension for locally compact Hausdorff spaces*, Duke Math. J. vol. 21 (1954) pp. 209–224.

7. Robert Ellis, *Continuity and homeomorphism groups*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 969–973.

8. W. M. Faucett, *Topological semigroups and continua*, Tulane University Dissertation, 1954.

9. L. Fuchs, *On semigroups admitting relative inverses*, Publ. Math. Debrecen vol. 1 (1950) pp. 227–231.

10. B. Gelbaum, G. K. Kalisch and J. M. H. Olmsted, *On the imbedding of topological semigroups and integral domains*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 807–821.

11. A. D. Wallace, *Struct ideals*, Proc. Amer. Math. Soc. (in press).

12. J. A. Green, *On the structure of semigroups*, Ann. of Math. vol. 54 (1951) pp. 163–172.

13. Einar Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, 1948.

14. J. W. Keesee, *On the homotopy axiom*, Ann. of Math. vol. 54 (1951) pp. 247–249.

15. R. J. Koch, *On topological semigroups*, Tulane University Dissertation, 1953.

16. ———, *Remarks on primitive idempotents in compact semigroups with zero*, Bull. Amer. Math. Soc. vol. 5 (1954) pp. 828–833.

17. R. J. Koch and A. D. Wallace, *Maximal ideals in topological semigroups*, Duke Math. J. vol. 21 (1954) pp. 681–685.

18. H. B. Mann, *On certain systems which are almost groups*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 879–888.

19. Deane Montgomery, *Continuity in topological groups*, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 879–882.

20. Mikaa Moriya, *Theorie der Halb-topologischen Gruppen und Körper*, Math. J. Okayama Univ. vol. 1 (1952) pp. 109–124.

21. K. Numakura, *On bicompact semigroups with zero*, Bull. Yamagata Univ. (Natural Science) vol. 4 (1951) pp. 405–412.

22. ———, *On bicompact semigroups*, Math. J. Okayama Univ. vol. 1 (1952) pp. 99–108.

23. J. E. L. Peck, *An ergodic theorem for a noncommutative semigroup of linear operations*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 414–421.

24. D. Rees, *On semi-groups*, Proc. Cambridge Philos. Soc. vol. 36 (1940) pp. 387–400.

25. Hans Samelson, *Topology of Lie groups*, Bull. Amer. Math. Soc. vol. 58 (1952) pp. 2–37.

26. E. H. Spanier, *Cohomology theory for general spaces*, Ann. of Math. vol. 49 (1948) pp. 407–427.

27. A. Suschkewitsch, *Über die endlichen Gruppen*, Math. Ann. vol. 99 (1928) pp. 30–50.

28. A. D. Wallace, *Outline for algebraic topology*, Tulane University, 1949 and 1952.

29. ———, *A theorem on endpoints*, Anais Acad. Brasil. Ci. vol. 22 (1950) pp. 29–33.

30. ———, *The map excision theorem*, Duke Math. J. vol. 19 (1952) pp. 177–182.

31. ———, *A note on mobs* I, Anais Acad. Brasil. Ci. vol. 24 (1952) pp. 329–334.

32. ———, *A note on mobs* II, ibid. vol. 25 (1953) pp. 335–336.

33. ———, *Indecomposable semi-groups*, Math. J. Okayama Univ. vol. 3 (1953) pp. 1–3.

34. ———, *Inverses in Euclidean mobs*, ibid. vol. 3 (1953) pp. 23–28.

35. ———, *Cohomology, dimension and mobs*, Summa Brasiliensis Mathematicae vol. 3 (1953) pp. 43–54.

36. J. G. Wendel, *Haar measure and the semigroup of measures on a compact group*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 923–929.

37. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28.

38. W. M. Faucett, *Compact semigroups irreducibly connected between two idempotents*, Proc. Amer. Math. Soc. (in press).

39. ———, *Topological semigroups and continua with cutpoints*, ibid. (in press).

40. W. M. Faucett and R. J. Koch, *Complements of maximal ideals in semigroups* (to appear).

41. E. J. McShane, *Images of sets satisfying the condition of Baire*, Ann. of Math. vol. 51 (1950) pp. 380–386.

42. B. J. Pettis, *On continuity and openness of homomorphisms in topological groups*, Ibid. pp. 293–308.

43. ———, *Uniform Cauchy points and points of equicontinuity*, Amer. J. Math. vol. 73 (1951) pp. 602–614.

44. ———, *Remarks on a theorem of E. J. McShane*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 166–171.

45. A. D. Wallace, *The position of C-sets in semigroups*, ibid. (in press).

46. T. Rado and P. V. Reichelderfer, *Continuous transformations in analysis*, Springer (in press).

THE TULANE UNIVERSITY OF LOUISIANA

THE ANNUAL MEETING IN PITTSBURGH

The sixty-first Annual Meeting of the American Mathematical Society was held at the University of Pittsburgh on Monday through Wednesday, December 27–29, 1954, in conjunction with meetings of the Mathematical Association of America, which extended through Thursday, December 30; of the Association for Symbolic Logic on Wednesday, December 29; and of the Society for Industrial and Applied Mathematics on Tuesday, December 28. The registration was 658 including 546 members of the Society.

The twenty-eighth Josiah Willard Gibbs Lecture, entitled *Asymptotic phenomena in mathematical physics*, was delivered by Professor K. O. Friedrichs of the Institute of Mathematical Sciences, New York University, on Monday evening. Professor G. T. Whyburn, President of the Society, presided.

By invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings, Professor Samuel Eilenberg of Columbia University addressed the Society on *Homological algebra and dimension* on Monday afternoon, and Professor Lipman Bers of the Institute of Mathematical Sciences, New York University, addressed the Society on *Pseudo-analytic functions* on Tuesday afternoon. Professor Saunders MacLane presided at the former address, and Professor T. H. Hildebrandt at the latter.

On Wednesday afternoon, December 29, the Frank Nelson Cole Prize in Algebra was awarded to Professor Harish-Chandra of Columbia University for his papers on representations of semisimple Lie algebras and groups, and particularly for his paper *On some applications of the universal enveloping algebra of a semisimple Lie algebra*, which appeared in volume 70 of the Transactions.

The ladies of the Department of Mathematics of the University of Pittsburgh entertained at tea on Monday afternoon in the Faculty Club of the Cathedral of Learning. A conducted tour of the Mellon Institute of Industrial Research was held on Tuesday afternoon, and visits were made to other points of interest in Pittsburgh.

A banquet was held at the Hotel Webster Hall on Wednesday evening. Professor J. S. Taylor of the University of Pittsburgh was the toastmaster. The speakers were Dean S. C. Crawford, College of Arts and Sciences, University of Pittsburgh, Professor Harold W. Kuhn, President of the Society for Industrial and Applied Mathematics, Professor W. V. Quine, President of the Association for Symbolic Logic, Professor E. J. McShane, President of the Mathematical Association of America, and Professor L. W. Cohen, Associate Secretary of the American Mathematical Society. A resolution of thanks to the University of Pittsburgh was presented by Professor C. B. Allendoerfer and adopted unanimously.

The Berkeley sessions of the Sixty-first Annual Meeting were held at the University of California, Berkeley, California, on Thursday and Friday, December 30 and 31, 1954, in conjunction with the meetings there of the American Association for the Advancement of Science. All sessions of the Society were joint sessions with section A of the A.A.A.S. The registration was 64, including 57 members of the Society.

The Program Committee for Section A of the A.A.A.S. and the Committee to Select Hour Speakers for Far Western Sectional Meetings of the Society invited two hour speakers. Professor Hans Lewy of the University of California, Berkeley, delivered on Thursday morning an address entitled *Reflection laws for linear elliptic equations*, while on Friday morning Professor H. F. Bohnenblust of the California Institute of Technology spoke on *Transition points in differential equations*. Professor Lewy was introduced by Professor J. W. Green and Professor Bohnenblust by Professor R. E. Langer.

On Thursday evening a dinner was arranged at the Shattuck Hotel for the mathematicians attending the meetings of the A.A.A.S., the Society, and the Institute of Mathematical Statistics. The principal speaker at the dinner was Dr. C. B. Tompkins of the Numerical Analysis Research at U.C.L.A. The Department of Mathematics at Berkeley entertained the visitors at tea following the meetings on Friday.

The annual Business Meeting of the Society was held on Wednesday, December 29, 1954. The Secretary reported that at this time the ordinary membership of the Society is now 4693, including 466 nominees of institutional members and 36 life members. The membership of the Society showed a greater increase in 1954 than in the last few preceding years. There are also 132 institutional members. The total attendance at all meetings in 1954 was 2197; the number of papers read was 823; there were 20 hour addresses; 1 Gibbs Lecture and 10 papers at an Applied Mathematics Symposium. The number of members attending at least one meeting was 1515.

At the annual election, in which over one thousand votes were cast, the following officers were elected:

Vice President, Professor Emil Artin.

Treasurer, Dean A. E. Meder, Jr.

Secretary, Professor E. G. Begle.

Associate Secretary, Professor R. D. Schafer.

Member of the Editorial Committee of the Proceedings, Professor Shizuo Kakutani.

Member of the Editorial Committee of the Transactions and Memoirs, Professor Samuel Eilenberg.

Member of the Editorial Committee of the Colloquium Publications, Professor Salomon Bochner.

Member of the Editorial Committee of the Mathematical Reviews, Professor J. L. Doob.

Member of the Editorial Committee of the Mathematical Surveys, Professor R. J. Walker.

Member of the Committee on Printing and Publishing, Professor E. F. Beckenbach.

Members of the Board of Trustees, Professors G. A. Hedlund, Deane Montgomery, and Dean Mina Rees.

Members-at-large of the Council, Professors E. F. Beckenbach, David Blackwell, R. H. Fox, K. O. Friedrichs, and G. de B. Robinson.

The Council met on Tuesday evening, December 28, 1954.

The Secretary announced the election of the following forty-four persons to ordinary membership in the Society:

Mr. Allan Ingvald Benson, Los Alamos Scientific Laboratory;
Professor Edward Charles Blom, State Teachers College, Fredonia, New York;
Professor William Antonio Carito, Boston College;
Professor Kenneth Scott Carman, Kansas Wesleyan University;
Mr. Wilton Roy Cooper, Columbia University;
Lieutenant Richard Edwin Cornish, Fort George G. Meade, Maryland;
Professor Frank Bigley Crippen, Fordham University;
Miss Betty Charles Detwiler, Washington University;
Mr. Lester Eli Dubins, University of Chicago;
Professor James Watson Ellis, Florida State University;
Dr. Marc Francis Fontaine, The Beacon Laboratories of The Texas Company, Beacon, New York;
Mr. Talmage Yates Hicks, International Business Machines Corp., Houston, Texas;
Mr. Lee Harmon Hook, University of Minnesota;
Professor Stephen Kulik, University of South Carolina;
Professor Norman Douglas Lane, McMaster University;
Mr. Melchiore Louis LaSala, Army Chemical Center, Maryland;
Mr. Bernard S. Levine, City College of New York;
Pfc. Richard Lawrence Liboff, Army Chemical Center, Maryland;
Mr. Louis Floyd McAuley, University of Maryland;
Dr. Robert Richard McCready, Cleveland-Hopkins Airport, Cleveland, Ohio;
Dr. Garner McCrossen, National Bureau of Standards, Boulder, Colorado;
Mr. Robert Gibson McDermot, University of Pittsburgh;
Mr. Jack Nicholas Medick, Cornell Aeronautical Laboratories;
Mr. Victor Julius Mizel, Massachusetts Institute of Technology;

Mr. George Aaron Paxson, United States Army;
Mr. Paul Anthony Penzo, University of Pittsburgh;
Professor William Morris Perel, Georgia Institute of Technology;
Dr. William Wesley Peterson, International Business Machines Corp., Poughkeepsie, New York;
Mr. Stanley Preiser, Nuclear Development Associates, White Plains, New York;
Mr. Henry Rainbow, Shell Oil Company, Houston, Texas;
Professor John William Riner, St. Peters College;
Professor Hortense Corbett Rogers, Winthrop College;
Pvt. Eugene Paul Rozycki, United States Army;
Professor Hanno Rund, University of Toronto;
Miss Joan P. Salatino, Marquette University;
Mr. Charles B. Shaw, Jr., Lockheed Aircraft Corp.
Mr. Thomas Harold Starks, Convair, Fort Worth, Texas;
Mr. John Kenneth Sterrett, Eglin Air Force Base, Florida;
Dr. George Herbert Swift, Jr., Duke University;
Mr. Robert Earl Thomas, Iola Senior High and Junior College, Iola, Kansas;
Mr. Robert J. Wernick, Army Chemical Center, Maryland;
Mr. Fred Walter Wolock, Iona College;
Professor Nelson Paul Yeardley, Thiel College.

It was reported that the following one hundred ninety-four persons had been elected to membership on nomination of institutional members as indicated:

University of Alabama: Mr. Theodor D. Sterling.

University of British Columbia: Mr. Eugene Butkov and Mr. Robert Charles Thompson.

Brown University: Mr. Abdulrazak Ali Huswan, Mr. Harry Lighthall, Mr. Donald Gene Malm, Professor Leonard C. Maximon, and Mr. J. Rainer Maria Radok.

California Institute of Technology: Mr. John Williams Lamperti, Mr. Lee Meyers Sonneborn, and Mr. Lloyd Richard Welch.

University of California, Berkeley: Mr. Israel Jacob Abrams, Mr. Satya P. Agarwal, Mr. David Wilson Bressler, Mr. Theodore Thomas Frankel, Mr. Richard Carlton MacCamy, Mr. Marvin Rosenblum, Mr. Warren Bernard Stenberg, Mr. Balkrishna Vasudeo Sukhatme and Mr. Adil Mohamed Yaqub.

University of California, Los Angeles: Mr. Robert Kenneth Froyd and Mr. Kenneth Myron Hoffman.

College of The City of New York: Mr. Irwin Stanley Bernstein and Mr. Ronald Jacobowitz.

Columbia University: Mr. Norman Alling, Mr. Yeaton Hopley Clifton, Mr. George Claggett Francis, Mr. Paul Alexander Gillis, Mr. Jacob Eli Goodman, Mr. Wendell Lyons Jones, and Mr. Laurence Edward Sigler.

Cornell University: Mr. Louis deBranges, III, Mr. Lonnie Cross, Mr. Stanislaw Leja, Mr. Richard E. Schwartz, and Mr. Paul Strauss.

Duke University: Mr. Robert Bruce Jackson, Jr.

Harvard University: Mr. Edward F. Assmus, Mr. Leonard E. Baum, Mr. Ira Ewen, Mr. Robert Gold, Mr. Solomon Wolf Golomb, Mr. Stevens Heckscher, Mr. John Arnold Kalman, Mr. Ralph Mack Krause, Mr. Karl Martin Kronstein, Mr. Henry Siggins Leonard, Jr., Dr. Donald Howard Menzel, Dr. Marvin Lee Minsky, Mr. Thomas Wilson Mullikin, Mr. Mason Miller Phelps, and Miss Lisa A. Steiner.

University of Illinois: Mr. Richard Solomon Ballance, Mr. William Clarence Bennewitz, Mr. Kenneth Allyn Brons, Mr. Yuan Shih Chow, Mr. Aubyn Freed, Mr. John Graham, Mr. Willard Donald James, Mr. Morris Wolfe Katz, Mr. Eugene Edmund Kohlbecker, Mr. Raymond Peter Polivka, Mr. Burnett Henry Sams, and Mr. Norman Emil Sexauer.

Institute for Advanced Study: Dr. Alfred V. Froelicher, Dr. William Martin Huebsch, Dr. Masatake Kuranishi, Mr. Christer Lech, Dr. Paul Malliavin, Dr. Georges H. Reeb, Dr. Peter Roquette, and Dr. Kosaku Yosida.

Iowa State College: Mr. Fowler Redford Yett.

Johns Hopkins University: Mr. William Henry Adams, Jr., Mr. Theodore Albert Bickart, Mr. George Backus Brown, Mr. Charles Vernon Coffman, Mr. Daniel Isaac Fivel, Mr. Gabriel M. Gerstenblith, Mr. Thomas Lewis Gibson, Jr., Mr. John Allan Henneberger, Mr. Donald William Kydon, Mr. Frank S. Levin, Mr. David Bowen McCandlish, Mr. Ganesan Sri Ram, Mr. Ray Ronald Rudolph, Mr. Jose Luis Soto-Alarcón, Mr. William Richard Webster, Mr. George Wend, and Professor Robert W. Zwanzig.

University of Kansas: Mr. George Alfred Ladner, and Mr. Eugene Kay McLachlan.

Kenyon College: Mr. Erwin Herman Knapp.

Lehigh University: Mr. Thomas Francis Green.

University of Maryland: Mr. James Henry Bramble, Mr. Hubert Whitman Lilliefors, and Mr. Charles Leonard Waldman.

Massachusetts Institute of Technology: Mr. Oma Hamara, Mr. Noel Justin Hicks, Mr. Harold Frazyer Mattson, Jr., Mr. Nevin William Savage, Mr. Gustave Solomon, and Mr. Walter E. Weissblum.

Michigan State College: Mr. James William Caltrider, Mr. Yousel Alavi, and Miss Elizabeth Karol Banks.

University of Michigan: Mr. Ervin Roy Deal, Dr. Richard C. W. Kao, Mr. James Edwin Keisler, Dr. Kyung Whan Kwun, Mr. Earl Edwin Lazerson, Mr. Gerald Otis Losey, and Mr. Stephen Smale.

University of Minnesota: Mr. Herbert Fantle, Mr. William Ashton Harris, Jr., Mr. Gerald Arthur Heuer, Mr. Roy Alfred Jorgensen, Jr., Mr. Richard Kent Juberg, Mr. Sherman James O'Neill, Mr. Roger Noel Pederson, and Mr. William Daniel Serbyn.

University of Missouri: Mr. John Homer DeHardt.

Northwestern University: Dr. José María González-Fernández, and Dr. Walter Thomas Kyner.

Ohio State University: Mr. Henry Hofman Diehl, Miss Betty Lou Bernhardt, Mr. Carl Christopher Maneri, Mr. Donald Edward Robison, and Mr. Thomas Aloys Willke.

Oklahoma Agricultural and Mechanical College: Mr. John Leroy Folks and Mr. Oliver Paulsanders.

University of Oregon: Mr. Calvin Thomas Long.

University of Pennsylvania: Mr. Samuel Abraham Hoffman and Mr. Wallace Smith Martindale, 3rd.

Princeton University: Mr. Dean Paul Benzecri, Mr. William Browder, Mr. Albert William Currier, Mr. George Howard Dinsmore, Jr., Mr. James Hugo Griesmer, Dr. Kiyoshi Ito, Dr. Ioan Mackenzie James, Mr. Harold Melvin Kaplan, Mr. James A. Lechner, Dr. Kenneth Rogers, Mr. Norman Shapiro, Mr. Richard Gordon Swan, and Mr. David Leon Yarmush.

Purdue University: Mr. William Austin Beck and Mr. Leonard Irvin Holder.

The Rice Institute: Mr. Robert Melvin McLeod and Mr. John Raymond Swaffield.

Rutgers University: Mr. Allen Howard Miller.

College of Saint Thomas: Mr. Hubert Ronald Walczak.

University of Southern California: Mr. Russell Verner Benson, Mr. Albrecht Ferling, Mr. Thomas William Kampe, and Mr. Jack Bard Stutesman.

Stanford University: Mr. James Sterling Ayars, Mr. Gerald Leland Davey, Mr. Adriano Mario Garsia, Mr. John Walker Gray, and Mr. Cyril Joseph Murphy.

Syracuse University: Professor James B. Munz and Mr. Wilbert Neil Prentice.

University of Toronto: Mr. Donald Trevor Bean, Mr. Hermes Andrew Eliopoulos, Mr. William Kahan, Mr. Rudolph Oscar Robinson, and Mr. Ralph Wormleighton.

University of Utah: Mr. Byron Leon McAllister.

University of Virginia: Mr. Walter Francis Davison, Mr. John Jay Greever, III, and Mr. Edward Malcolm Wyatt.

University of Washington: Mr. Robert Trull Ives.

Washington University: Mr. Richard Allen Askey, Mr. Paul Richard Beesack, Dr. Edgar Henry Brown, Jr., and Mr. Ning Sheng Fan.

University of Wisconsin: Mr. Eugene Lang Albright, Mr. John A. Fibiger, Mr. Harry Melvin Friedman, Mr. Thomas I. Gilroy, Mr. Horace Clark Hearne, Jr., Mr. Orville John Marlowe, Mr. Robert Kirkpatrick Meany, and Dr. Roy Clifford Townsend Smith.

Yale University: Mr. Victor Elihu Bach, Mr. Leon Greenberg, Mr. Leo Hellerman, Mr. David Carter McGarvey, Mr. Allen Lewis Morton, Jr., Mr. Gian Carlo Rota, Mr. Dallas Wilton Sasser, and Miss Maria Josefa Wonenburger.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematiker Vereinigung: Professor Friedrich Lösch, Technical University of Stuttgart, Stuttgart, Germany, and Professor Oskar Perron, University of Munich, Munich, Germany; Indian Mathematical Society: Mr. Srinivasa Swaminathan, University of Madras; London Mathematical Society: Professor Abraham Robinson, University of Toronto; Société Mathématique de France: Professor Sunouchi Gen-ichirô, Tôhoku University, Sendai, Japan, and Professor Tadao Tannaka, Tôhoku University, Sendai, Japan; Suomen Matemaattinen Yhdistys: Professor Gunnar Johannes af Hällström, Abo Akademi, Abo, Finland, and Mr. Carl Gustaf Wolff, Ekehäs Samlyuum, Ekenäs, Finland.

The following appointments by the President were reported: as a committee to nominate a representative of the Society on the Policy Committee for Mathematics: J. R. Kline, Chairman, Marston Morse, and Antoni Zygmund; as a joint committee of the Society and the Mathematical Association of America on the Employment Register: J. S. Frame, Chairman, H. M. Bacon, T. R. Hollcroft, Morris Ostrofsky, and J. A. Ward; as Chairman of the Committee on Places of Meetings for 1955: E. R. Lorch; as a new member of this Committee: C. B. Morrey (Committee now consists of E. R. Lorch, Chairman,

W. M. Whyburn, and C. B. Morrey); as a representative of the Society on the Advisory Board of the Applied Mechanics Reviews for a term of three years beginning July 1, 1954: Dr. Eleazer Bromberg; as a committee on arrangements for the 1955 Summer Meeting to be held at the University of Michigan: Wilfred Kaplan, Chairman, P. S. Dwyer, H. M. Gehman, P. S. Jones, W. J. LeVeque, R. M. Thrall, R. L. Wilder, and J. W. T. Youngs; as members of Committees to Select Hour Speakers (terms to expire December 31, 1956): Summer and Annual Meetings: D. H. Lehmer (Committee now consists of E. G. Begle, Chairman, Wilfred Kaplan, and D. H. Lehmer); Eastern Sectional Meetings: Lipman Bers (Committee now consists of R. D. Schafer, Chairman, Nelson Dunford, and Lipman Bers); Western Sectional Meetings: P. V. Reichelderfer (Committee now consists of J. W. T. Youngs, Chairman, M. E. Shanks, and P. V. Reichelderfer); Far Western Sectional Meetings: Ivan Niven (Committee now consists of J. W. Green, Chairman, J. L. Kelley, and Ivan Niven); Southeastern Sectional Meetings: G. B. Huff (Committee now consists of J. H. Roberts, Chairman, W. V. Parker, and G. B. Huff); as Chairman of the Committee on Applied Mathematics for 1955: F. J. Murray; and as members for terms of three years beginning January 1, 1955: A. S. Householder and F. J. Murray (Committee now consists of F. J. Murray, Chairman, R. E. Bellman, Garrett Birkhoff, R. S. Burington, A. S. Householder, and Shizuo Kakutani); as Chairman of the Committee on Visiting Lectureships for 1955: Einar Hille; and as a member for a term or three years beginning January 1, 1955: J. M. Thomas (Committee now consists of Einar Hille, Chairman, P. R. Halmos, and J. M. Thomas); as tellers for the 1954 election: F. M. Stewart and John Wermer.

The following appointments to represent the Society were reported: at the inauguration of Herrick Black Young as President of Western College for Women on October 9, 1954: Professor H. S. Pollard; at the inauguration of Carl Cluster Bracy as President of Mount Union College on October 15, 1954: Professor R. P. Gosselin; at the inauguration of Owen Meredith Wilson as President of the University of Oregon on October 19, 1954: Professor W. E. Milne; at the dedication ceremonies of New York University's Institute of Mathematical Sciences on November 29, 1954: Dr. J. H. Curtiss.

The Secretary reported that Professors H. F. Bohnenblust and Hans Lewy had accepted invitations to deliver addresses at the joint meeting of the Society and Section A of the AAAS at Berkeley, California, December 1954; that Professor Harish-Chandra had accepted an invitation to deliver an address at the February 1955 meeting in New York; that Professor I. S. Cohen had accepted an invitation to deliver an address at the April 1955 meeting in Brooklyn, and that Professor Salomon Bochner had accepted an invitation to deliver the Colloquium Lectures in 1956.

The Council voted to elect Professor André Weil as a representative of the Society on the Board of Editors of the American Journal of Mathematics to complete the term of Professor Samuel Eilenberg, who resigned from this position December 31, 1954.

The Council voted to elect Professor Charles Loewner a representative of the Society on the Board of Editors of the Annals of Mathematics for the period 1955–1957.

The Executive Director reported that the Bulletin of Information was being rewritten. He also reported that so far we have had a sixty percent return on the questionnaires sent out for the National Register of Scientific and Technical Personnel. It is hoped that by a follow-up the response will reach at least eighty percent.

The Council voted, in view of the appointment of Professor Paul Erdös as Visiting Lecturer of the Society in 1955–56, to request the President and Secretary to make inquiries in the appropriate offices in Washington as to the visa of Professor Erdös.

The Bulletin Editorial Committee reported that 607 pages had been used in 1954. The Council voted to recommend to the Board of Trustees that the Bulletin be authorized for 1955 to print 690 pages, which would include 65 pages for the 10 volume index.

The Transactions and Memoirs Editorial Committee reported that the interval between receipt of a manuscript and publication is approximately fifteen months. The Council voted to recommend to the Board of Trustees that three volumes of 550 pages each be published in the Transactions during 1955.

The Proceedings Editorial Committee reported that the interval between receipt of a manuscript and publication was approximately eight months. The Council voted to recommend to the Board of Trustees that 1006 pages be authorized for the 1955 Proceedings. Professors Walter Rudin and R. H. Bruck were reported as new Associate Editors for the Proceedings.

The Council voted to recommend to the membership of the Society an amendment to the by-laws which would increase the number of editors of the Proceedings from three to four.

On recommendation of the Colloquium Editorial Committee the Council voted to invite Professor Norman Steenrod as the Colloquium Speaker for 1957.

The Mathematical Reviews Editorial Committee reported that 1012 pages had been published in 1954. The subscription list as of

November 1954 was 2353.

The Committee on the Employment Register reported that over 100 listings appeared in the Employment Register at this meeting, about half of which listed academic positions.

The Council voted to accept an invitation from the University of Washington to hold the 1956 Summer Meeting at Seattle, August 20–25, and to accept an invitation from the University of Rochester to hold the 1956 Annual Meeting at Rochester, December 27–29. The Council voted to request the Presidents of the Society and the Mathematical Association of America to appoint a committee to investigate the possibility of holding the annual meeting at a time other than the week between Christmas and New Years.

The Council approved the following times and places of sectional meetings in 1955: April 22–23, University of Chicago; October 29, University of Maryland; November 18–19, University of Tennessee.

The Council voted to co-sponsor a Midwestern Conference on Solid and Fluid Mechanics to be held at Purdue University, September 8–10, 1955.

The Council voted to recommend to the membership of the Society an amendment to the by-laws which would make the term of office of the Trustees five years instead of the present two, one Trustee being elected each year.

There were sixteen sessions for contributed papers at Pittsburgh, presided over by Professors R. D. Anderson, Reinhold Baer, J. L. Brenner, Bernard Friedman, O. G. Harrold, J. J. L. Hinrichsen, R. E. Johnson, W. T. Martin, D. D. Miller, E. E. Moise, C. G. Mumford, Everett Pitcher, P. C. Rosenbloom, Lowell Schoenfeld, Seymour Sherman, R. M. Thrall. The sessions for contributed papers at Berkeley were presided over by Professors G. C. Evans, D. H. Lehmer, Charles Loewner, and J. L. Kelley.

The abstracts of the papers are appended. Those having the letter "*t*" following the abstract number were read by title. Those having the letter "B" following the abstract number were read at the Berkeley sessions. Where a paper, presented in person, has more than one author, the symbol (p) follows the name of the author who presented it. Dr. Kyner was introduced by Professor S. P. Diliberto, Dr. Stoll by Professor W. H. Gottschalk, Dr. McAuley by Professor Dick Wick Hall, Mr. Arms by Dr. R. L. Jeffery, Dr. Klingenberg by Professor J. W. T. Youngs, Professor Allen by Professor L. W. Johnson, Dr. Saworotnow by Professor C. T. Taam, Professor Kulik by Professor W. L. Williams, Dr. Rogers by Professor Mary P. Dolciani, and Dr. Swift by Professor Edwin Hewitt.

ALGEBRA AND THEORY OF NUMBERS

166. J. E. Adney, Jr.: *On the power of a prime dividing the order of a group of automorphisms.*

Let G be a group of finite order, $A(G)$ its group of automorphisms, and p a prime. I. N. Herstein and the author (Amer. Math. Monthly vol. 59 (1952)) proved if p^2 divides the order of a group G, then at least p divides the order of the group of automorphisms $A(G)$. W. Scott (Proc. Amer. Math. Soc. vol. 5 (1954)) extended this result to the following: If p^3 divides the order of G, then p^2 divides the order of $A(G)$. The main result of this paper is the following theorem. If G has order $p^n m$, $(m, p) = 1$, and the Sylow subgroup Sp associated with the prime p is abelian, then at least p^{n-1} divides the order of $A(G)$. The proof requires the use of the transfer of G into Sp. The author knows of no counter-example to the general conjecture: if p^n is the highest power of p that divides the order of G, then at least p^{n-1} divides the order of $A(G)$. (Received November 12, 1954.)

167. Maurice Auslander: *Commutator subgroup of free groups.*

Let F be a non-abelian free group, R a proper normal subgroup of F, and $[R, R]$ the commutator subgroup of R. The following theorems are proved. (1) The center of $F/[R, R]$ is a proper subgroup of $R/[R, R]$. (2) If $G = F/R$ is a finite group such that $G \neq [G, G]$, then the center of $F/[R, R]$ is not $\{1\}$. An immediate consequence of (1) is that if F is a non-abelian free group and R is a normal subgroup such that $[R, R] = [F, F]$, then $F = R$. The proofs of (1) and (2) are entirely cohomological in nature, the principal tool being the cup-product reduction theorem. (Received November 12, 1954.)

168*t*. Maurice Auslander: *On the dimension of modules and algebras. III. Global dimension.*

This paper is concerned with the homological dimension of a ring Λ with unit (see H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, for notations and definitions). First, two results of a general nature are proved. (1) It is shown that l.gl.dim $=$ sup l.dim A, where A ranges over all left Λ-modules generated by a single element. (2) If Λ is both left and right Noetherian, l.gl.dim $\Lambda =$ r.gl.dim Λ. The rest of the paper is concerned with more specialized results. Assume Λ satisfies the descending chain condition for left ideals. (3) If N denotes the radical of Λ, then l.gl.dim $\Lambda =$ l.dim Λ/N. (4) If every simple left Λ-module is isomorphic to a left ideal of Λ, then l.gl.dim $\Lambda = 0$, ∞ (completely primary rings and quasi-Frobenius algebras are examples of rings satisfying the hypothesis of (4)). (5) If Λ is commutative, then gl.dim $\Lambda = 0$, ∞. Assume Λ is a commutative ring. (6) If every ideal of Λ is generated by a single element, then gl.dim $\Lambda = 0, 1$ if and only if Λ has no nilpotent elements; otherwise, gl.dim $\Lambda = \infty$. (7) If Λ is Noetherian, then gl.dim $\Lambda \leqq 1$ if and only if Λ is the direct sum of a finite number of Dedekind rings. (Received November 12, 1954.)

169*t*. H. W. Becker: *Alliston's problem: Hero triangles with* 1, 2 *or* 3 (?) *sides squares.*

These are resp. of type I, II, or III. Corresponding to various general solutions for $H\blacktriangle$, there are various gen. sols. for type I. It is conjectured that: general solution for type II is a chimera; type III is impossible. Any Pyth. tet. $v^2+x^2=u^2+z^2=t^2$,

$v^2-z^2=u^2-x^2=y^2\rightarrow$a type II $H\triangle u^2$, v^2, $x^2+z^2=$base; alt.$=2txyz/(x^2+z^2)$. A corollary of the impossibility of type III would be the imp. of Martin vectors (Pyth. tets. with $x^2+z^2=\square$). Working from a table, or the 10 known basic parametric solutions, of $4abcd(a^2-b^2)(c^2-d^2)=\square=(\text{alt.})^2$, Bull. Amer. Math. Soc. vol. 60 (1954) p. 250, or their transforms, one derives all type II of Alliston's first method, Math. Snack Bar (1936) p. 119. Or else, try $(a^2+b^2)^2$, $(c^2+d^2)^2e^2$, $a^4-6a^2b^2+b^4\pm 4cd(c^2-d^2)e^2$; $4ab(a^2-b^2)=(c^4-6c^2d^2+d^4)e^2$, as $a, b, c, d, e=16, 9, 2, 1, 120$. Alliston's next method, top p. 120, generalizes to $(4r^4+s^4)^2$, $(4r^4+s^4)^2+\{2rs(2r^2-s^2)\}^2$, $\{2rs(2r^2+s^2)\}^2$; $4rs(4r^4+s^4)(2r^2-s^2)$. His last method may be formulated $(r^4-2r^2s^2+9s^4)^2$, $(r^4-s^4)\cdot(r^4-81s^4)$, $\{2rs(r^2+3s^2)\}^2$; $8rs(r^2-s^2)(r^2-3s^2)(r^2-9s^2)$. Working from a table or the general solution of $T^2=Y^2+X^2+Z^2$, a type I is $(X^2+Z^2)^2$, $(T^2+Y^2)(X^2-Z^2)$, $2(TX\pm YZ)(YX\mp TZ)$; $(T^2-Y^2)(X^2-Z^2)$. The base$=\square$ for the upper and lower signs resp. if $T, Y, X, Z=3, 2, 2, 1$ and $7, 6, 3, 2$. $(T^2+Y^2)(X^2-Z^2)=\square$ when $T, X, Y, Z=9r^4\pm 8r^2s^2+4s^4$, $4rs(3r^2\pm 2s^2)$, so the $H\triangle$ sides ⧈ are $(81r^8-56r^4s^4+16s^8)^2$, II$(9r^4\pm 16r^2s^2+4s^4)^2$, as 41^2, 87^2, 5920 or 9200; 369. (Received November 12, 1954.)

170*t*. H. W. Becker: *3-D Diophantine vectors with elements squares.* Preliminary report.

Solutions of $t^2=x^2+y^2+z^2$ (t, x odd) are type I, II, III or IV according as 1, 2, 3 or 4 of the elements t, x, y, z are ⧈. Euler conjectured that no type IV exist; none exist with $t<10^4$, Morgan Ward, Duke Math. J. vol. 15 (1948) p. 827. Parametric solutions of type III. 1, 2, 3 (with respectively $t, x, y\neq$⧈) were given by Diophantus, Fauquembergue, and Escott, Dickson's *History* II, p. 657–658; but these cannot contribute to a type IV sol. Type II. 1, 2, 3, 4 sols. have resp. t, x; t, y; x, y; y, z both ⧈. Para. sols. are resp. $t, x, y, z=\{m^2+n^2\pm(p^2+q^2)\}^2$, $2(mp+nq)(m^2+n^2+p^2+q^2)+2(mq-np)(m^2+n^2-p^2-q^2)$, $2(mq-np)(m^2+n^2+p^2+q^2)-2(mp+nq)(m^2+n^2-p^2-q^2)$; the same, except $x=(m^2+n^2)^2-(p^2+q^2)^2+4(mq-np)(mp+nq)$, $y=\{2(mp+nq)\}^2$; $9r^4+48r^3s+16r^2s^2-128rs^3+264s^4$, $(3r^2+8rs-8s^2)^2$, $\{2s(3r+4s)\}^2$, $8s^2(6r^2+16rs-31s^2)$, adapted from Gerardin, L'Int. des Math. vol. 26 (1919) p. 17; $m^4+n^4\pm 4p^4$, $(2pm)^2$, $(2pn)^2$. The above para. sols. for type II.1, 2 might be general, hence contributory to the settlement of type IV. The above para. sols. for type II. 3, 4 are not general, Ð resp. 65, 25, 36, 48 and 69, 67, 16, 4. Sols. $q^2, m^2, n^2, 2p^2\rightarrow q^4$, $m^4+n^4-4p^4$, $(2pm)^2$, $(2pn)^2$, a type III.2→III.3 transform. Thus 9, 1, 4, 8 →81, 47, 16, 64; 49, 9, 36, 32→2401, 353, 2304, 576. In type I.1, 2, 3 sols., t, x, y resp.$=\square$. Special para. sols. are indicated by Bills, Dickson ibid. p. 666, and R. D. Carmichael, *Diophantine analysis* (1915) p. 46, which generalize readily to t, x, or $y=k$th power. (Received November 12, 1954.)

171. S. G. Bourne: *On semirings.* Preliminary report.

A semiring is a system consisting of a set S and two binary operations in S called addition and multiplication such that (a) S together with addition is a semigroup, (m) S together with multiplication is a semigroup, (d) The left and right hand distributive laws $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ hold. Semirings arise naturally when we take the totality of endomorphisms E of an arbitrary commutative semigroup $\mathfrak{S}$. An additive idempotent of a semiring S is an element a such that $2a=a$ and a multiplicative idempotent is an element m such that $m^2=m$. An additive identity, called zero, is an element 0 such that $0+s=s+0=s$, for all s in S. If S possesses a zero, then $a\cdot 0$ and $0\cdot a$, if defined, are additive idempotents of S, which are not necessarily equal. If m_1 and m_2 are distinct multiplicative idempotents of a semiring

S with zero, such that $m_1m_2+m_2m_1=m_1m_2+m_2=m_2m_1+m_2=0$, then $m_3=m_1+m_2$ is a multiplicative idempotent orthogonal to m_2 and $m_3m_1+m_2=m_1m_3+m_2=m_1$. (Received November 12, 1954.)

172*t*. Barron Brainerd: *A commutative semigroup theorem.*

Let A be a commutative semigroup containing an identity e. If K is a sub-semigroup of A, then the following condition in A defines a congruence relation ϕ_K: $a\equiv b \bmod \phi_K$ if and only if there exist elements l and k in K such that $al=bk$. ϕ_K is called a reversible congruence relation by some authors. Let $O(\phi_K)$ be the kernel of the congruence relation ϕ_K. If $O(\phi_K)=K$ for all sub-semigroups K of A, then A is a group. This result follows from the fact that the set $A'=\{a \mid \text{for all } x,\ ax\neq e\}$ is a sub-semigroup of A. (Received October 22, 1954.)

173*t*. Leonard Carlitz: *Some class number formulas for quadratic forms over* $GF[q, x]$.

Byers (Duke Math. J. vol. 21 (1954) pp. 445–461) has proved several class number formulas by setting up a correspondence between classes of definite quadratic forms and of classes of bilinear forms. In the present paper these as well as more general results are proved by using a "singular series" similar to that for sums of squares in $GF[q, x]$ (Duke Math. J. vol. 14 (1947) pp. 1105–1120). If $h(\Delta)$ represents the number of classes of discriminant Δ, then for example the sum $\sum h(\Delta-\alpha_1U_1^2-\cdots-\alpha_sU_s^2)$ is evaluated; here $\alpha_1, \cdots, \alpha_s\in GF(q)$ and the summation is over all $U_i\in GF[q, x]$ of degree $<m$ and $\deg \Delta=2m+1$. (Received November 8, 1954.)

174*t*. Leonard Carlitz: *Some class number relations.*

Using some well-known theta formulas due to Kronecker and Hurwitz, a number of class number relations are obtained that seem to be new. Typical formulas are $4\sum_{i+j=n}F(4i+1)F(4j+1)=(-1)^nE_2(2n+1)$ and $\sum_{i+j=n}F(8i+2)F(8j-2)=2E_2^1(n)$; here $E_2(n)$ denotes the sum of the squares of the divisors of n which are of the form $4k+1$ diminished by the sum of the squares of the divisors of n which are of the form $4k+3$; $E_2^1(n)$ denotes the sum of the squares of the divisors of n whose conjugates are of the form $4k+1$ diminished by the sum of the squares of the divisors of n whose conjugates are of the form $4k+3$. (Received November 8, 1954.)

175*t*. Leonard Carlitz and J. H. Hodges: *Distribution of bordered symmetric, skew and hermitian matrices in a finite field.*

Let $q=p^f$, $p>2$, and let A, M, U denote matrices with elements in $GF(q)$. If A is symmetric and non-singular of order m and U is $m\times t$ put $M=\left[\begin{smallmatrix}A & U\\ U' & 0\end{smallmatrix}\right]$. Then the number of matrices M of rank $m+r$ and given invariant is determined. Similar results are obtained in the skew and hermitian cases. (Received November 8, 1954.)

176*t*. Eckford Cohen: *The finite Goldbach problem in algebraic number fields.* II.

Let F be a finite extension of degree n of the rational field and let A be a proper ideal of F. This paper establishes the necessary and sufficient conditions for the representability of all elements of the residue class ring $R(A)$ as sums of at most s primes in $R(A)$. In conjunction with the theorem contained in Bull. Amer. Math. Soc. Abstract 59-3-271, this theorem completes the generalization of results proved previously in the rational case (Proc. Amer. Math. Soc. vol. 5, pp. 478–483). The extended results show the rational field to be quite special for this particular problem. A term referred to as the *finite Goldbach property* is introduced to describe fields whose residue class rings possess a property analogous to the statement of the Goldbach conjecture. This property is shown to hold if and only if 2 is the nth power of a prime ideal of F, and in particular if F is the rational field. (Received November 8, 1954.)

177. W. E. Deskins: *On finite Abelian groups having isomorphic group algebras.*

Perlis and Walker, in their paper, *Abelian group algebras of finite order*, Trans. Amer. Math. Soc. vol. 68 (1950) pp. 420–426, gave conditions that two Abelian groups have isomorphic group algebras over fields having characteristic relatively prime with the order of the groups. Here we consider the problem where the field characteristic divides the order of the groups. Let G and H be abelian groups of order gp^a, $(g, p)=1$. Then $G=G_0\times G_1$, $H=H_0\times H_1$, where G_0 and H_0 are of order g and G_1 and H_1 are of order p^a. Let F be a field of characteristic p. *Theorem. The group algebras of G and H over F are isomorphic if and only if both of the following conditions are satisfied*: (a) G_0 *and* H_0 *have isomorphic group algebras over* F. (b) G_1 *and* H_1 *are isomorphic groups.* This reduces the problem to the case solved in the Perlis-Walker paper. (Received November 5, 1954.)

178. Mary P. Dolciani: *An analogue of the Hardy-Littlewood method.*

Let $K[t]$ be the domain of all polynomials in an indeterminate t with coefficients in a fixed field K of p elements, p odd. p is assumed to be a prime, for a more general assumption leads only to a parallel discussion. Let $S(X)=A_1X_1^2+\cdots+A_mX_m^2$ be a quadratic form in 5 or more variables with each A_i in $K[t]$. For any fixed element C of $K[t]$, $I(2n)$ is the number of solutions of $S(X)=C$ in polynomials X_i, such that for each i the degree in t of $A_iX_i^2$ is less than the even integer $2n$. $S(X)$ is said to represent C congruentially relative to an irreducible polynomial P of $K[t]$, if the congruence $S(X)\equiv C \pmod{P^j}$ is solvable for each positive integer j, where, if $C=0$, at least one X_i of the solution must be prime to P. The following result is proved: $I(2n)=p^{n(m-2)}G+O(p^{mn/2})$ where $G=G(A_1,\cdots,A_m,C)>0$, provided $m\geq 5$ and S represents C congruentially relative to all irreducible polynomials P. Following the basic idea of the Hardy-Littlewood method, $I(2n)$ is expressed as a multiple sum which is divided into two parts, one of which leads to the main term involving a singular series and the other to an error term. The proof requires the consideration of certain exponential sums and the singular series. (Received November 12, 1954.)

179t. L. K. Durst: *On some numbers of Pólya.*

For $s=3$ and any integer $n>10g_3\,3$, the numbers $N_n^{(s)}$ (cf. Pólya, J. Symbolic Logic vol. 5 (1940) pp. 98–103) have been evaluated. The result is: $N_n^{(3)}=\sum_k\{[k/3]+\sum_r[(k-3r)/2]\}$, $2\leq k\leq n$, $0\leq r\leq[k/3]$. (Received October 25, 1954.)

180. M. P. Epstein: *On Liouvillian extensions.* Preliminary report.

Let F be an ordinary differential field of characteristic 0 with field of constants C. Kolchin considered a Liouvillian extension G of F using the assumption that C is algebraically closed. He obtained the equivalence of 10 conditions on the group of automorphisms of G over F with 10 conditions on the generators of G over F. Here the situation is considered after removal of the assumption that C is algebraically closed. It is seen that Kolchin's equivalences fail (in general) and that one set of conditions on the group is necessary while a different (but similar) set is sufficient. These serve to characterize G "up to" a possible algebraic extension. (Received December 8, 1954.)

181. Trevor Evans: *The isomorphism problem for multiplicative systems.*

A general method is exhibited for determining whether or not two finitely generated and related multiplicative systems are isomorphic. Here, by multiplicative system, is meant loop, quasigroup, groupoid, inverse property loop, etc. This decision procedure is a consequence of the following theorem. Two finitely generated and related multiplicative systems are isomorphic if, and only if, the minimal closed sets of generators and relations obtained from the sets of generators and relations defining the two multiplicative systems are isomorphic. The concept of a minimal closed set of generators and relations obtained from a finite generator and relation presentation of a multiplicative system is a simple extension of the concept of a closed set of generators and relations introduced in *Multiplicative systems*, I, Proc. Cambridge Philos. Soc. vol. 47 (1951) pp. 637–649. (Received November 12, 1954.)

182. Leonard Gillman and Melvin Henriksen (p): *Concerning adequate rings and elementary divisor rings.* II.

A *commutative* ring S with unit is called an *Hermite* ring if for every matrix M over S there exists a nonsingular matrix K such that KM is a triangular matrix, an *elementary divisor* ring if there exist nonsingular K, L such that KML is diagonal. S is *adequate* if all finitely generated ideals are principal, and if for every $a, b\in S$, with $a\neq 0$, there exist $r, s\in S$ such that $a=rs$, $(b, r)=S$, and, for every non-unit divisor t of s, $(b, t)\neq S$. (See Bull. Amer. Math. Soc. Abstract 60-4-463, and Kaplansky, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 464–491.) *Theorem* 1. S is Hermite *iff* for every $a, b\in S$, there exist a_1, b_1, d such that $a=a_1d$, $b=b_1d$, and $(a_1, b_1)=S$. *Theorem* 2. An Hermite ring S is an elementary divisor ring *iff* for all $a, b, c\in S$, with $(a, b, c)=S$, there exist $p, q\in S$ such that $(pa, pb+qc)=S$. *Theorem* 3. Every adequate Hermite ring is an elementary divisor ring. Examples are given of rings S of continuous functions in which (i) S is not adequate, but is an elementary divisor ring, (ii) every finitely generated ideal of S is principal, but S is not Hermite, (iii) S is Hermite, but not an elementary divisor ring. This work was supported in part by the National Science Foundation. (Received November 12, 1954.)

183. Leonard Gillman (p) and Melvin Henriksen: *On rings of continuous functions in which every finitely generated ideal is principal.* II.

Let X be a completely regular space; $\beta X=$ Stone-Čech compactification of X; $C=$ ring of all continuous real functions on X; $C^*=$ subring of bounded functions; $Z(f)=$ set of zeros of $f\in C$; $\overline{P}(f)=$ closure (in X) of $\{x\in X: f(x)>0\}$; $Z(f)^\beta=$ closure of $Z(f)$ in βX; for $p\in\beta X$, $M^p=\{f\in C: p\in Z(f)^\beta\}$; $N^p=\{f\in C$: there exists a neighborhood Σ of p such that $f(\Sigma\cap X)=0\}$. It is known that $M\subset C$ is a maximal ideal *iff* there is a (unique) $p\in\beta X$ such that $M=M^p$ (see Proc. Amer. Math. Soc. vol. 60 (1954) p. 448, Theorem 1). The following improve Bull. Amer. Math. Soc. Abstract 60-4-462. *Theorem* 1. For every maximal ideal M^p of C, the intersection of all the prime ideals contained in M^p is N^p. *Theorem* 2. The following are equivalent. (a) Every finitely generated ideal of C is principal. (b) For all $f\in C$, $(f, |f|)$ is principal. (c) For all

$f, g \in C$, $(f, g) = (|f| + |g|)$. (d) For all $f \in C$, the sets $\bar{P}(f)$, $\bar{P}(-f)$ are completely separated. (e) For all $f \in C$, $f = k|f|$ for some $k \in C$. (f) For every $p \in \beta X$, the ideal N^p is prime (see Theorem 1). (g) Every finitely generated ideal of C^* is principal. (h) For every $f \in C$, every bounded non-negative continuous function defined on $X - Z(f)$ has a continuous extension over X. This work was sponsored in part by the National Science Foundation. (Received November 12, 1954.)

184. Emil Grosswald: *On a theorem of Erdös and Szekeres.*

For any integer $k>0$, denote by M_k the set of all integers n such that, if $p|n$, then also $p^k|n$; let $M_k(x)$ stand for the number of integers $n \leq x$ in M_k. Erdös and Szekeres (Acta Univ. Szeged. vol. 7 (1934) pp. 95–102) proved that $M_k(x) = A_k x^{1/k} + O(x^{1/(k+1)})$, where A_k depends on k alone. This result can be improved to $M_k(x) = A_k x^{1/k} + B_k x^{1/(k+1)} + O(x^{1/(k+2)})$, where $A_k = \prod_p \{1 + \psi_1(p)\}$, $B_k = 6\pi^{-2}\zeta(k/(k+1)) \cdot \prod_p \{1 + \psi_2(p)\}$. Here $\zeta(s)$ stands for the Riemann zeta-function, $\psi_1(p) = p^{-(k+1)/k} \cdot (1 - p^{-(k-1)/k})(1 - p^{-1/k})^{-1}$, and $\psi_2(p)$ has a similar expression. Formally, this result follows immediately, applying Cauchy's theorem on residues to the function $f_k(s)x^s/s$, where $f_k(s) = \prod_p (\sum_{r=0}^{\infty} p^{-rks})(1 + \sum_{m=1}^{k-1} p^{-(k+m)s}) = \sum_{n=1}^{\infty} a_n n^{-s}$ is the generating function for the integers in M_k. It remains to justify the procedure, by showing that, for $c = 1/(k+2)$, $\int_{c-iT}^{c+iT} f_k(s) x^s s^{-1} ds$ stays bounded when $T \to \infty$. This is achieved by considering the function $F_k(s) = \prod_p \{(\sum_{r=0}^{\infty} p^{-rks}) \prod_{m=1}^{k-1} (1 + p^{-(k+m)s})\} = \sum_{n=1}^{\infty} b_n n^{-s}$. The sum $N_k(x) = \sum_{n \leq x} b_n = A_k' x^{1/k} + B_k' x^{1/(k+1)} + C_k' x^{1/(k+2)} + o(x^{1/(k+2)})$ can be obtained by elementary methods; a comparison with the results of contour integration applied to $F_k(s)x^s/s$ then shows that $\int_{c-iT}^{c+iT} F_k(s) x^s s^{-1} ds$ stays bounded for $c \geq 1/(k+2)$. The conclusion now follows, by proving that $f_k(s) = \alpha_k F_k(s) + \beta_k F_{k+1}(s) + \gamma_k F_{k+2}(s) + G_k(s)$, with α_k, β_k, γ_k constants and $G_k(s)$ regular for $s > 1/(k+3)$. (Received October 15, 1954.)

185. Franklin Haimo: *Some identities in semi-direct products.*

Certain identities related to normal nilpotent subgroups of semi-direct products with abelian normal factor are constructed. Work of W. H. Mills is extended. (Received November 15, 1954.)

186*t*. Isidor Heller: *A characterization of determinants.*

If V is a finite-nonzero-dimensional vectorspace over a field L and W is the set of linear transformations on V, the determinant $\Delta(A)$ associates an element of L to each A of W. The present note formulates a characterization of Δ in terms of linear transformations, avoiding the customary detour over matrices or inner product. Notation. I and 0 denote the identity and the null transformation respectively. E denotes an idempotent element of W which is not 0 and which is not the sum of two idempotent elements distinct from E and 0 (in other words, E is a projection of rank 1). E and $I - E = F$ are called an elementary pair. For the function Δ from W to L the following relations are postulated to hold for any A, B and an elementary pair E, F of W, and for any λ of L: I. $\Delta(AB) = \Delta(A) \cdot \Delta(B)$, II. $\Delta(A+B) = \Delta(FA+B) + \Delta(A+FB)$, III. $\Delta(\lambda A) = \lambda\Delta[(\lambda F + E)A]$, IV. $\Delta(A) \neq 0$. It is then shown that the above relations uniquely determine Δ. (Received October 25, 1954.)

187B. Edwin Hewitt (p) and H. S. Zuckerman: *Semicharacters of semigroups.*

Let G be a semigroup (i.e., a set with an associative binary operation). A semicharacter of G is a bounded complex function χ on G such that χ is not identically 0 and $\chi(x)\chi(y)=\chi(xy)$ for all $x,y\in G$. Let G be finite and commutative. Let $G'=\{x \mid x\in G,\ x^m=x$ for some pos. int. $m\}$. Let H be the set of all idempotents in G, and, for $a\in H$, let $T_a=\{x \mid x^s=a$ for some pos. int. $s\}$. Then $T_a\cap G'$ is a group. A function χ on G is a semicharacter if and only if there exists an $a_0\in H$ and a character ψ of $T_{a_0}\cap G'$ such that $\chi(x)=0$ for $x\in T_a$ and $a_0a\neq a_0$ and $\chi(x)=\psi(a_0x)$ for $x\in T_a$ and $a_0a=a_0$. Let $G^{\frown}$ denote the set of all semicharacters of G with ordinary multiplication as a binary operation (not necessarily defined for all pairs). $G^{\frown}$ is a semigroup if and only if H has a unit. A finite commutative semigroup Q is isomorphic to some $G^{\frown}$ if and only if $Q'=Q$ and Q has a unit. G is isomorphic to $G^{\frown\frown}$ if and only if $G'=G$ and G has a unit. The effects on $G^{\frown}$ of adding a unit and zero to G are determined, as well as the effect of admitting 0 as a semicharacter. Thus, let $G^{-}=G^{\frown}\cup\{0\}$. Then $G^{-}\cong(G_e')_z$, where the subscripts e and z denote the addition of a unit and a zero, respectively. This work is connected with results obtained independently by Š. Schwarz (Proc. Int. Cong. Math. 1954). (Received October 20, 1954.)

188. J. H. Hodges: *Representations by bilinear forms in a finite field.*

Let $GF(q)$, where $q=p^f$ and p is a rational prime, denote the Galois field of order p^f. This paper is concerned with the problem of determining the number $N_t^s(A,B)$ of pairs of matrices U, V, such that $UAV=B$, where U is $s\times m$, V is $n\times t$, A is $m\times n$ of rank r, B is $s\times t$ of rank ρ, and all matrices have elements in $GF(q)$. A reduction formula is first obtained, giving $N_t^s(A, B)$ in terms of $N_{t-\rho}^{s-\rho}(A_\rho, 0)$, where A_ρ is $(m-\rho)\times(n-\rho)$ of rank $r-\rho$. $N_{t-\rho}^{s-\rho}(A, 0)$ is then computed directly, giving $N_t^s(A, B)$ explicitly. A connection is indicated between $N_t^t(A, B)$, with A square and nonsingular, and bilinear forms in $GF[q, x]$. Finally, a certain exponential sum $H(B, z)$, which for $B=0$ reduces to the number of $s\times t$ matrices of rank z, is evaluated and applied to the solution of several matric equations. (Received October 28, 1954.)

189. Arno Jaeger: *On representations of differential polynomials with change of field characteristic.*

Let I be the ring of integers, R the field of rationals, and J be an integral domain over I, integrally closed in its quotient field Q and satisfying $J\cap R=I$. Assume that Q has a finite degree of transcendency over a subfield C, let Δ be a nontrivial derivation in Q over C satisfying $\Delta J\subseteq J$, and denote by $\Omega(\Delta, J)$ the ring of differential polynomials in Δ over J. If P is the prime ideal of J generated by the rational prime number p the operator Δ induces a derivation δ in J/P as well as a two-sided ideal $\mathfrak{X}$ of relations in the ring $\Omega(\delta, J/P)$ of differential operators in δ over J/P. $\Omega(\delta, J/P)/\mathfrak{X}$ is isomorphic with a ring M_p of matrices over the subring $J^p(C\cap J)/P$ of J/P having prime number characteristic, and consequently the canonical mappings $\Omega(\Delta, J)\to\Omega(\delta, J/P)/\mathfrak{X}$ lead to representations of $\Omega(\Delta, J)$ by matrices. Now local investigations in the rings M_p are sufficient to determine many properties of the elements of $\Omega(\Delta, Q)$, such as irreducibility and solvability. (Received November 15, 1954.)

190. R. E. Johnson: *Structure theory of faithful rings.* I. *Closure operations on lattices.*

A faithful ring is a ring that has no nonzero right or left annihilator. This paper gives the lattice-theoretic background of subsequent papers on the structure theory of faithful rings. It has to do mainly with properties of the lattice $C(\mathcal{L})$ of all closure operations on a complete lattice $\mathcal{L}$ satisfying the chain condition $(\bigcup_i A_i)\cap A$

$=\bigcup_i(A_i\cap A)$ for every $A\in\mathcal{L}$ and every chain $\{A_i\}\subseteq\mathcal{L}$. If $I(\mathcal{L})$ is the lattice of all subsets of $\mathcal{L}$ that contain I and are $\cap$-closed, and if, for $a\in C(\mathcal{L})$, $\mathcal{L}^a$ is the set of all $A\in\mathcal{L}$ such that $A^a=A$, then $C(\mathcal{L})$ and $I(\mathcal{L})$ are dual isomorphic under the correspondence $a\to\mathcal{L}^a$. The set $C_m(\mathcal{L})$ of all elements of $C(\mathcal{L})$ that are also $\cap$-endomorphisms of $\mathcal{L}$ is a complete sublattice of $C(\mathcal{L})$. If each $A\in\mathcal{L}^a$ contains an atom of $\mathcal{L}^a$, a is called homogeneous. The homogeneous elements of $C_m(\mathcal{L})$ form a dual ideal of $C_m(\mathcal{L})$. If $\mathcal{L}$ and $\mathcal{M}$ are two lattices and x is a $\cap$-homomorphism of $\mathcal{M}$ into $\mathcal{L}$, necessary and sufficient conditions are given for $ax=xb$, where $a\in C(\mathcal{L})$ and $b\in C(\mathcal{M})$. If for each $A\in\mathcal{L}^c$, $A\neq I$, there exists a nonzero $B\in\mathcal{L}$ such that $A\cap B=0$, then c is called reducible. If $\mathcal{L}$ is modular, and if $\mathcal{L}$ has a reducible $c\in C_m(\mathcal{L})$, then c is unique. Furthermore, if c is homogeneous, $\mathcal{L}^c$ is an atomic lattice. (Received November 10, 1954.)

191. Irving Kaplansky: *Any orthocomplemented complete modular lattice is a continuous geometry.*

The paper is devoted to proving the theorem stated in the title. The method is first to introduce coordinates from a regular ring, with the usual exception of things like non-Desarguian projective planes. The ring will possess an involution that makes it resemble closely a ring of operators. The arguments then needed are slight modifications of known ones—except for the proof of "finiteness." To prove finiteness it is argued that otherwise there is an infinite set of matrix units giving rise to infinite matrices. It turns out that the rows and columns can be arbitrarily prescribed. This bizarre state of affairs leads ultimately to a contradiction. (Received November 4, 1954.)

192. Erwin Kleinfeld: *Primitive alternative rings.*

An alternative ring R is defined to be primitive in case it contains a regular maximal right ideal which contains no proper two-sided ideal of R. It is then shown that a primitive alternative ring is either associative, in which case the well known Jacobson theory applies (Amer. J. Math. vol. 67 (1945) pp. 300–320), or else a Cayley-Dickson algebra of dimension 8 over its center. The proof depends on the author's previous result (Ann. of Math. vol. 58 (1953) pp. 544–547) that a simple alternative ring S is either associative or a Cayley-Dickson algebra, S being defined as simple provided it has no proper two-sided ideal and is not nil. Incidental to the proof is the following easily established result: A non-associative, alternative ring which has no proper two-sided ideal also has no proper one-sided ideal. This verifies the known fact that a Cayley-Dickson algebra has no proper one-sided ideal. It also adds weight to the conjecture that there are no alternative, non-associative nil rings without proper ideals. (Received August 30, 1954.)

193t. Erwin Kleinfeld: *Semi-simple alternative rings.*

The radical N of an alternative ring R is defined to be the intersection of the regular maximal right ideals of R. Then using the above classification of primitive, alternative rings (Bull. Amer. Math. Soc. Abstract 61-2-192), as well as Jacobson's density theorem it is easily established that N is also the intersection of the right-primitive ideals of R and hence an ideal. A ring with zero radical is called semi-simple. The quotient ring R/N is then semi-simple and consequently a subdirect sum of Cayley-Dickson algebras and associative, primitive rings. Another characterization of N, in terms of quasi-regular elements, has been given by Bailey Brown (Proc.

Amer. Math. Soc. vol. 2 (1951) pp. 114–117). Irving Kaplansky obtained the above structure theory under the additional assumption of π-regularity and conjectured the present result (Portugaliae Math. vol. 10 (1951) pp. 37–50). An immediate application shows that squares of associators and commutators of a semi-simple alternative ring are in its nucleus. For arbitrary alternative rings one can only assert that fourth powers of commutators are in the nucleus at the present time (Ann. of Math. vol. 58 (1953) pp. 544–547). (Received August 30, 1954.)

194t. L. A. Kokoris: *On u-stable commutative power-associative algebras.*

A stable simple commutative power-associative algebra of degree two over a center whose characteristic is prime to 30 is known to be a Jordan algebra (cf. A. A. Albert, *On commutative power-associative algebras of degree two*, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 323–343). It is also known (cf. the above reference and *Power-associative commutative algebras of degree two*, Proc. Nat. Acad. Sci. U. S. A. vol. 38 (1952) pp. 534–537) that simple u-stable, that is, stable with respect to the idempotent u, algebras of degree two and characteristic $p>5$ need not be Jordan algebras. This paper extends these results by showing that a u-stable central simple commutative power-associative algebra of degree two and characteristic zero is a Jordan algebra. (Received November 5, 1954.)

195. D. J. Lewis: *Cubic congruences.*

Let $\mathfrak{O}$ be the ring of integers of an algebraic number field, and $\mathfrak{m}$ any ideal in $\mathfrak{O}$. Let $F(X)=\sum_{i=1}^{n} a_i x_i^3$, where the a_i are in $\mathfrak{O}$. It is shown that the congruence $F(X)\equiv 0 \pmod{\mathfrak{m}}$ has a solution in $\mathfrak{O}$ which is nontrivial modulo every prime factor of $\mathfrak{m}$, provided $n\geqq 7$. Furthermore 7 is the smallest such integer. The result is obtained first for $\mathfrak{m}=\mathfrak{p}^r$, where $\mathfrak{p}$ is a prime ideal. Here, except for when $\mathfrak{p}$ divides 3, the usual methods apply. The final conclusion is then obtained by means of the approximation theorem. This result is comparable with that of the author's in *Cubic homogeneous polynomials over $\mathfrak{p}$-adic fields*, Ann. of Math. vol. 56 (1952) pp. 473–478. (Received November 12, 1954.)

196t. B. N. Moyls and M. D. Marcus: *Field convexity of a square matrix.*

Let C be a complex n-square matrix. Set $f(x)=(Cx, x)$ for $\|x\|^2=\sum_{i=1}^{n} x_i\bar{x}_i=1$; $F(C)$ = range of f; $P(C)$ = smallest convex polygon in the plane containing all the eigenvalues of C; N^n = set of all complex n-square normal matrices; Q^n = set of all complex n-square matrices with $P(C)=F(C)$. It is known that $F(C)$ is convex and $P(C)\subset F(C)$. Results: (i) $n\leqq 4$ implies $Q^n=N^n$ and for $n>4$, $Q^n\neq N^n$; (ii) If at most one eigenvalue lies in the interior of $P(C)$ then $C\in Q^n$ implies $C\in N^n$. By Schur's Lemma we need only consider triangular matrices. For such we give nasc. that $C\in Q^n$ for arbitrary n in terms of a quadratic form associated with C. (Received November 12, 1954.)

197. E. T. Parker: *A bound on transitivity of groups.*

A theorem of Miller (Bull. Amer. Math. Soc. vol. 22, pp. 68–71) is generalized: If $n=kp+r$, p a prime, k and r integers, $k<p^2$, $k<r(r-2)/2$, $r\geqq 13$, then each $(r+1)$-fold transitive permutation group of degree n contains the alternating group of degree n. The argument is begun like Miller's. Lemmas proved are: (1) No transitive

group of degree p or p^2 (p a prime), with a normal p-subgroup transitive on all the letters, has a composition-factor isomorphic with any alternating group of degree >5. (2) For $r\geqq 13$, the alternating group of degree r has no subgroup of index strictly between r and $r(r-1)/2$. It is shown that the hypothesis implies existence of an element of the group fixing enough letters to satisfy the hypothesis of a theorem of Bochert (Math. Ann. vol. 40, pp. 176–193). Then it may be shown, for example, that any group of degree $n>6592$ is less than $16n^{1/2}/5$ times transitive or contains the alternating group. (Received November 12, 1954.)

198. G. W. Patterson: *A uniqueness theorem concerning generalized notation for integers.*

Consider an infinite sequence of nonzero integers $\{b_n\}$, $n\geqq 1$, and another infinite sequence of sets of integers $\{R_n\}$, $n\geqq 0$. For each b_i, the corresponding set R_{i-1} is a complete set of residue class representatives in the ring $I/(b_i)$, where I is the ring of integers. By iterating a simple generalization of the division algorithm, any integer, a, can be expressed $a=q_{n+1}\prod_{i=1}^{n+1} b_i+\sum_{i=1}^{n} r_i\prod_{k=1}^{i} b_k$, with $r_i\in R_i$ for $0\leqq i\leqq n$. The infinite sequence $\{r_n\}$ obtained in this way is called the expansion of a. It may happen that for some n, q_{n+1} vanishes. In such a case we say that a has a finite or terminating expansion (with respect to $\{b_n; R_n\}$). The present theorem concerns the uniqueness of such an expansion: $\sum_{i=0}^{n} r_i\prod_{k=1}^{i} b_k=\sum_{i=0}^{n} r'_i\prod_{k=1}^{i} b_k$ & $\bigwedge_{i=0}^{n-1} (r_i, r'_i\in R_i)$: $\rightarrow\bigwedge_{i=1}^{n} (r_i=r'_i)$, for all $n\geqq 0$, where $\bigwedge_{i=0}^{n}$ signifies the conjunction of the statements of the form indicated. Verbally, two terminating expansions are identical if their corresponding coefficients are selected from the same complete sets of residue classes. The proof is by induction on n and utilizes the fact that $r_i, r'_i\in R_i\rightarrow(r_i=r'_i \pmod{b_i}\rightarrow r_i=r'_i)$. If $b_i=b$ and $R_i=(k\mid 0\leqq k\leqq b_i)$, then we obtain the uniqueness theorem for ordinary positional notation; decimal for $b=0$, binary for $b=2$. If $b_i=5$ for i even and $b_i=2$ for i odd, with R_i as before, the theorem applies to biquinary notation. If $b_i=b$ (odd) but $R_i=(k\mid -b+1\leqq 2k\leqq b-1)$, then the theorem applies to symmetric notation. (Received November 12, 1954.)

199t. Rimhak Ree: *A class of simple Lie algebras of characteristic p.* Preliminary report.

Let $\mathfrak{A}$ be an associative, commutative algebra over a field Φ of characteristic p, and let $D_1, D_2, \cdots, D_m$ be derivations on $\mathfrak{A}$ such that $D_iD_j=D_jD_i$. Denote by $\mathfrak{L}=\mathfrak{L}(\mathfrak{A}; D_1, \cdots, D_m)$ the subalgebra of the derivation algebra of $\mathfrak{A}$ consisting of all derivations of the form $f_1D_1+\cdots+f_mD_m$, where $f_i\in\mathfrak{A}$. This class of Lie algebras $\mathfrak{L}$ contains the generalization of the Witt-Zassenhaus-Jacobson algebras mentioned by Kaplansky [Bull. Amer. Math. Soc. vol. 60 (1954) pp. 470–471]. If $\mathfrak{A}$ is a field of finite degree over Φ and if $\sum f_iD_i=0$ implies $f_i=0$, $i=1, 2, \cdots, m$, then $\mathfrak{L}$ is shown to be simple. If Φ is not algebraically closed, this result gives a class of new simple Lie algebras. Other simple Lie algebras of the form $\mathfrak{L}(\mathfrak{A}, D_1, \cdots, D_m)$ are being investigated. (Received November 15, 1954.)

200. Irving Reiner: *Automorphisms of the symplectic modular group.*

The symplectic modular group Γ_{2n} consists of all integral $2n\times 2n$ matrices M satisfying $MFM'=F$, where F is the matrix of the bilinear form $\sum_{i=1}^{n} (x_iy_{n+i}-x_{n+i}y_i)$. It is proved that every automorphism of Γ_{2n} is given by $X\rightarrow f(X)\ AXA^{-1}$, where A is an integral matrix satisfying $AFA'=\pm F$, and f is a character of Γ_{2n}, that is, a homomorphism into $\{\pm 1\}$. In the course of the proof, canonical forms are obtained for involutions under similarity transformations in Γ_{2n}, and certain types of involutions are characterized by inner properties. (Received November 1, 1954.)

201*t*. Irving Reiner: *Characters of the symplectic modular group.*

The symplectic modular group Γ_{2n} consists of all integral $2n\times 2n$ matrices M satisfying $MFM' = F$, where F is the matrix of the bilinear form $\sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i)$. It is shown that Γ_{2n} has no nontrivial characters (defined to be homomorphisms into $\{\pm 1\}$) for $n>2$, and that Γ_2 and Γ_4 each have exactly one nontrivial character. The proof depends on a new identity connecting generators of Γ_{2n}, and on some known results on symplectic groups over $GF(p)$. (Received November 1, 1954.)

202. C. E. Rickart and P. C. Curtis, Jr. (p): *A characterisation of AW* algebras.*

In an unpublished paper Loomis has considered B^* algebras which satisfy the following axiom: if $\{x_i\}$ is a uniformly bounded family of elements of the algebra A such that $x_i x_j^* = x_i^* x_j = 0$ for all $i, j, i\neq j$, then there exists an element x in A with the following property: if y is an element of A and $yx_i = 0$ for all $i\neq j$, then $yx = yx_j$ and if $x_i y = 0$ for all $i \neq j$ then $xy = x_j y$. Loomis has established that B^* algebras satisfying this axiom are a subclass of AW^* algebras as defined by Kaplansky (Ann. of Math. vol. 53). In this note it is shown that the Loomis axiom is satisfied in every AW^* algebra. (Received November 12, 1954.)

203. R. F. Rinehart: *The derivative of a matric function.*

The only definition of the derivative of a function of a square matrix which has appeared in the literature is: $f^I(A) = \lim_{h\to 0} \{[f(A+hE) - f(A)]/h\}$ where h is a scalar and E is the identity matrix. This definition restricts severely the manner in which the "incremental matrix" shall approach zero. The following less restrictive definition is proposed. Let $f(z)$ be a scalar function of a complex variable and let $f(A)$ be defined. Let $H\neq 0$ be a matrix commutative with A. If $f(A+H)$ is defined for all H whose elements are sufficiently small in absolute value, if for arbitrary such H, $f(A+H)-f(A)$ can be expressed as $H\cdot M$, and if $\lim_{H\to 0} M$ exists independently of the manner in which $H\to 0$ subject to the above conditions, then this limit is defined to be the derivative, $f^I(A)$. The following theorem is proved for $f^I(A)$ so defined. If $f(z)$ is a function of a complex variable and $f(A)$ is defined, then a necessary and sufficient condition that $f^I(A)$ exist is that $f(z)$ be analytic at the characteristic roots of A. Further, $f^I(A) = g(A)$, where $g(z) = f'(z)$. (Received November 8, 1954.)

204. G. de B. Robinson (p) and R. M. Thrall: *The content of a Young diagram.*

The purpose of this paper is to throw light on the *graph* $G[\lambda]$ of a Young diagram $[\lambda]$, discussed elsewhere, and the significance of Frobenius' notation for a partition in this connection. Consider a doubly infinite matrix $G = (g_{ij})$ where $g_{ij} = j-i$, and imagine a given $[\lambda]$ superimposed upon G so that the (i, j) node of $[\lambda]$ covers g_{ij}. The *content* of $[\lambda]$ corresponds to the set of elements of G covered by $[\lambda]$. The authors obtain the necessary and sufficient condition that a given content should be *admissible*, i.e. should correspond to a diagram $[\lambda]$, and show how to construct $[\lambda]$ when its content is given. Replacing the g_{ij} by their non-negative residues modulo q, they obtain a criterion that $[\lambda]$ should be a q-core in terms of Frobenius' notation, the criterion having already been given in Young's case. If one represents the horizontal and vertical translation of $[\lambda]$ over G by two operators T, S, one may adapt the familiar partition generating function to yield the content of $[\lambda]$ for all λ. (Received November 8, 1954.)

205*t*. Kenneth Rogers: *Indefinite binary Hermitian forms.*

For an indefinite binary Hermitian form $f(x, y)$ of determinant $-d\neq 0$ the inhomogeneous minimum $M(f)$ with respect to $k(i)$ is defined, and it is shown that $M(f)<2d^{1/2}/5$ except for f equivalent to a multiple of $e^{\pi i/4}\bar{x}y+e^{-\pi i/4}x\bar{y}$, $x\bar{x}-3y\bar{y}$, $x\bar{x}-y\bar{y}$ or $x\bar{y}+\bar{x}y$, $x\bar{x}-21y\bar{y}$, $|x+((1+i)/2)y|^2-3y\bar{y}/2$ or $x\bar{x}-6y\bar{y}$: for these $M(f)=(d/2)^{1/2}$, $(d/3)^{1/2}$, $d^{1/2}/2$, $2(d/21)^{1/2}$, $(d/6)^{1/2}$. If $|f|$ takes arbitrarily small nonzero values for integers x, y of $k(i)$, then $M(f)=0$. The method of proof is first to show that $M(f)<2d^{1/2}/5$ if Gaussian integers x, y exist with $0<|f(x, y)|<d^{1/2}/6$. Null and non-null forms are then treated, the latter by the detailed knowledge of the possible homogeneous minima of $f(x, y)$ obtained by Professor Oppenheim and the author. Except for a small range this method leads to a finite set of forms whose $M(f)$ is found or estimated. Standard reduction transformations are used, as well as an analogue of Cassels' method for obtaining the Markoff chain. (Received November 9, 1954.)

206. Alex Rosenberg: *The Cartan-Brauer-Hua theorem for matrix and local matrix rings.*

Let $R=A_n$ be an $n\times n$ matrix ring over a ring with unit A, and let Z be the center of R. Let S be a subring of R setwise invariant under all inner automorphisms of R and not in Z. If $n\geq 3$, $S\supset\{Ie_{ij},\ i\neq j;\ I(e_{ii}-e_{jj});\ [A, I]e_{ii}\}$ for some nonzero (two-sided) ideal I of A. In particular if A is simple this means $S=R$. If the only nilpotent ideals of A are in the center this implies that either $S\supset K_n$ for some nonzero ideal K of A or $S\supset$ all matrices with trace 0 and entries from an ideal of square zero in the center of A. If $n=2$ and $2a\neq 0$ for each a in A and if A is generated by its quasi-regular elements or commutative the same results hold. If now R is locally matrix of degree ≥ 3 in the sense that every finite set of elements is contained in a subring which is a matrix ring of degree ≥ 3 over a simple ring with unit, then the only subrings setwise invariant under all quasi-inner automorphisms are subrings of the center and R. These results generalize those of Hattori (Jap. J. Math. vol. 21 (1952) pp. 120–128) and Kasch (Arch. Math. vol. 4 (1953) pp. 182–190). (Received October 20, 1954.)

207. Alex Rosenberg and Daniel Zelinsky (p): *Cohomology of infinite algebras.*

Let A be an algebra over a field F and, for any two-sided A-module N, let $H^n(A, N)$ denote the n-dimensional cohomology group of A with coefficients in N as defined by Hochschild (Ann. of Math. vol. 46 (1945) pp. 58–67). Then it is shown that if $H^1(A, N)=0$ for each N, A is finite-dimensional over F. Combined with results of Hochschild (loc. cit.) this shows that $H^1(A, N)=0$ for every N if and only if A is a finite-dimensional separable algebra. Now let A be a field with nonzero transcendence degree over F. Then $H^2(A, N)=0$ for each N if and only if A is a separable finite extension of a rational function field in one variable over F. (Received November 8, 1954.)

208*t*. Abraham Seidenberg: *On separating transcendency bases for differential fields.*

Let F be an arbitrary ordinary differential field of characteristic p, $p\neq 0$, and let $F\langle u_1, \cdots, u_n\rangle$ be a differential extension field of the field F. It is proved that if $F\langle u_1, \cdots, u_n\rangle/F$ is separable, then any transcendency basis of $F\langle u_1, \cdots, u_n\rangle/F$ is also a separating transcendency basis. (Received November 10, 1954.)

209*t*. Seymour Sherman: *Doubly stochastic matrices and complex vector spaces.*

A doubly stochastic (d.s.) matrix is a matrix P such that $P_{ij} \geq 0$, $\sum_i P_{ij} = \sum_j P_{ij} = 1$ for all i and j. A. Horn has proved (Amer. J. Math. vol. 76 (1954) pp. 620–630): Theorem 1. *If $y = Px$, where x, y are complex n-vectors, and P is a d.s. matrix, and $c_1, c_2, \cdots, c_n$ are any complex numbers, then $\sum_{i=1}^n c_i y_i$ lies in the convex hull of all the points $\sum_{i=1}^n c_i x_{\alpha i}$, $\alpha \in R^n$, where R^n is the set of all the permutations of* $(1, \cdots, n)$, and conjectured the truth of: Theorem 2. *If x, y are complex n-vectors and $c_1, c_2, \cdots, c_n$ are any complex numbers imply that $\sum_{i=1}^n c_i y_i$ lies in the convex hull of the vectors $\sum_{i=1}^n c_i x_{\alpha i}$, $\alpha \in R^n$, then $y = Px$ where P is a d.s. matrix.* This note establishes the validity of Theorem 2. (Received November 12, 1954.)

210. E. C. Smith, Jr.: *A distributivity condition for Boolean algebras.*

Let n be an infinite cardinal number and ν the least ordinal number such that $\bar{\nu} = n$. Let B be a Boolean algebra. B has the property (P_n) if, for every set $\{a_{\xi,\eta} \mid \xi, \eta < \nu\} \subset B$ such that all the $\sum_{\eta<\nu} a_{\xi,\eta}$ and $\prod_{\xi<\nu} \sum_{\eta<\nu} a_{\xi,\eta}$ exist and the latter is >0, there is a function f on $\{\xi \mid \xi < \nu\}$ to $\{\eta \mid \eta < \nu\}$ such that $\prod_{\xi<\kappa} a_{\xi f(\xi)} = 0$ is false for every $\kappa < \nu$. B is an n-quotient algebra if, for some n-additive ideal I in an n-additive field of sets F, B is isomorphic to F/I. Results: I. If B has the property (P_n) and the power of every set of pairwise disjoint elements of B is $<n$, then B is atomistic. II. If B has the property (P_n) and every set of at most n elements of B has a sum in B, then B is an n-quotient algebra. (This generalizes a theorem of Loomis, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 757–760.) III. If $n^m = n$ whenever $0 < m < n$ and B is an n-quotient algebra, then B has the property (P_n). IV. If n is regular, there exists a complete Boolean algebra with the property (P_n) which is not n-distributive. (Received November 1, 1954.)

211. R. R. Stoll: *A rank function for certain semigroups.* Preliminary report.

Let S be a semigroup with the following properties: (i) S contains a zero and a unit element. (ii) Each element of S is regular, i.e., for each element $a \in S$ there exists an x such that $axa = a$. (iii) For each idempotent $e \in S$ there exists an element e' orthogonal to e and such that $xe' = 0$ if and only if $xe = x$ and $e'x = x$ if and only if $ex = 0$. Then R, the set of principal right ideals of S, when partially ordered by inclusion, is a complemented modular lattice. If R has finite length, then the rank of a principal right ideal and consequently of an element in S is defined. (Received November 15, 1954.)

212. E. G. Straus and J. D. Swift (p): *The representation of integers by certain rational forms.*

The results of a previous paper (Arch. Math. vol. 5 (1954) pp. 12–18) are extended to the consideration of representation of integers by quotients $N(x)/D(x)$ where $x = \{x_1, x_2, \cdots, x_n\}$, N and D are at most quadratic in any x_i, and the total degree of D is at least that of N. The solutions of $z = N(x)/D(x)$ are reduced to two classes.

The regular solutions satisfy simpler Diophantine equations obtained by consideration of the critical cone, $D_0(x)=0$, where $D_0(x)$ consists of the terms of highest total degree in $D(x)$. The remaining solutions are called exceptional and under certain divisibility criteria can be shown to be finite in number. These criteria provide that to any lattice point $\{x_1, \cdots, x_n, z\}$ of $z=N/D$ there exists a conjugate lattice point differing only in that one or more of the x_i are replaced by conjugate values obtained by considering the equation $zD-N=0$ as a quadratic in x_i. Where the divisibility criteria are not met, it is frequently possible to show the existence of an infinite number of exceptional solutions. Diophantine equations which can be reduced by substitution to the form considered may frequently be handled more easily than by other methods. (Received November 4, 1954.)

213. Leonard Tornheim: *The minimum of a product of four homogeneous linear forms.*

Let $L_i(u)=a_{i1}u_1+\cdots+a_{in}u_n$ $(i=1, \cdots, n)$ be n real forms of determinant $\Delta=|\det(a_{ij})|$. Let Δ_n be the greatest lower bound of the Δ for which (A) $|L_1\cdots L_n|<1$ has only the trivial solution $u=0$. Korkine and Zolotareff [Math. Ann. vol. 6 (1873) p. 581] proved that $\Delta_2=5^{1/2}$ and Davenport [Proc. London Math. Soc. (2) vol. 44 (1938) p. 414] that $\Delta_3=7$. The conjecture of Hofreiter [Monatshefte für Mathematik vol. 49 (1941) p. 295] that $\Delta_4=(725)^{1/2}$, the least discriminant of totally real quartic fields, is proved. Furthermore all sets of forms having this property (A) and with $\Delta=(725)^{1/2}$ are equivalent. Also the minimum is isolated. The proof is geometric. All admissible lattices of determinant <26.93 of the region $|x_1x_2x_3x_4|\leqq 1$ and containing (1, 1, 1, 1) are considered. A great multiplicity of cases occur and are eliminated, except for one, by various conditions on the generators. Part of this work was accomplished under a grant from the National Science Foundation. (Received November 12, 1954.)

214. A. W. Tucker: *A skew-symmetric matrix theorem.*

Construct in Euclidean vector n-space: (a) the convex hull of the columns of a real n-rowed matrix A, (b) the convex cone spanned by the columns of a real n-rowed matrix B, and (c) the closed convex set of all vectors $X+Y$, x in a, Y in b. *Either* (1) the set c contains the null vector, i.e. $AP+BQ=0$ for some parameter vectors $P\geqq 0$ and $Q\geqq 0$, the component-sum of P being one, *or* (2) the set c contains a shortest vector ($\neq 0$). Such a vector, transposed into a row U, determines a half-space $UX\geqq 0$ which contains the hull a in its interior and the cone b in its closure, i.e. $UA>0$ and $UB\geqq 0$. This is a geometric form of an "alternative" or "transposition" theorem, related to theorems of Farkas, Stiemke, Motzkin, von Neumann and Morgenstern. Now let $A=S+I$ and $B=(S, I)$, where S is skew-symmetric and I the identity, both of order n. Alternative (1) proves impossible, so (2) must hold. Hence the theorem: A system of homogeneous linear inequalities $US\geqq 0$ with skew-symmetric matrix S has a solution $U(\neq 0)$ such that $U\geqq 0$ and $US+U>0$. Applied to a symmetric zero-sum two-person game in normalized form (to which any normalized zero-sum two-person game can be reduced), this existence theorem yields a form of the von Neumann minimax theorem sharpened to include the theorem of Bohnenblust, Karlin and Shapley on "essential" or "active" pure strategies. Also, through Dantzig's program-game equivalence, it implies the basic theorems of linear programming, including the detailed duality between "tight" and "slack" constraints. (Received November 12, 1954.)

215. George Whaples: *Axiomatic characterization of the classical local fields.*

The fields k which are complete under a discrete rank 1 valuation with a Galois field as residue class field can't be characterized by the fact that local class field theory holds over them, since this theory holds over a much wider set of fields. But they can be characterized by the axioms of local class field theory and the further axioms: (a) The residue class field is of characteristic p; (b) The norm residue symbol maps the group of units of k into a compact subgroup of the Galois group of the maximal Abelian extension of k. (Received November 15, 1954.)

ANALYSIS

216. M. I. Aissen: *Difference equations with periodic coefficients.*

Some standard results concerning solutions of linear homogeneous difference equations with constant coefficients are generalized to analogous results concerning solutions of linear homogeneous difference equations with periodic coefficients. The methods used are also analogous to the methods used for the classical "constant coefficients" case. Some applications are then made to display connections between continued fraction and other approximations to certain quadratic irrationals. (Received November 12, 1954.)

217. R. J. Arms: *Convex bodies, tangential points.*

The author considers a real linear Hausdorff space E satisfying: (i) every point in E is the intersection of a denumerable set of open sets, (ii) addition is continuous with respect to both factors simultaneously, (iii) scalar multiplication is continuous with respect to each factor separately. *Theorem* 1. *Let E be separable. Let K be a convex subset of E having inner points. Let B denote its boundary and T its set of tangential points (i.e. those boundary points through which there is not more than one support to K). Then, relative to B, T is an F_σ set, $B-T$ is a G_δ set, $B-T$ is a set of the first category.* Let E be a normed space. E is said to be *uniformly smooth* if, given $\epsilon>0$, there exists a $\delta=\delta(\epsilon)$ such that x_0, $x\in E$, $|x_0|=1$, $|x|<\delta$ implies $|x_0+x|+|x_0-x|\leq 2+\epsilon|x|$. It is easily verified that $L^{(p)}$ $(1<p<\infty)$ spaces are uniformly smooth. *Theorem* 2. *If E is uniformly smooth, $B-T$ is a set of the first category relative to B.* Theorem 2 does not hold for all normed spaces. Mazur (see Studia Math. vol. 4, pp. 70–84) states Theorem 1 for normed separable spaces. (Received July 29, 1954.)

218. Nachman Aronszajn: *Computation of eigenvalues in ordinary differential problems.*

The basic principle underlying the application of the generalized Rayleigh-Ritz and Weinstein approximation methods is the following fact: If L is a completely continuous self-adjoint operator in a Hilbert space and if $\mathcal{L}\subset\mathcal{L}'$ are two closed subspaces with $\dim(\mathcal{L}'\ominus\mathcal{L})=n<\infty$, and if the eigenvalue problem in the subspace $\mathcal{L}$ (R.R. method) or $\mathcal{L}'$ (W. method) has been solved, then a determinant $D_n(\lambda)$ can be written whose elements are explicitly given as sums of simple fractions in λ and whose poles and zeros determine the eigenvalues in the other subspace. The above statement can be extended to the case when it is only supposed that $\dim[(\mathcal{L}\oplus\mathcal{L}')\ominus\mathcal{L}']=m$ and $\dim[(\mathcal{L}\oplus\mathcal{L}')\ominus\mathcal{L}]=n$; m, n finite: if the problem has been solved in $\mathcal{L}$, a determinant of order $n+m$ can be written explicitly which gives the solution in $\mathcal{L}'$. Applying this to ordinary differential equations the general result is obtained that if the solution of the eigenvalue problem is known for some boundary conditions, then it can be solved explicitly with any other boundary conditions. This fact is of interest even for equations with constant coefficients for which the eigenvalue problem with periodic boundary conditions is explicitly solvable (by simple exponentials). The classical method in this case for equations of higher order leads to inextricable problems of elimination. (Received November 12, 1954.)

219*t*. W. G. Bade: *On functions of spectral measures.*

A Boolean algebra (B.A.) of projections $\mathfrak{B}$ in a real or complex Banach space is complete if $\mathfrak{B}$ is complete as an abstract B.A. and for each subset $\{E_\alpha\} \subseteq \mathfrak{B}$, $(\bigvee E_\alpha)\mathfrak{X} = \overline{sp}\ \{E_\alpha \mathfrak{X}\}$. We denote by $\mathfrak{A}(\mathfrak{B})$ and $\mathfrak{W}(\mathfrak{B})$ the uniformly and weakly closed algebras generated by $\mathfrak{B}$ respectively. If $\mathfrak{B}$ is σ-complete, then $\mathfrak{W}(\mathfrak{B}) = \mathfrak{A}(\overline{\mathfrak{B}}^s)$, where $\overline{\mathfrak{B}}^s$, the strong closure of $\mathfrak{B}$, is complete. One says $\mathfrak{B}$ has simple spectrum if for some $x_0 \in \mathfrak{X}$, $\mathfrak{X} = \overline{sp}\ \{Ex_0 | E \in \mathfrak{B}\}$. If $\mathfrak{B}$ is complete and has simple spectrum, then $\mathfrak{A}(\mathfrak{B})$ equals the commutant of $\mathfrak{B}$. If $\mathfrak{X}$ is a complex Banach space and A is a scalar type spectral operator of class $\mathfrak{X}^*$ (cf. N. Dunford, *Spectral operators*, Pacific J. Math. vol. 4, no. 3) then each operator in the weakly closed algebra generated by A is again of this type. The proofs use the lemma: If $\mathfrak{B}$ is σ-complete and $x \in \mathfrak{X}$, there exists an $x^* \in \mathfrak{X}^*$ such that $x^*Ex \geqq 0$, $E \in \mathfrak{B}$, and if $x^*Ex = 0$ then $Ex = 0$. Certain of these results extend theorems proved by a different method for reflexive spaces in the Pacific J. Math. vol. 4, no. 3. (Received November 15, 1954.)

220. A. V. Balakrishnan: *An operational calculus for closed operators.*

E. Hille has described (E. Hille, *Functional analysis and semi-groups*, American Math. Soc. Colloquium Publications, vol. 31, p. 303) an operational calculus in which the operand space is a B-space, each operator A is the infinitesimal generator of a one-parameter semigroup of linear bounded operators, and the functions $\phi(\cdot)$ for which the operator extension $\phi(A)$ is to be defined are Laplace-Stieltjes transforms. The present paper extends the calculus to the case where the functions $\phi(A)$ are closed operators, the numerical valued functions $\phi(\cdot)$ being now ratios of Laplace transforms. The extension preserves many of the properties of the previous calculus. (Received November 17, 1954.)

221. J. H. Barrett (p) and F. E. Lilley: *Second order matrix differential equations with symmetric coefficients.*

Let $P(x)$ and $Q(x)$ be symmetric square matrices of continuous functions on $x \geqq a$ and $P(x)$ be positive definite. Consider the second-order matrix differential equation (I) $(PY')' + QY = 0$. A theory of this equation is developed based on the matrix "Wronskian" of two solutions U and V: (II) $\mathrm{Wr}[U, V] = U*PV' - U*'PV$, where the $(*)$ denotes the transpose of the matrix. A study is made of the asymptotic behavior of (matrix) solutions of (I) including oscillation of solutions. A (matrix) solution of (I) is said to be oscillatory for large x provided that its determinant, $|Y|$, is oscillatory in the usual sense for scalar functions. Theorem: Let $P = (p_{ij})$, $Q = (q_{ij})$, $P^{-1} = (\bar{p}_{ij})$ and $\overline{P}_k$ be the matrix P^{-1} with the diagonal element $\bar{p}_{kk}$ replaced by zero. If for some integer k the scalar equation (III) $(y'/\bar{p}_{kk})' + q_{kk}y = 0$ is oscillatory for large x and $\overline{P}_k$ is positive semi-definite for $x \geqq a$, then (I) is oscillatory for large x. (Received November 10, 1954.)

222. R. G. Bartle: *A general bilinear vector integral.*

Let X, Y, and Z be Banach spaces and let there be a continuous bilinear function on $X \times Y$ to Z. Let S be an abstract set and $\mathfrak{S}$ a field of subsets of S. This paper develops a Lebesgue-type theory of integration of (unbounded) functions $f: S \rightarrow X$ with respect to a (finitely) additive set function $\mu: \mathfrak{S} \rightarrow Y$. This integral has many of the familiar properties. In particular, it is seen that the Vitali and the bounded convergence theorems remain valid if almost everywhere convergence is replaced by convergence in measure. The countably additive case is also studied. The approach is essentially that of F. Riesz [Acta Math. vol. 42 (1920)]. Special cases of this integral reduce to well-known integrals, and a discussion of the relation is made. (Received December 21, 1954.)

223*t*. R. G. Bartle: *Newton's method in Banach spaces.*

Let f be a map between Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$. In order to find a solution of $f(x)=0$, one may employ the iterations $x_{n+1}=x_n-[f'(x_n)]^{-1}\cdot f(x_n)$ or $x_{n+1}=x_n-[f'(x_0)]^{-1}\cdot f(x_n)$, $n=1, 2, \cdots$, termed Newton's method, or Newton's modified method. It is shown that if f is in class C' in the sense of T. H. Hildebrandt and L. M. Graves [Trans. Amer. Math. Soc. vol. 29 (1927)] and if $|f(x_0)|$ is sufficiently small, then either of the above methods converge to a locally unique solution. This extends results of L. V. Kantorovič and I. P. Mysovskih [Trudy Mat. Inst. Steklov vol. 28 (1949)] and M. L. Stein [Proc. Amer. Math. Soc. vol. 3 (1952)], although these authors establish faster convergence for the ordinary method. These results appear to be new even in the case of the real numbers. (Received November 12, 1954.)

224. F. E. Browder: *Asymptotic formulae for elliptic eigenvalue problems.*

In extension of the author's previous results, the eigenvalue problem considered is $Ku=\lambda Bu$ with K and B elliptic operators of orders $2m$ and $2r$ respectively ($m>r$) on the C^∞ bounded precompact domain D of Euclidean n-space under general boundary conditions, B having no null solutions. If K and B are self-adjoint with self-adjoint boundary conditions, the asymptotic formulae are established for the eigenvalues λ_i, the asymptotic average of the square of the eigenfunction as well as of their various derivatives. All results are expressed in terms of n, m, r, and the characteristic forms $a(x, \xi)$, $b(x, \xi)$ of K and B. Extensions are given to manifolds and to strongly elliptic systems. (Received November 12, 1954.)

225. A. H. Brown: *Absolute equivalence of projections.*

Call two projections E and F on complex Hilbert space *absolutely equivalent* if they are equivalent (in the sense of Murray-von Neumann, *On rings of operators*, Ann. of Math. vol. 37 (1936) pp. 116–229) with respect to every ring containing them. A criterion for absolute equivalence is set forth along with the appropriate generalization of a formula for the implementing partial isometry previously established by sz. Nagy in the case $\|E-F\|<1$. (Received November 15, 1954.)

226*t*. A. H. Brown: *On an equivalence relation for operators.*

It is shown that rank, co-rank and the (Hermitian) operator A^*A form a complete set of invariants for an operator A under transformations UAV where U and V are unitary, thus generalizing the familiar theorem for finite matrices. The proof is entirely elementary, depending on nothing more profound than polar resolutions. This permits the construction of a straightforward geometric proof that A^*A is a complete set of unitary invariants for an operator A with $A^2=0$ and without normal kernel, a result previously obtained by the author (*The unitary equivalence of binormal operators*, A. Brown, Amer. J. Math. vol. 76 (1954) pp. 414–434) by much more involved methods. (Received November 15, 1954.)

227B. F. H. Brownell: *Extended asymptotic eigenvalue distributions for domains in n-space.*

Let D be a bounded, open, connected set with boundary B in R_n, n dimensional euclidean space for integer $n \geqq 2$. Let λ_j, $0<\lambda_j \leqq \lambda_{j+1}$, be the eigenvalues of the problem $-\nabla^2 u=\lambda u$ on D, $u=0$ on B. It is assumed that B has a finite number of components, that each of these is sufficiently smooth to possess at every point a tangent hyperplane and a unit normal vector, and that this unit normal satisfies a Hölder condition with respect to distance in B. Then, by using the heat equation Green function for D, one can show that $N(y)=\sum_{\lambda_j \leqq y} 1$ has $\int_0^\infty e^{-yt}dN(y)=\sum_{j=1}^\infty e^{-\lambda_j t}=\mu_n(D)/(2\pi^{1/2})^n t^{n/2} -\sigma_n(B)/4(2\pi^{1/2})^{n-1}t^{(n-1)/2}+O(1/t^{(n-1-\eta)/2})$ over $t>0$, with $0<\eta \leqq 1$. From this it is concluded that $N(y)=\mu_n(D)/(2\pi^{1/2})^n\Gamma(n/2+1)y^{n/2}-(\sigma_n(B)/4(2\pi^{1/2})^{n-1}\Gamma((n-1)/2+1))y^{(n-1)/2}+\tilde{O}(y^{(n-1-\eta)/2} \ln y)$, where this $\tilde{O}$ estimate on the remainder has the sense that a certain Gaussian average, which drops out oscillating parts, has the indicated order. For D an n dimensional parallelopiped, whose boundary of course would not satisfy our smoothness conditions because of the corners, this formula for $N(y)$, with $\tilde{O}$ replaced by O, is a trivial consequence of the standard result for a corresponding lattice point problem of number theory. (Received November 12, 1954.)

228t. F. H. Brownell: *Extended asymptotic eigenvalue distributions for domains in 3-space.*

Let D be a bounded, open, connected set with boundary B in R_3, three-dimensional euclidean space; let λ_j, $0<\lambda_j \leqq \lambda_{j+1}$, be the eigenvalues of the problem $-\nabla^2 u=\lambda u$ on D, $u=0$ on B. Assume that $s \in B$ is given locally with one coordinate of s a function of the other two coordinates which has continuous second partials, and for which these second partials satisfy a uniform Hölder condition of order η, $0<\eta \leqq 1$. Under these conditions one has over $\omega \geqq 1$ the asymptotic result $\sum_{j=1}^\infty 1/\lambda_j(\lambda_j+\omega^2) =(\mu_3(D)/4\pi)(1/\omega)-(\sigma_2(B)/16\pi)\ (\ln\ (\omega^2)/\omega^2)+(C/\omega^2)-(1/24\pi)\int_B(q(s)d\sigma_2(s))(1/\omega^3) +O(1/\omega^{3+\eta})$ where μ_3 is 3-dimensional Lebesgue measure, σ_2 is 2-dimensional area measure, and $q(s)$ is twice the mean curvature of B at s (i.e. the trace of the curvature matrix defined as the derivative of the outward unit normal). From this asymptotic behavior follows $N(\lambda)=\sum_{\lambda_j \leqq \lambda} 1=(\mu_3(D)/6\pi^2)\lambda^{3/2}-(\sigma_2(B)/16\pi)\lambda+((1/12\pi^2) \cdot \int_B q(s)d\sigma_2(s))\lambda^{1/2}+\tilde{O}(\lambda^{(1-\eta)/2} \ln \lambda)$. This $\tilde{O}$ estimate on the remainder $R(\lambda)$ has the sense that a certain Gaussian average of $R(\lambda)$, which drops out oscillating parts, is of the indicated order. Stronger smoothness conditions on B yield additional terms in these asymptotic results. (Received October 25, 1954.)

229t. R. C. Buck: *Positive sequences and tauberian conditions.*

A sequence $\{S_n\}$ is said to be of bounded increase if there are numbers M and $\lambda>1$ such that $0 \leqq S_k \leqq MS_n$ for all k and n with $n \leqq k \leqq \lambda n$. Theorem: if $(C, 1)$ $\lim S_n=0$ and $\{S_n\}$ is of bounded increase, then $\lim S_n=0$. In this tauberian type theorem, the limit 0 cannot be replaced with any other positive value. Corollary: if $\sum a_n$ converges and $\{a_n\}$ is of bounded increase, then $\lim na_n=0$. (Received November 15, 1954.)

230B. Thelma M. Chaney: *On the existence of invariant means and measures on certain semigroups.*

Let G be a semigroup. A nonvoid set $S\subset G$ is a right ideal if S is closed under right multiplication by elements of G. For a finite semigroup the following are proved: a necessary and sufficient condition for the existence of a right invariant mean on G is the existence of a right ideal in G which is a group; a necessary and sufficient condition for the existence of right invariant measure on G is the existence of a right ideal $S\subset G$ which is a group such that $S'\cap G$ is also a right ideal; structure theorems for all invariant means and measures on G; the existence of more than one right invariant mean on G implies the nonexistence of a left invariant mean; a necessary and sufficient condition for the existence of both a right and left measure on G is that G be a group; if the complex linear space of all linear functionals on G is a semi-simple algebra, then there exists a unique invariant mean on G. By an application of the Markoff-Kakutani fixed-point theorem the existence of an invariant mean on any topological, Abelian semigroup is shown. (Received November 15, 1954.)

231t. Harvey Cohn: *Perturbation modular functions.* Preliminary report.

Starting with a fundamental domain in the upper half plane touching the real axis only at cusp-points, one constructs a circle of radius ϵ tangent to the real axis at one cusp point, and with it the images under the corresponding group. The perturbation formed by mapping the remaining z-plane onto itself with fixed point and direction is of the form $\epsilon^2P(z)$ where $P(Sz)(cz+d)^2-P(z)$ is a quadratic in z, depending on $S=(az+b)/(cz+d)$ (where $ad-bc=1$, and S belongs to the group in question). Conversely, under rather broad assumptions, any $P(z)$ satisfying the above transformation is attainable by such a perturbation, and furthermore such a (cusp) perturbation is the most general perturbation producing a $P(z)$ regular in the upper half plane. The decisive formalism is the fact that $P'''(z)[J'(z)]^{-1}$ is a modular invariant and that the fixed point of the perturbation mapping affects $P(s)$ by an added quadratic polynomial. In unpublished work, Schiffer has shown how the theta functions (with poles) can be produced by (interior) variations of the type used in M. Schiffer and D. C. Spencer, *Functionals of finite Riemann surfaces*, Princeton University Press, 1954. (Received November 12, 1954.)

232. V. F. Cowling: *On the summability of Dirichlet series.*

In this paper a study of the application of S_α methods of summabilty introduced by W. Meyer-König (Math. Zeit. vol. 52 (1949) pp. 252–304) to Dirichlet series is considered. It is shown, for example, that if for $0\leqq\alpha\leqq(3-5^{1/2})/2$, $\sum_1^\infty a_nn^{-s}$ is absolutely summable T_α (see the reference above) and S_α at $s=s_0$ then $\sum_1^\infty a_nn^{-s}$ is absolutely summable S_α, $0\leqq\alpha\leqq(3-5^{1/2})/2$, for Re $(s)>$ Re (s_0). The application of S_α to other classes of series is also studied. (Received November 15, 1954.)

233. R. B. Deal (p) and E. W. Titt: *The relationship between Green's ideas and Hadamard's finite part as applied to the normal hyperbolic equation with an odd number of independent variables.*

Recently one of the authors presented an alternative method of evaluating Hadamard's finite part in which integration by parts replaced the Taylor expansion. In considering the extension of these ideas to multiple integrals the multiplicity of possible directions of integration presents a problem of uniqueness. This paper presents a method of integrating the normal hyperbolic equation without a formal discipline in finite parts. The source of the idea lies in a closer imitation of the geometry of Green's method in which the ϵ-sphere is an ϵ-hyperboloid and the null sphere is the characteristic cone. Integration by parts is used to extract infinite line integrals from surface integrals over both the ϵ-hyperboloid and the data bearing surface. In fact, these line integrals cancel before passing to the limit and the result is a perfect analogue of the corresponding Green relation. The use of integration by parts to extract the line integrals leads one to the type of integration formula found in McCully and Titt (Journal of Rational Mechanics and Analysis vol. 2 (1953) pp. 443–484). (Received November 15, 1954.)

234. R. B. Deal and E. W. Titt (p): *An application of finite parts to a study of the boundary values for an elliptic equation.*

Our problem is to compute the order of the infinite discontinuity in the boundary values of the normal derivative u_N at a finite jump discontinuity in the boundary values of the solution u. For Laplace's equation over a circular region an explicit formula for the boundary values of u_N in terms of tangential derivatives of u is given in the lecture notes of one of the authors. The formula for the case of discontinuous u is obtained as a limit of the corresponding formula for continuous u since in a direct attack divergent integrals would have entered the work due to the order of the infinite discontinuity in u_N. In the theses of Faircloth and Eason it is found that for an arbitrarily shaped boundary the explicit formula becomes an integral equation. In this paper these formulas are derived by the use of a limiting region rather than limiting boundary values u. The method of the preceding paper by the same authors is used to extract the infinities and the balancing of these infinities in the resulting limiting relation leads to a relationship between a known infinite discontinuity and the unknown infinite discontinuity in u_N. (Received November 15, 1954.)

235. W. F. Donoghue: *Continuous function spaces isometric to Hilbert space.*

A well-known theorem of Banach provides that the space $C(0, 1)$ of continuous functions on the unit interval with norm $\|x\| = \sup_t |x(t)|$ has a subspace isometric to any pre-assigned separable Banach space, in particular therefore to separable Hilbert space. It is shown that if such a subspace is isometric to a Hilbert space of dimension $N>2$, then any $k<N$ linearly independent elements $x_i(t)$, $i=1, \cdots, k$, define a curve in R_k which covers a sphere. No such result holds for $N \leqq 2$. For infinite-dimensional spaces it is shown that the mapping of the unit interval into Hilbert space determined by the reproducing kernel covers a sphere in Hilbert space. (Received November 12, 1954.)

236. R. J. Duffin and D. H. Shaffer (p): *Asymptotic expansion of double Fourier transforms.*

The problem considered is the asymptotic expansion at infinity of the two-dimensional Fourier transform of a function which has a singularity at the origin of the form $x^m y^n (x^2+y^2)^k$ where m and n are positive integers and k is an arbitrary real constant. The addition of a function having all partial derivatives everywhere does not affect the formulae obtained. If $2k+m+n \leqq -2$, convergence of the integral is assured by employing a suitable generalized Fourier transform. In the theory of discrete harmonic functions and discrete analytic functions the fundamental solutions may be defined as two-dimensional Fourier series coefficients of functions with a singular point. As an application, the asymptotic expansions of these fundamental solutions are obtained. (Received November 15, 1954.)

237. Albert Edrei: *On a conjecture of Pólya.*

Let $f(z)$ be a real entire function of finite order ρ (>1) and mean type. The author shows that if $\limsup x^{-\rho} \log |f(\pm x)| < 0$ as $x \to +\infty$, then every point of the real axis is a point of accumulation of zeros of the successive derivatives of $f(z)$. This result follows from a recent theorem of the author [Bull. Amer. Math. Soc. Abstract 60-2-221]; it throws some light on a conjecture of Pólya [Bull. Amer. Math. Soc. vol. 49, p. 183]. (Received November 9, 1954.)

238*t*. Ky Fan and I. L. Glicksberg: *Fully convex normed linear spaces.*

A normed linear space X is fully convex (*weakly fully convex*) if every sequence $\{x_n\}$ of elements in X satisfying (a) $\lim \|x_n\| = 1$, (b) $\|(x_n + x_m)/2\| \geqq 1$ for $n, m = 1, 2, 3, \cdots$ is a Cauchy sequence (converges weakly to some element). Every uniformly convex normed linear space is fully convex. There exist fully convex Banach spaces which are not uniformly convex in any topologically equivalent norm. Every separable reflexive Banach space has a topologically equivalent norm which makes the space weakly fully convex. A normed linear space X is reflexive if and only if every sequence $\{x_n\}$ in X satisfying (a), (b) has a subsequence which converges weakly to some element. G. Birkhoff's mean ergodic theorem for uniformly convex Banach spaces [Duke Math. J. vol. 5 (1939) pp. 19–20] has the following generalization: Let $\{x_n\}$ be a sequence of elements in a fully convex (weakly fully convex) Banach space. If $\|(1/n)\sum_{i=1}^{n} x_{i+j} + (1/m)\sum_{i=1}^{m} x_{i+j}\| \leqq \|(1/n)\sum_{i=1}^{n} x_i + (1/m)\sum_{i=1}^{m} x_i\|$ holds for every three integers, n, m, j, then the sequence $\{(1/n)\sum_{i=1}^{n} x_i\}$ converges (converges weakly) to some element. All above results can be extended if, for any integer $k \geqq 2$, we replace (b) by (b′) $\|(1/k)\sum_{i=1}^{k} x_{n_i}\| \geqq 1$ for $n_1 \leqq n_2 \leqq \cdots \leqq n_k$. (Received November 10, 1954.)

239. Ky Fan, Olga Taussky, and John Todd (p): *Discrete analogs of inequalities of Wirtinger.*

This paper contains discrete analogs of several known integral inequalities involving functions and their derivatives. All results are concerned with $m(\geqq 3)$ vectors $x_1, x_2, \cdots, x_m$ in a unitary space U^n of arbitrary dimension n, and in each case the best possible constant (which depends on m) is determined. In the three typical results stated below, it is assumed that $\sum_{i=1}^{m} x_i = 0$. (i) If $x_{m+1} = x_1$, then $\sum_{i=1}^{m} \|x_i - x_{i+1}\|^2 \geqq 4 \sin^2 (\pi/m) \sum_{i=1}^{m} \|x_i\|^2$. The equality sign holds if and only if there exist two vectors y, z (not necessarily linearly independent) in U^n such that $x_i = (\cos (2i\pi/m)) \cdot y + (\sin (2i\pi/m)) \cdot z$ for $1 \leqq i \leqq m$. (ii) If $x_0 = x_1$, $x_{m+1} = x_m$, then $\sum_{i=0}^{m-1} \|x_i - 2x_{i+1} + x_{i+2}\|^2 \geqq 16 \sin^4 (\pi/2m) \sum_{i=1}^{m} \|x_i\|^2$. The equality sign holds if and only if there exists a vector $y \in U^n$ such that $x_i = (\cos ((2i-1)\pi/2m)) \cdot y$ for $1 \leqq i \leqq m$. (iii) If $x_{m+1} = x_1$, then $\mathrm{Max}_{1 \leqq i \leqq m} \|x_i - x_{i+1}\| \geqq a(m) \cdot \mathrm{Max}_{1 \leqq i \leqq m} \|x_i\|$, where the best possible constant $a(m) = 4/m$ for even m and $a(m) = 4m/(m^2 - 1)$ for odd m. Result (i) is the discrete analog of Wirtinger's inequality (cf. G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, 1934, p. 185). Result (iii) is the discrete analog of an inequality of D. G. Northcott [J. London Math. Soc. vol. 14 (1939) pp. 198–202]. (Received November 3, 1954.)

240. M. K. Fort, Jr.: *Research problem number 22.*

Richard Bellman (Bull. Amer. Math. Soc. vol. 60, p. 501) has proposed the following research problem: Solve the functional equation $f(x) = \mathrm{Max}\ (g(x)+f(ax), h(x)+f(bx))$, given that $0<a, b<1$; $h(x), g(x)>0$; $h(0)=g(0)=0$; $h'(x), g'(x)>0$; $h''(x), g''(x)>0$. It is shown that the equation has a unique solution f with the following properties: $f(x)$ is defined for all $x \geqq 0$; f is continuous; $f(0)=0$. If $g'(0)/(1-a) > h'(0)/(1-b)$, there exists $\epsilon>0$ such that $f(x)=g(x)+g(ax)+g(a^2x)+\cdots$ for $0 \leqq x \leqq \epsilon$. This condition together with the given equation then determines $f(x)$ for all $x \geqq \epsilon$. If $g'(0)/(1-a)<h'(0)/(1-b)$, a similar solution is obtained. If $g'(0)/(1-a) = h'(0)/(1-b)$, then it can be shown that $f_\lambda(x) \to f(x)$ as $\lambda \to 1$, where f_λ is the solution corresponding to $g, \lambda h, a, b$. (Received November 8, 1954.)

241. Bernard Friedman and Joel Franklin: *A convergent asymptotic representation for integrals.*

A new method of obtaining a series asymptotic in p for integrals of the form $\int_0^\infty e^{-px}x^{c-1}f(x)dx$ is proposed. It is shown that if $f(x)=\int_0^\infty e^{-xt}d\phi(t)$ for Re $x>0$, where $\phi(t)$ is a complex-valued function which is of bounded variation in each finite interval $0 \leqq t \leqq T$ and which satisfies the inequality $|\Psi(t)| \leqq M$ for $t \geqq 0$, then the asymptotic series converges to the value of the integral if $p>0$ and Re $c>0$. Numerical calculation seems to show that the first term of the series gives a close approximation to the value of the integral for a wide range of values of p. (Received November 23, 1954.)

242. T. M. Gallie, Jr.: *Essential sets of singularities of Dirichlet series.*

It is known that $s=1$ is an isolated singular point of a uniform function $f(s)$ if and only if there exists an entire function $F(s)$ of minimal exponential type such that $f(s)-\sum_{n=0}^{\infty} F(n)s^n$ is regular at $s=1$. In the light of this fact Mandelbrojt has suggested the definition: a set E is called an essential set of singularities of a uniform function $f(s)$ with respect to a sequence $\{\Lambda_n\}$ if there exists an $F(s)$ as above such that: (a) $\sum_{n=1}^{\infty} F(\Lambda_n)e^{-\Lambda_n s}$ is uniform and its set of singular points is E; (b) $f(s) -\sum_{n=1}^{\infty} F(\Lambda_n)e^{-\Lambda_n s}$, called a regular part of $f(s)$, is regular on E. Let $f(s)$ and $g(s)$ be Dirichlet series whose exponents are complementary with respect to a sequence $\{\Lambda_n\}$. It is proved that if $f(s)$ has an essential set of singularities with respect to $\{\Lambda_n\}$ and if a regular part of $f(s)$ together with $g(s)$ satisfy the hypotheses of Agmon's composition theorem (Journal d'Analyse Mathématique vol. 1 (1951) pp. 232–243), then $g(s)$ can have no accessible singularities (see Pólya, Göttingen Nach., 1927, pp. 187–195) in the composition domain of Agmon's theorem. This result contains results of Mandelbrojt and Pólya (C. R. Acad. Sci. Paris vol. 184 (1927) pp. 502–504). (Received November 12, 1954.)

243. R. K. Getoor: *On semi-groups of unbounded normal operators.*

In some recent investigations of the author semi-groups of unbounded normal operators in Hilbert space arose in a natural fashion. Such semi-groups do not seem to have been considered in the literature although semi-groups of unbounded self-adjoint operators have been considered by A. Devinatz, *A note on semi-groups of unbounded self-adjoint operators*, Proc. Amer. Math. Soc. vol. 5 (1954). The following theorem is proved. Let $N(t)$, $t>0$, be a semi-group of normal operators on a Hilbert space H satisfying: (1) if $D(t)$ is the domain of $N(t)$ then $D=\bigcap_{t>0}D(t)$ is dense in H; (2) if $R(t)$ is the range of $N(t)$ then $R(t)\subset D$ for all $t>0$; (3) if $x, y\in D$ then $(N(t)x, y)$ is continuous at $t>0$ and moreover $(N(t)x, y)\to(x, y)$ as $t\to 0$; under these conditions there exists a complex spectral resolution $K(\lambda)$, $\lambda=\lambda_1+i\lambda_2$, such that if $\Delta\subset\{\Re(\lambda)<0\}$ then $K(\Delta)=0$ and $N(t)=\int_{\Re(\lambda)\geq 0}\lambda_1^t \exp(i\lambda_2 t)dK(\lambda)$. (Received November 12, 1954.)

244. A. W. Goodman: *Functions typically-real and meromorphic in the unit circle.*

Let TM denote the class of functions which are typically-real and meromorphic in the unit circle, with $f(z)=z+\sum_{n=2}^{\infty} b_n z^n$ in a neighborhood of the origin. The condition $\Im f(z)\Im(z)>0$ for $|z|<1$ and z not real implies that all the poles are simple and lie on the real axis. By quite elementary means it is possible to find the precise domain in variability of $f(z)$ for each fixed z and $f(z)\in TM$. Further for this class of functions sharp bounds for $|b_n|$ in terms of the distance of the closest pole to the origin are found. (Received November 12, 1954.)

245. R. P. Gosselin: *On the convergence of subsequences of Fourier series in an L^p class.*

The following notation will be used: $s_{m_\nu}(x)$ for the m_νth partial sum of the Fourier series of a function $f(x)$; $\{n_k\}$ for a lacunary sequence of integers, i.e. $n_{k+1}/n_k\geq\lambda>1$; and $[y]$ for the greatest integer $\leq y$. A sequence $\{m_\nu\}$ is said to have upper density 1 if lim sup $(\sigma(n)/n)=1$ where $\sigma(n)$ is the number of terms of the sequence $\leq n$. The following theorems are proved: (i) If $f(x)$ belongs to L^2, with $\{n_k\}$ as above, then there exists a sequence $\{m_\nu\}$ of positive integers containing $[n_{k+1}-n_k/\log n_{k+1}]$ consecutive terms in each interval (n_k, n_{k+1}) such that $s_{m_\nu}(x)$ converges to $f(x)$ almost everywhere. (ii) If $f(x)$ belongs to L^2, then there is a sequence $\{m_\nu\}$ of upper density 1 such that $s_{m_\nu}(x)$ converges to $f(x)$ almost everywhere. There is also a theorem analogous to (i) for functions of class L^p, $1<p<2$. The main tools of proofs are two theorems of Kolmogoroff on the convergence of Fourier series (cf. A. Zygmund, *Trigonometrical series*, Warsaw, 1935, pp. 251–254). (Received November 12, 1954.)

246B. L. W. Green: *Remark on ergodic transformations.*

Let Ω be a compact metric space with a measure μ defined on the Borel field $\mathcal{E}$. Assume that $\{\phi_t\}$ is an ergodic one-parameter group of measure-preserving homeomorphisms of Ω onto itself. It is proved that the physical problem of finding the time average of a given L_1-phase function $f(\omega)$ by discrete observations is "well-set" in the following sense: $(1/n)\sum_{i=0}^{n-1} f(\phi_{t_i}\omega)$ converges to the same constant for "almost all" choices of the sequence $t_0<t_1<\cdots\to\infty$, provided the interval between observations is uniformly bounded.. Convergence is a simple consequence of the Random Ergodic Theorem; proof of the constancy of the limit is a little more involved. (Received December 13, 1954.)

247. R. T. Herbst: *The equivalence of linear and nonlinear differential equations.*

The nonlinear equation considered is (*) $y''-(\ln w(x))'y'+q(x)Z(y)=A(y)y'^2+w(x)^2C(y)$, where Z, A, C are given functions satisfying $Z\neq 0$, $Z'-AZ=1$, $ZC'+(3-AZ)C=0$. This paper shows that under suitable continuity conditions (*) can be solved by putting $y=F(u, v)$, where u, v are independent solutions with Wronskian

w of the linear equation $Y''-(\ln w)'Y'+qY=0$, and where F satisfies a passive system of total differential equations. Moreover (*) is the only equation of the second order which can be solved by an F which is not homogenous of degree 0. This answers in part the question raised by J. M. Thomas (Proc. Amer. Math. Soc. vol. 3 (1952) pp. 899–903) regarding the determination of equations equivalent to a linear differential equation in the above sense. (Received November 12, 1954.)

248. I. I. Hirschman, Jr.: *Weak quadratic norms.*

Define $N_a[f]=[\int_{-\pi}^{\pi}|f(\theta)|^2|\theta|^{2a}d\theta]^{1/2}$ $(-1/2<a<1/2)$, $L_b[f]=[\int_{-\pi}^{\pi}|f(\theta)|^2\cdot(\log(|\theta|/2\pi))^{b}d\theta]^{1/2}$ $(-\infty<b<\infty)$. Let $f(\theta)\sim\sum_1^\infty a_ne^{in\theta}$ (the restriction to Fourier series of power series type being made only for convenience). The author has previously shown that if T is a multiplier transformation, $Tf(\theta)\sim\sum_1^\infty t_na_ne^{in\theta}$ and if $|t_n|\leq A$ $(n=1,2,\cdots)$, $\sum_{2^{n-1}}^{2^n-1}|t_{k+1}-t_k|\leq A$ $(n=1,2,\cdots)$, then $N_a[Tf]\leq AC_1(a)\cdot N_a[f]$ where $C_1(a)$ depends only upon a $(-1/2<a<1/2)$. It is shown here that if $|t_n|\leq A$ $(n=1,2,\cdots)$, $\sum_{(n-1)^2}^{n^2-1}|t_{k+1}-t_k|\leq A$ $(n=1,2,\cdots)$, then $L_b[Tf]\leq AC_2(b)\cdot L_b[f]$ where $C_2(b)$ depends only upon b $(-\infty<b<\infty)$. (Received November 15, 1954.)

249B. J. C. Holladay: *On convergence to eigenvectors for iterates of homogeneous transformations.*

A theorem due to O. Perron, although usually attributed to Frobenius, states that given a matrix whose elements are positive, there exists a unique positive vector v_0 such that if v is a positive vector, then v_0 is the limit as $j\to\infty$ of $M^j(v)/\|M^j(v)\|$. The main purpose of this paper is to generalize from linearity to homogeneity. Also, positiveness of the transformation is generalized to positiveness of some iterate of the non-negative transformation. For homogeneity of degree α where $0<\alpha\leq 1$, one obtains uniform convergence to a unique vector while for homogeneity of degree greater than one, the convergence need not be unique, nor uniform, nor need it even exist. Some observations are made concerning continuity of the dependence of the eigenvector on the transformation. Also, an observation is made concerning positiveness of iterates of non-negative matrices. (Received November 15, 1954.)

250. Meyer Jerison: *Martingale convergence theorems on rings of functions.* Preliminary report.

Let $\{X_n\}$ be an increasing sequence of orthonormal sets of bounded real functions on a finite measure space, such that $X=\bigcup X_n$ is complete in L^2. For $f\in L^1$, define f_n as the function (if it exists) satisfying $\int xf=\int xf_n$ if $x\in X_n$ and $\int xf_n=0$ if $x\in X-X_n$. Theorem. If the set of linear combinations of functions in X_n is closed under pointwise multiplication (almost everywhere) for each n, then for every $f\in L^1$, f_n exists, is in L^1, and $f_n\to f$ almost everywhere as well as in the L^1 norm. The theorem is proved by combining a measure-theoretic analogue of the Stone-Weierstrass theorem [Segal, Amer. J. Math. vol. 76 (1954) p. 725] with a martingale convergence theorem [Doob, *Stochastic processes*, New York, 1953, p. 331]. (Received November 15, 1954.)

251. M. L. Juncosa and D. M. Young (p): *On the accuracy of finite difference methods for solving the diffusion equation.*

Solutions of the implicit six-point difference equation analogue (J. Crank and P. Nicolson, Proc. Cambridge Philos. Soc. (1947)) of the diffusion equation $u_t=u_{xx}$ with boundary conditions $u(0+,t)=u(1-,t)=0$, $t\geq 0$, and $u(x,0+)=f(x)$, $0<x<1$, are shown to converge to the exact solution of the differential equation in R_0: $0 \leqq x \leqq 1$, $t \geqq t_0 > 0$, as $\Delta x = M^{-1} \rightarrow 0$ provided $f(x)$ is piecewise continuous and $\Delta t < M^{-1} \cdot (t_0/3 \log M)^{1/2}$. This requirement on Δt allows the use of many fewer time intervals than the usual condition, $\Delta t = O(M^{-2})$. Previous results of the authors (Bull. Amer. Math. Soc. Abstract 60-2-240) on the order of convergence in R_0 are shown to hold under the same conditions on $f(x)$ and Δt. Moreover, if $\Delta t = O(M^{-2})$, if $f(x)$ has a bounded kth derivative ($k = 2, 3$) except at a finite number of values of x, and if $f^{(k-1)}(x)$ is piecewise continuous, then as $M \rightarrow \infty$ the error vanishes like $O(M^{-(k-1)})$ uniformly in $0 \leqq x \leqq 1$, $0 \leqq t$. This is also true for the four-point forward difference equation provided $\Delta t M^2 = r$, where r is a constant and $r \leqq 1/2$. For the latter results the methods are similar to those used by Wasow (Journal of Research (1952)) and by Walsh and Young (Journal of Research (1953)). (Received November 15, 1954.)

252. W. B. Jurkat: *An absolute Fatou-Riesz theorem for Laplace integrals.*

In collaboration with A. Peyerimhoff a theorem on the absolute convergence of power series in points of regularity was proved, which corresponds to the well-known theorem of Fatou for ordinary convergence (Archiv d. Math. vol. 4 (1953)). Continuing the collaboration the following theorem for Laplace integrals is now proved, which corresponds to M. Riesz's extension of Fatou's theorem mentioned above: If $b(t)$ is integrable on every bounded interval, $\int_T^\infty |db(t)| < \infty$ for some $T \geqq 0$, and $f(s) = \int_0^\infty e^{-st} b(t)\,dt$ regular for $s = 0$, then $\int_0^\infty |b(t)|\,dt < \infty$ holds. There are certain natural generalizations and also applications to Dirichlet series. (Received November 9, 1954.)

253t. R. V. Kadison: *Multiplicity theory for operator algebras.*

If $\mathfrak{A}$ is a C^*-algebra and ϕ is a *-representation of $\mathfrak{A}$ as an algebra of operators on a Hilbert space, H, the author associates with ϕ a "multiplicity function," f_ϕ, which assigns to 0 a class of ideals of Borel sets in the pure state space, X, of $\mathfrak{A}$, and to each positive real number or infinite cardinal, a, not exceeding the dimension of H, an ideal, N_a, of Borel sets in X (the N_a form a descending chain of ideals). Two *-representations ϕ_1, ϕ_2 of $\mathfrak{A}$ are unitarily equivalent (i.e., there exists a unitary transformation, U, of H_1 onto H_2 such that $U\phi_1(A)U^{-1} = \phi_2(A)$ for each A in $\mathfrak{A}$) if and only if their associated multiplicity functions, f_{ϕ_1}, f_{ϕ_2}, are identical. (Received November 12, 1954.)

254B. Tosio Kato: *Notes on perturbation theory.*

In perturbation theory of linear operators there occurs the problem of constructing a regular transformation W of a Banach space which maps the range $\mathfrak{M}_0$ of a projection P_0 onto the range $\mathfrak{M}_1$ of a projection P_1 in a one-to-one fashion, where e.g. $\|P_1 - P_0\| < 1$ is assumed. Such a transformation was first given by Sz.-Nagy in the form $W = P_1[I + P_0(P_1 - P_0)P_0]^{-1/2}P_0 + (I - P_1)[I + (I - P_0)(P_0 - P_1)(I - P_0)]^{-1/2} \cdot (I - P_0)$. In the present note a simpler and symmetric form will be given to this W: $W = [P_1P_0 + (I - P_1)(I - P_0)][I - (P_1 - P_0)^2]^{-1/2}$. Also a different transformation W_1 was given by the author for the case where P_0 and P_1 are connected by a continuously (in the norm) changing projection P_t, $0 \leqq t \leqq 1$, as the solution of the differential equation $W_t' = (P_t'P_t - P_tP_t')W_t$. In the present note this method is applied to the case where only P_0 and P_1 are given ($\|P_1 - P_0\| < 1$) by first constructing a continuous path P_t and then solving the ensuing differential equation. It is interesting that the result W_1 is again identical with W stated above. (Received November 15, 1954.)

255. N. D. Kazarinoff: *Asymptotic solution with respect to a parameter of a differential equation having an irregular singular point.*

The asymptotic solutions for $|\lambda|$ large of the differential equation (*): $(z-z_0)^\nu \cdot d^2w/dz^2+\lambda(z-z_0)^{\nu/2}P_1(z, \lambda)dw/dz+\lambda^2P_2(z, \lambda)w=0$ with ν real and greater than 2 are considered, the complex variable z being confined to a bounded neighborhood of the irregular singular point z_0. An analogous problem for a class of nth order differential equations containing (*) for integral $\nu \geqq 6$ has been discussed by C. C. Hurd [Tohoku Math. J. vol. 44 (1938) pp. 243–274], but it appears that a more direct investigation of solutions of (*) is possible. This is accomplished by adapting to this situation the classical theory of the asymptotic solution of (*) in the neighborhood of an ordinary point. A simple algorithm is given for the determination of formal solutions of (*) in the form of infinite series in descending powers of λ. Finite segments of these series are shown to approximate solutions of (*). The solutions are found to be subject to the Stokes' phenomenon. (Received November 12, 1954.)

256t. Jacob Korevaar: *Sequences of polynomials whose zeros lie in a given set.*

R denotes a closed set in the z-plane. Any polynomial whose zeros belong to R is called an R-polynomial. The author called a set R regular (Duke Math. J. vol. 18, p. 573) if every entire function is the limit of a sequence of R-polynomials which converges uniformly in every bounded domain. Problem 1. Given a bounded domain D. How should R be in order that every $f(z)$ analytic in D, with its zeros in R, is the limit of a sequence of R-polynomials which converges uniformly in every compact subset of D? Two classes of sets R are known or easily shown to be admissible. A. The sets R which contain a rectifiable Jordan curve going around D (see G. R. MacLane, Duke Math. J. vol. 16, p. 461). B. The regular sets. By solving a second problem the author shows that these are essentially the only possibilities. Problem 2 (posed by Ganelius, Ark. Mat. vol. 3, p. 1). For which sets R does the uniform convergence of a sequence of R-polynomials in $|z|<1$ always imply its uniform convergence in every bounded domain? Answer given in this paper: for all non-regular sets R whose complement is connected and contains part of $|z|<1$. (Received Novembr 15, 1954.)

257. Bertram Kostant: *An extension of the Gårding space.* Preliminary report.

Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, and $x\rightarrow U_x$ a strongly continuous representation on a Banach space V. For any $X\in\mathfrak{g}$ let $x(t)=\exp tX$ for all real t, $D_X=[v\in V|\lim_{t\rightarrow 0} (U_{x(t)}-1)/tv=u$ exists$]$ and let $\pi(X)$ be the operator defined on D_X by $\pi(X)v=u$. For a Radon measure μ (resp. a measurable function $f(x)$) for which $\int\|U_x\|d\mu(x)<\infty$ $(\int|f(x)|\|U_x\|dx<\infty)$ define $L_\mu=\int U_xd\mu(x)$ $(L_f=\int f(x)U_xdx)$. Finally let $C_k^0(G)$ be the set of functions of class C_k on G having compact support. Gårding [Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) p. 331] has shown that for any $X\in\mathfrak{g}$, if $f\in C_k^0(G)$ $(k\geqq 1)$ then L_f: $V\rightarrow D_X$ and $\pi(X)L_f=L_{Xf}$ where $Xf(x)=\lim [f(x(t)x)-f(x)]/t$ as $t\rightarrow 0$. In the case where V is a Hilbert space and the U_x are unitary the author shows that this holds for any $f\in L_1(G)$ such that $f(x(t)x)$ is locally absolutely continuous in t for each x and $Xf\in L_1(G)$. More generally if μ and ν are bounded Radon measures and $[T_{x(t)}-1]/t\mu\rightarrow\nu$ weakly with respect to $C_\infty^0(G)$ [weakly in the space of distributions on G], then L_μ: $V \rightarrow D_X$ and $\pi(X)L_\mu = L_\nu$ where $T_{x(t)}\mu$ is left translations of μ by $x(t)$. (Received November 15, 1954.)

258. Stephen Kulik: *Application of a new method for the solution of equations.*

A root z_0 of an equation $f(z)=0$ is expressed as a limit of a function $F_n(z)$ with $n \rightarrow \infty$ and z any number which is closer to z_0 than to any other root of the equation. By this method the largest and smallest roots of an algebraic equation with all real roots can be calculated with stating the lower and upper limits of the approximations; in case of a cubic equation the solutions are expressed in terms of its coefficients. (Received November 10, 1954.)

259. W. T. Kyner, Jr.: *An existence proof for periodic k-surfaces.*

In 1952, S. P. Diliberto proposed a theory of periodic k-surfaces in order to obtain qualitative information about the behavior of solutions of perturbed systems of nonlinear ordinary differential equations. This paper contains the proof of a general existence theorem for these periodic k-surfaces (in the so-called nondegenerate case). G. Hufford, in 1953, proved an existence theorem for periodic 2-surfaces. The first results for periodic k-surfaces, for k any integer, were obtained by M. D. Marcus. Both Hufford and Marcus defined their transformations in terms of differential equations. The above results have been generalized. The problem is first considered as a fixed point theorem in a function space (under a class of transformations that includes those used previously). The elements of the function space can be interpreted as hypersurfaces in some Euclidean space. The hypothesis is used that a certain linear transformation, defined as a translation operator minus a matrix operator, has an inverse. This hypothesis is weaker than that used by either Hufford or Marcus. The proof offered is a generalization of the method used by Hufford. (Received November 15, 1954.)

260B. R. E. Langer: *The asymptotic forms of the solutions of a class of ordinary differential equations of the third order about a multiple turning point.*

The paper gives a derivation of the asymptotic forms, for $|\lambda|$ large, of the solutions of differential equations $L(u)+\lambda^2[b_1(x)u'+b_0(x)u]=0$, with $L(u)\equiv u'''+a_2(x)u''+a_1(x)u'+a_0(x)u$, in regions of the complex x in which $b_1(x)$ has a simple zero and is otherwise different from zero. It shows that by transformations of the variables such an equation can be normalized to the extent that its solutions are asymptotically expressible in terms of those of the "base" equation $w'''+\lambda^2 xw'+3\lambda^2\mu w=0$, in which μ is a constant. The asymptotic solutions of this base equation are derived for all values of μ by use of the theory of residues. (Received November 10, 1954.)

261. Benjamin Lepson: *On the inner and outer logarithmic capacities of plane sets.*

The identity of the inner and outer Newtonian capacities for analytic sets in Euclidean space of three dimensions has been shown recently by Choquet [C. R. Acad. Sci. Paris vol. 234 (1952) pp. 498 and 784]. One of his principal tools is the property of "strong convexity" for the Newtonian capacity of compact sets, i.e., the inequality (1) $C(A\cup B)+C(A\cap B)\leq C(A)+C(B)$. In order to obtain, by Choquet's method, a similar relation between the inner and outer logarithmic capacities of plane sets, it seems necessary to verify (1) in some form for plane compact sets. It is known that (1), and in fact even the less restrictive condition of subadditivity, fails to hold when $C(A)$ is taken to be Fekete's transfinite diameter (which is equivalent to logarithmic capacity in the sense of Frostman). However, when the capacity is taken in the sense of De La Vallée Poussin or Carleson (i.e., the reciprocal of Robin's constant), and when both sets are assumed to lie in a fixed circle of unit diameter, the condition of subadditivity does hold [see Nevanlinna, *Eindeutige analytische Funktionen*, p. 120 or Fekete, *On transfinite radius*, Proceedings of the International Congress of Mathematicians 1950, pp. 380–381]. It therefore seems reasonable to attempt to verify (1) under the same assumptions. However, this cannot be done, as is shown by the following counterexample: A and B are semicircles each of unit diameter, $A\cup B$ is a circle of unit diameter, and $A\cap B$ is a line of unit length. The numerical values, which are taken from the recent monograph by Pólya and Szegö, are: $C(A) = C(B) = 1.04737$, $C(A\cup B) = 1.44270$, and $C(A\cap B) = .72135$. (Received November 12, 1954.)

262. Lee Lorch (p) and Peter Szego: *A singular integral whose kernel involves a Bessel function.*

Let $t^{\lambda}f(t)$ be Lebesgue integrable over $(0, A)$, $A>1$ and constant, for some constant λ; let $f(1-)$ exist and suppose that $f(t)$ is of bounded variation over $[1, A]$. Then, for $\nu\to\infty$, (i) $\lim \nu\int_0^A f(t)J_\nu(\nu t)dt = f(1-)/3+2f(1+)/3$, and (ii) $\lim \nu \int_0^{1+p(\nu)} f(t) \cdot J_\nu(\nu t)dt = [f(1-)/3+f(1+)/3]\int_0^{\sigma}[J_{1/3}(t)+J_{-1/3}(t)]dt$, where $\sigma = \lim \nu[2p(\nu)]^{3/2}/3$, provided this last limit exists and $p(\nu)\geqq 0$ for all large ν. As usual, $J_\nu(t)$ is the Bessel function of order ν. Continuity alone of $f(t)$ at $t=1$ does not suffice for the validity of (i), since the Lebesgue constants for this transform are unbounded, being in fact exactly of order $\nu^{1/3}$. [Thus, the kernel $\nu J_\nu(\nu t)$ behaves, so to speak, like the Fejér kernel in Fourier series when $t<1$ and like the Dirichlet kernel when $t>1$. This peculiarity is a consequence of the change in behavior of $J_\nu(\nu t)$ at $t=1$ from nonoscillatory to oscillatory.] Using (ii), the limit, as $\nu\to\infty$, of $\nu^{\lambda}\int_0^{\nu}t^{-\lambda}J_\nu(t)dt$ is found for fixed $\lambda\geqq -1/2$; the value of the limit is independent of λ. For certain ranges of fixed λ, ν it is also shown that (iii) $\gamma(\lambda, \nu)<\int_0^{\nu}t^{-\lambda}J_\nu(t)dt<\gamma(\lambda, \nu)+A_\lambda \nu^{-\lambda}$, where A_λ ($\geqq .27$) depends only on λ and $\gamma(\lambda, \nu) = 2^{-\lambda}\Gamma[(\nu+1-\lambda)/2]/\Gamma[(\nu+1+\lambda)/2]$. (Received November 12, 1954.)

263. G. G. Lorentz: *Properties of Banach lattices.*

Let X be a Banach lattice of measurable functions on a measure space E, for simplicity with separable measure and with $\mu E<+\infty$. Let (a) $f\in X$, $0\leqq |g| \leqq f$ imply $g\in X$, and (b) $0\leqq f_n\uparrow f$, $\|f_n\|\leqq M$ imply $f\in X$. Then X is separable if and only if $\|f\chi_e\|\to 0$ for $f\in X$, $me\to 0$. For instance, an Orlicz space L_Φ is separable if and only if $\Phi(2u)\leqq C\Phi(u)$. The author discusses the question under what conditions a Banach lattice of measurable functions X is a dual $X(C)$, i.e. when there is a representation $\|f\| = \sup \int_E |f(x)| C(x)d\mu(x)$, $c\in C$. The necessary and sufficient conditions are (a) and (b′) $\|f\|\leqq \liminf \|f_n\|$ if $0\leqq f_n\uparrow f$. Further, the duals of the spaces $X(C_1C_2)$ are described, where C_1C_2 consists of all products c_1c_2, $c_1\in C_1$, $c_2\in C_2$. Essential use is made of the minimax theorem of H. Kneser-Ky Fan. (Received November 15, 1954.)

264. John McCarthy: *Stability of invariant manifolds.*

Let T: $(W, V)\to(W, V)$ be a differentiable homeomorphism where V is a compact differentiable submanifold of the manifold W. V is called an invariant manifold of

T and is said to be stable if for any differentiable family $T_\lambda: W \to W$ such that $T_0 = T$ there is a continuous family $V_\lambda(|\lambda| < c)$ of submanifolds such that $T_\lambda: V_\lambda \to V_\lambda$ and $V_0 = V$. Let $\alpha = T | V$. A Riemannian metric is put on W and a 1-1 correspondence between the space of neighboring manifolds of V and the space N of normal vector fields to V is established. T defines a linear map of N into itself which in a pair (β, γ) of coordinate systems covering a point v and its image $\alpha(v)$ is described by a matrix $A_{\beta\gamma}(v)$. It is shown that V is stable if $\|A_{\beta\gamma}(v)\| < 1$ and $\|\alpha'(v)^{-1} A_{\beta\gamma}(v)\| < 1$ for all β, γ, v. (The norms are in the sense of Banach and $\alpha'(v)$ is the Jacobian matrix of $\alpha = T | V$.) This theorem is applied to give necessary and sufficient conditions for stability in some special cases. (Received November 12, 1954.)

265. G. R. MacLane: *A convergence theorem and Riemann surfaces.*

Let G be a Riemann surface over the w-sphere such that (1) there exists a set Γ, of arcs on G, which splits G into two simply-connected components, G' and G'', (2) there exists a 1-1 inversely conformal map of G onto G which carries w into $\bar{w}$ and leaves Γ pointwise invariant, (3) all points of G over ∞ and all algebraic branch points of G lie on Γ, and (4) all singularities of G lie over $\Im w = 0$. Then G is the image of $D = Z - E$ by $w = f(z)$ where Z is the z-sphere, E is a closed proper subset of the real axis, $f(z)$ is meromorphic in D, $f'(z)$ has only real zeros and poles, and any arc boundary value of $f(z)$ is real. Also, $f(z)$ is the limit of a naturally defined sequence of rational functions $R_n(z)$, where $R_n'(z)$ has only real zeros and poles. $R_n(z)$ may be chosen so that D is the maximal domain of convergence. Some results are obtained concerning the possible singularities of $f(z)$. Despite the restrictions on the surface G, the class of functions $f(z)$ is remarkably wide, including, for example, a simple transformation of the modular function. (Received November 12, 1954.)

266*t*. J. S. MacNerney: *Weakly reproducing kernels.*

Let S be a linear orthogonal complete space, with inner product function Q, and let E be a set. Suppose S_1 is a LOC space of functions from E to S, with inner product function Q_1, and suppose that for each t in E and each x in S the linear function L_{tx}, where $L_{tx}(f) = Q(f(t), x)$, from S_1 to the numbers is continuous. *Theorem* A: For each t in E the linear function L_t, where $L_t(f) = f(t)$, from S_1 to S is continuous. *Theorem* B: There is a function K from $E \times E$ such that (i) for s and t in E, $K(s, t)$ is a continuous linear transformation from S to S with $K(s, t)^* = K(t, s)$ and $K(t, t) \geqq 0$, and (ii) for t in E and x in S the function h, where $h(s) = K(s, t)x$ for s in E, belongs to S_1 and $Q_1(f, h) = Q(f(t), x)$ for all f in S_1. *Theorem* C: For each t in E and each x in S, $|L_{tx}|^2 = Q(K(t, t)x, x)$ and $|L_t|^2 = |K(t, t)|$. (Received November 15, 1954.)

267. Arne Magnus: *On polynomial solutions of a differential equation.*

Let $u = u(x, y)$ and $v = v(x, y)$ denote two polynomials of degrees $m \geqq 2$ and $n \geqq 2$, respectively. If the Jacobian $u_x v_y - u_y v_x = k =$ constant and m and n are relatively prime, then $k = 0$ and there exists a polnomial h of first degree in x and y such that u and v are polymonials in h. The proof employs a formula developed in a previous paper (Proc. Amer. Math. Soc. vol. 5 (1954) p. 258) expressing u in terms of v and a recurrence relation between generalized Legendre polynomials. (Received November 15, 1954.)

268*t*. J. L. Massera: *Contributions to stability theory.* I.

Lyapunov proved that if A is a real matrix whose characteristic roots have negative real parts and if $U(x)$ is a positive definite algebraic form of degree m in the components of the vector x, there is one and only one negative definite form $V(x)$ of degree m such that $(\partial V/\partial x)\cdot Ax = U$ (where $\partial V/\partial x$ represents the gradient and the dot, scalar product); if at least one characteristic root has a positive real part there is a form V assuming positive values and a number $a>0$ such that $(\partial V/\partial x)\cdot Ax = aV+U$; as a consequence, Lyapunov's theorems on asymptotic stability and on instability admit reciprocals for the class of linear autonomous systems. The following generalization is proved: if the real parts of the characteristic roots are $\leqq 0$ and the elementary divisors of those roots having vanishing real parts are linear, then a quadratic positive definite form $V(x)$ exists such that $(\partial V/\partial x)\cdot Ax$ is negative semidefinite; the reciprocal of Lyapunov's theorem on (simple) stability for linear autonomous systems follows. (Received November 18, 1954.)

269*t*. J. L. Massera: *Contributions to stability theory*. II.

Četaev [Doklady Akad. Nauk SSSR. vol. 1 (1934) pp. 529–531 and Učenye Zap Kazanskogo Gos. Univ. vol. 98 (1938) pp. 43–58] proved certain instability criteria which are generalized as follows. Let $\dot{x} = f(x, t)$, where x, f are n-vectors and $\dot{x} = dx/dt$, be a system of differential equations defined in the set $S_n(a)\times J$, where $S_n(a)$ is the solid sphere $\|x\| \leqq a$ and J the interval $0 \leqq t < +\infty$; f is assumed to be sufficiently regular and $f(0, t) = 0$. Let $G \subset S_n(a)\times J$ be an open set and $V(x, t)$ a function defined in the closure of G, bounded and positive in G, $V(0, t) = 0$, and such that, in G, $dV/dt = (\partial V/\partial t)+(\partial V/\partial x)\cdot f(x, t) \geqq \beta(\alpha)\cdot\gamma(t)$ whenever $V(x, t) \geqq \alpha > 0$; $\beta(\alpha)$ is any positive continuous nondecreasing function and $\gamma(t) > 0$, $\int^{+\infty}\gamma(t)dt = +\infty$. Then the solution $x = 0$ is unstable if one of the following complementary assumptions is satisfied: (a) $V(x, t) = 0$ for any boundary point of G such that $\|x\| < a$ and G has at least one boundary point $(0, T)$, $T>0$; (b) G is bounded by smooth hypersurfaces and at any boundary point (x, t), $0<\|x\|<a$, the vector f points towards the interior of G; moreover, G has a boundary point $(0, T)$, $T>0$; (c) G is bounded by smooth hypersurfaces and at any boundary point (x, t), $0<\|x\|<a$, the vector f points outwardly; moreover, a sequence $t_n \to +\infty$ and a sequence $G_n \subset G$ exist such that $(x, t) \in G_n$ implies $t \geqq t_n$, and the sets $G_n \cup \{(0, t): t \geqq 0\} \cup \{(x, t): \|x\| = a, t \geqq 0\}$ are connected. Theorems (a) and (b) are true in infinite-dimensional spaces. (Received November 18, 1954.)

270*t*. J. L. Massera: *Contributions to stability theory*. III.

The solution $x = 0$ of the system $\dot{x} = f(x, t)$ is said to be ***uniformly stable*** if given $\epsilon>0$ there exists a $\delta(\epsilon)>0$ (independent of $t_0 \geqq 0$) such that $\|x_0\| < \delta$ implies $\|F(t, x_0, t_0)\| < \epsilon$ for $t \geqq t_0$, where $x = F(t, x_0, t_0)$ is the solution through (x_0, t_0); it is ***uniform-asymptotically stable*** if, in addition, there exist a $\delta_0>0$ (independent of ϵ and t_0) and a $T(\epsilon)>0$ (independent of t_0) such that $\|F(t, x_0, t_0)\| < \epsilon$ for any $\|x_0\| < \delta_0$ and $t \geqq t_0 + T(\epsilon)$. The following are generalizations of previous results of Malkin [Prikl. Mat. Meh. vol. 18 (1954) pp. 129–138], Barbašin [Mat. Sbornik vol. 29 (1951) pp. 233–280] and the author [Ann. of Math. vol. 50 (1949) pp. 705–721]. 1. If the system is autonomous or periodic, stability (asymptotic stability) implies uniform stability (uniform-asymptotic stability). 2. If f is bounded and of class C^r, $r \geqq 1$, and if $x = 0$ is uniform-asymptotically stable, a positive definite function $V(x, t)$ of class C^r exists, having bounded partials, such that dV/dt (see II) is negative definite; if f is periodic in t (independent of t), V may be chosen periodic in t (independent of t). The proof of Theorem 2 is based on a generalization of a lemma proved by the author (loc. cit.). (Received November 18, 1954.)

271*t*. J. L. Massera: *Contributions to stability theory*. IV.

The solution $x=0$ is said to be *asymptotically stable in the large* [Barbašin-Krasovskiĭ, Doklady Akad. Nauk SSSR. vol. 86 (1952) pp. 453–456] if it is stable and furthermore every solution tends to zero as $t\to+\infty$. The following are generalizations of results of Barbašin-Krasovskiĭ. 1. If a positive definite function $V(x, t)$ exists having an infinitely small upper bound (i.e. given $\epsilon>0$ there is a $\delta>0$ such that $V(x, t)<\epsilon$ for $\|x\|<\delta$, $t\geqq 0$) and which is infinitely large (i.e. given $M>0$ there is an $N>0$ such that $V(x, t)\geqq M$ when $\|x\|\geqq N$) and such that dV/dt (see II) is negative definite, then $x=0$ is asymptotically stable in the large. 2. If the system is autonomous or periodic, $f\in C^r$, $r\geqq 1$, $x=0$ is asymptotically stable in the large and if every solution exists in the past, then a positive definite function $V(x, t)$ exists, $V\in C^r$, the partials of V being bounded in any cylinder $\|x\|\leqq a$ (whence V admits an infinitely small upper bound), V infinitely large, such that dV/dt is negative definite (V is independent of or periodic in t, respectively). (Received November 18, 1954.)

272*t*. J. L. Massera: *Contributions to stability theory*. V.

The following theorems precise previous results on stability in the first approximation due to Malkin [Doklady Akad. Nauk SSSR. vol. 76 (1951) pp. 783–784] and Lyapunov [see Četaev, Prikl. Mat. Meh. vol. 12 (1948) pp. 639–642]. 1. Let $f(x)$ be a homogeneous vector function of degree m; if the solution $x=0$ of the system $\dot{x}=f(x)$ is asymptotically stable, a number $M>0$ exists such that for any function g satisfying $\|g(x, t)\|\leqq M\cdot\|x\|^m$, the solution $x=0$ of the system $\dot{x}=f(x)+g(x, t)$ is uniform-asymptotically stable (cf. III). 2. Suppose that the order numbers $\chi_1, \cdots, \chi_n$ (negatives of the "characteristic numbers" of Lyapunov) of the linear system $\dot{x}=A(t)\cdot x$ (A: bounded matrix) satisfy the condition $\chi_i<-\sigma/(m-1)$, $m>1$, where $\sigma=\sum\chi_i-\liminf_{T\to+\infty}(1/T)\int_{t_0}^{T}\operatorname{tr}A(\tau)d\tau\geqq 0$ is the degree of irregularity of the linear system; then for any g such that $\|g(x, t)\|\leqq\|x\|^m$ (uniformly in t) the solution $x=0$ of the system $\dot{x}=A(t)\cdot x+g(x, t)$ is asymptotically stable. (Received November 18, 1954.)

273*t*. J. L. Massera: *Contributions to stability theory*. VI.

The following theorem includes results by Dini-Hukuwara [Hukuwara, Journal Fac. Sci. Hokkaido Univ. (1) vol. 2 (1934) pp. 13–88] and Caligo [Atti II Congresso Unione Mat. Italiana, Bologna, 1940, pp. 177–185]. If a fundamental matrix solution $R(t)$ of the linear system $\dot{x}=A(t)\cdot x$ satisfies $\|R(t)R^{-1}(t_0)\|\leqq N\cdot e^{-\nu(t-t_0)}$, $\nu>0$, $t\geqq t_0$, and if $\int^{+\infty}[\|B(t)\|-(\nu/N)]dt<+\infty$, the system $\dot{x}=[A(t)+B(t)]\cdot x$ is stable; if the integral above diverges to $-\infty$, the system is asymptotically stable. The condition on R has been previously considered by Malkin, Persidskiĭ and Kreĭn. (Received November 18, 1954.)

274*t*. J. L. Massera: *Contributions to stability theory*. VII.

The solution $x=0$ of $\dot{x}=f(x, t)$ is said to be *totally stable* ("stable under constantly acting perturbations" of the Soviet mathematicians) if given $\epsilon>0$ there is a $\delta>0$ such that $\|G(t, x_0, t_0)\|<\epsilon$ for $t\geqq t_0$, $\|x_0\|<\delta$, where $x=G$ is the general solution of the system $\dot{x}=f(x, t)+g(x, t)$, $\|g(x, t)\|<\delta$. The following are partial reciprocals of a theorem due to Goršin and Malkin [Goršin, Izvestiya Akad. Nauk Kazahskoĭ SSR, Ser. Mat. Meh. vol. 2 (1948) pp. 46–73; Malkin, Prikl. Mat. Meh. vol. 18 (1954) pp. 129–138]. 1. If the solution $x=0$ of the linear system $\dot{x}=A(t)\cdot x$ is totally stable, it is asymptotically stable; 2. If the solution $x=0$ of the autonomous or periodic system $\dot{x}=f(x, t)$ is totally stable, it is asymptotically stable. (Received November 18, 1954.)

275. W. A. Michael: *Singular integral equations with normal kernels.*

Let $K(x, y)$ be a complex-valued kernel, measurable in the square $0 \leq x, y \leq 1$, which is *singular* in the sense of Carleman, i.e., $|K(x, y)|^2$ is integrable in each variable separately but not necessarily in (x, y). If $K(x, y)$ is the mean-square limit in the separate variables x and y of a sequence $\{K_n(x, y)\}$ of Fredholm kernels which are *normal*, i.e., $\int_0^1 K_n(x, s)\overline{K_n(y, s)}ds = \int_0^1 \overline{K_n(s, x)}K_n(s, y)ds$ for almost all (x, y), and if the $K_n(x, y)$ satisfy a certain continuity condition, then $K(x, y)$ has spectral representations $K(x, y) \sim \int \lambda dE(x, y; B_\lambda)$ of Carleman type which lead to formulas for solutions of the integral equation $u(x) - \mu\int_0^1 K(x, s)u(s)ds = f(x)$. The spectral function $E(x, y; B)$, defined for the Borel sets B of the complex plane, is obtained by means of a compactness argument inspired by a theorem of Frostman to the effect that every uniformly bounded sequence of measures (on a suitable space) has a subsequence which converges (in a certain sense) to a measure. (Received November 15, 1954.)

276*t*. E. P. Miles, Jr. and Ernest Williams: *General solutions for homogeneous second order partial differential equations in k variables.*

Consider the equation E: $\sum_{i,j=1}^{k} a_{ij}D_{x_i x_j}U = 0$ with real coefficients having matrix $\|a_{ij}\|$ of rank k. By a suitably chosen nonsingular linear transformation T on the set $\{D_{x_i x_j}\}$, E becomes E^*: $\sum_{i=1}^{k} b_i D_{x_i^* x_i^*}U = 0$ with all b_i either plus or minus 1. The author's basic sets of polynomial solutions for the k variable Laplace or wave equation are readily generalized to provide solutions for any sign distributions in E^* of the form $U^* = \sum_{m=0}^{\infty}\sum_{i=1}^{C_{m+k-1,k-1}-C_{m+k-3,k-1}} MW_i^m(x_1^*, \cdots x_k^*, E^*)$. Using the relation $x_i^* = \sum_{j=1}^{k} c_{ij}x_j$, the inverse of the nonsingular linear transformation induced on the x_i's by T, U^* is converted to the desired solution U. (Received November 15, 1954.)

277B. G. M. Muller: *On certain infinite integrals involving Bessel functions.*

By the use of a well-known lemma from the theory of hypergeometric functions, it is possible to express various classes of such integrals in terms of elementary, or at least tabulated, functions. As an illustration, it is shown that the integral $\int_0^\infty K_p(t)I_q(ts)t^r dt$ can be expressed in terms of the complete elliptic integrals of the first and the second kind of parameter s, provided the three numbers q, $(q+r\pm p)/2$ are positive integers or zero. A particular case of this result is used in the evaluation of a certain improper integral. (Received November 8, 1954.)

278. O. G. Owens: *A wave-equation with an integral-condition.*

Let Ω denote the 2-dimensional cartesian plane with coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$; let $F(\theta)$ denote any prescribed function of class C^2 and with period 2π; finally, l will denote an arbitrarily chosen, but hereafter fixed, constant with $0 < l < 3/2$. Then, there exists a unique $u(x_1, x_2)$ on Ω, which thereon is a solution of the 2-dimensional wave-equation, $u_{x_1x_1} + u_{x_2x_2} + u = 0$, and is such that $\int_0^\infty u(r, \theta)r^{l-1}dr = F(\theta)$, the convergence of the integral being uniform in θ. (Received November 12, 1954.)

279*t*. Pasquale Porcelli: *A Stieltjes integral representation for bounded analytic functions.*

The following theorem is proved: If $f(z)$ is analytic and bounded for $|z|<1$, then there exists a unique continuous function γ (possibly complex valued) of bounded variation on $[0, 1]$ such that $f(z)=[(1+z)/(1-z)]\int_0^1(1/(1+(1-z)^{-2}4zu))d\gamma(u)$, for each $|z|<1$. The proof involves results of H. S. Wall. (Received November 12, 1954.)

280. R. A. Raimi: *Equicontinuity of linear transformations.*

Let (E, E') and (F, F') be linear systems (Mackey, Trans. Amer. Math. Soc. vol. 57 (1945) pp. 155–207), and let $L(E, F)$ be the space of weakly continuous linear transformations on E to F. For various topologies on E and F, consistent with the linear systems, criteria are given for equicontinuity of a subset $U\subset L(E, F)$, using the Arens-Bourbaki scheme of defining topologies by means of dual sets (e.g. Duke Math. J. vol. 14 (1947) pp. 787–794). If E and F are given weak topologies, U is equicontinuous if and only if $U'y'$ is bounded and finite-dimensional for each $y'\in F'$, where U' is the set of operators adjoint to the elements of U. When E and F are given their k-topologies (Arens, op. cit.), equicontinuity of U may be related to compactness of U' in a suitable topology for $L(F', E')$. The method also produces the theorem: All the topologies of E consistent with the linear system (E, E') may be distinguished by the specification of the equicontinuous subsets of $L(E, E)$. If for two topologies on E, the space of continuous linear transformations is the same, then the stronger will admit the larger equicontinuous sets. (Received November 10, 1954.)

281. L. V. Robinson: *Lie infinitesimal transformation groups of one parameter and differential transforms.*

From differential transforms of the type $e^{t[\xi(x,y)D_x+\eta(x,y)D_y]}$, $D_x=\partial/\partial x$, $D_y=\partial/\partial y$, the Lie infinitesimal transformation groups of parameter t are shown to have their origins. The natures and limitations of these are pointed out then. (Received November 15, 1954.)

282*t*. W. G. Rosen: *On invariant means over compact semigroups.*

A *mean* on a topological semigroup Σ is a positive element of norm one in $C(\Sigma)^*$, where $C(\Sigma)$ is the space of real-valued bounded continuous functions on Σ. Let Σ be a compact semigroup. 1. The right and left regular representations of Σ over $C(\Sigma)$ are w^*-continuous. 2. There is a right invariant mean if and only if Σ contains a unique minimal left ideal. 3. There is a two-sided invariant mean if and only if the kernel of Σ (see Numakura, Math. J. of Okayama Univ. vol. 1 (1952) pp. 99–108) is a group. If such a mean exists, it is unique. 4. If Σ has a right invariant mean, then the kernel of Σ is a direct product as regards its topology, its algebraic structure, and its measure. 5. If a right invariant mean on Σ is unique, then it is two-sided invariant. (Received October 21, 1954.)

283*t*. W. G. Rosen: *On invariant means over topological semigroups.* Preliminary report.

A *mean* on a topological semigroup Σ is a positive element of norm one in $C(\Sigma)^*$, where $C(\Sigma)$ is the space of real-valued bounded continuous functions on Σ. Σ is called *weakly continuously representable* (abbreviated WCR) if its right and left regular representations over $C(\Sigma)$ are weakly continuous. Discrete and compact semigroups are WCR. A theorem by Day (Trans. Amer. Math. Soc. vol. 69 (1950) pp. 276–291) is extended from discrete semigroups to WCR semigroups. If Σ is a WCR semigroup, then there is an invariant mean if and only if every bounded w^*-continuous representation of Σ is ergodic, and if and only if the regular representations of Σ over $C(\Sigma)$ are ergodic. Also, if Σ is a WCR semigroup with an invariant mean in $C(\Sigma)^*$, then all invariant means assume the same value at an element $x\in C(\Sigma)$ if and only if x lies in the ergodic subspace of the regular representations. (Received October 21, 1954.)

284*t*. Walter Rudin: *On a problem of Bloch and Nevanlinna.*

If f is bounded and analytic in the open unit disc U, is the derivative f' of bounded characteristic in U? This is the problem referred to in the title. A counter-example (a Blaschke product) has been found by Forstman. It is now shown that the answer is negative even if f is the sum of a power series which converges absolutely on the boundary of U. More precisely, a function $f(z)=\sum c_k z^{n_k}$ is constructed such that $\sum |c_k| < \infty$, $n_k/n_{k-1}\to\infty$, and $f'(re^{i\theta})\to\infty$ as $r\to 1$, for almost all θ. (Received November 8, 1954.)

285. Walter Rudin: *Square roots of absolutely convergent Fourier series.*

Let A be the class of functions defined on the unit circumference C whose Fourier series converge absolutely. A well-known theorem of Wiener and Lévy implies that if $f\in A$ and $f(x)>0$ for all $x\in C$, then $f^{1/2}\in A$. It is shown, by complex variable methods, that there exists a function $f\in A$ such that $f(x)>0$ on C except at one point x_0, and such that $f^{1/2}\notin A$. This leads to the following conjecture: If G is an infinite locally compact abelian group with dual group G^*, there exists a non-negative function f on G which is the Fourier transform of a function $f^*\in L^1(G^*)$, although $f^{1/2}$ is not such a transform. Apart from the case $G=C$, the conjecture is proved if G is the real line, and if G is discrete. (Received November 8, 1954.)

286. P. P. Saworotnow: *On the generalisation of the notion of H^*-algebra.* Preliminary report.

An H^*-algebra has a property that the orthogonal complement of every ideal is an ideal of the same kind. The author considers the converse problem. It is proved that if a Banach algebra $\mathfrak{a}$ is a semi-simple dual ring, is a Hilbert space, and has the property that the orthogonal complement of every right ideal is a right ideal, then $\mathfrak{a}$ is a right H^*-algebra as well as a left H^*-algebra (though the involutions $x\to x^r$ such that $(yx, z)=(y, zx^r)$ and $x\to x^l$ such that $(xy, z)=(y, x^l z)$ may not coincide). The author also shows that every simple algebra of the type described above is isomorphic to the algebra of all linear transformations (a_{ij}) from $L^2(J)$ into $L^2(I)$ of the Hilbert-Schmidt type (where $I=(i)$, $J=(j)$) with the multiplication $(ab)(k, l)=\sum_{j,i} a_{kj}\lambda_{ji}b_{il}$ where (λ_{ji}) is a nondecreasing linear operator from $L^2(I)$ onto $L^2(J)$ which has an inverse transformation. Both results are obtained by proving the structure (Wedderburn) theorems. (Received November 10, 1954.)

287*t*. H. M. Schaerf: *Metric transitivity and existence of means.* Preliminary report.

Let m be a σ-finite content on a σ-algebra S of subsets of a set X. Let H be a group of mappings of X into X which preserve both measurability and null sets of m. A mapping k of X into X is termed *H-pieced* if there is both a partition of X into a sequence $\{X_i\}\subset S$ and a sequence $\{h_i\}\subset H$ such that the sequence $\{h_iX_i\}$ is disjoint and $kx=h_ix$ for $x\in X_i$ $(i=1, 2, \cdots)$. A number M is called a *uniform* (H, m)*-mean* of a function f defined on X, if, for every number $\epsilon > 0$, there is a null set N_ϵ of m and a finite sequence of H-pieced mappings $h_1, \cdots, h_n$ such that $\left| M - (1/n) \sum_{i=1}^{n} f(h_i x) \right| < \epsilon$ for $x \notin N_\epsilon$. *Theorem*: In order that every bounded measurable function on X have a uniform (H, m)-mean it is necessary and sufficient that H be metrically transitive for m. If there is a finite measure invariant under H and weaker than m, then each uniform mean is unique. (Received November 26, 1954.)

288t. H. M. Schaerf: *Metric transitivity in product spaces*. I. Preliminary report.

Generalizing the usual terminology, a binary relation R on a σ-algebra S of subsets of a set X will be called ***metrically transitive*** for a measure m on S if every set $A \in S$ which m-almost contains every R-image of any of its subsets is either a null set or the complement of a null set of m. Let S be the cartesian product of a family of σ-algebras S_i of subsets of sets X_i. Then every measure m_i on S_i of the form $m_i(E) = m(E \times F)$ with $E \in S_i$, $F \in \mathrm{P}_{j \in I-(i)} S_j$, will be called a ***projection of m on*** S_i. Let R_i be a binary relation on S_i which preserves finite partitions. Write $(\mathrm{P}_{i \in I} A_i) R (\mathrm{P}_{i \in I} B_i)$ if there is $j \in I$ with $A_j R_j B_j$ and $A_i = B_i$ for $i \neq j$. *Theorem* 1. If each projection m_i of m on S_i is invariant under R_i and R_i is metrically transitive for m_i, then R is, in general, metrically transitive for m. (Certain weak regularity conditions for m are required.) *Corollary*. If m is the direct product of measures m_i invariant under R_i, then R is metrically transitive for m if and only if each R_i is metrically transitive for m_i. (Received November 26, 1954.)

289t. H. M. Schaerf: *Metric transitivity in product spaces*. II. Preliminary report.

The notations of the preceding report are kept. A set H of mappings of X into X is called ***metrically transitive*** for the measure m if so is the binary relation R where ARB means $A = hB$, $h \in H$. *Theorem* 2. For every i in I, let H_i be an at most countable set of 1-to-1 mappings of X_i into X_i which preserve both measurability and null sets of a measure m_i on S_i. Let H be the set of all sequences $\{h_j\}$ where $h_i \in H_i$ for some $i \in I$ while h_j is the identity mapping of X_j for $j \in I-(i)$. Then H is metrically transitive for the direct product of the measures m_i if each H_i is metrically transitive for m_i. All measures mentioned in these reports are assumed to be σ-finite. (Received November 26, 1954.)

290. Morris Schreiber: *Generalized spectral resolution for operators in Hilbert space*.

A "spectral" decomposition in terms of positive-operator-valued set functions is developed which holds for arbitrary bounded operators. Proceeding from a theorem of Nagy (Acta Szeged. vol. 15 (1953) pp. 87–92) stating that the powers of an operator A with $\|A\| \leq 1$ are simultaneously compressions of the powers of a unitary operator on a larger space, it follows that the compression of the spectral measure of the unitary operator associated to A is a set function of the desired type ("operator measure") and is unique. It is proved that the support of the operator measure of A (always a subset of $|z| = 1$) does not divide the plane if and only if A is unitary. A functional calculus of the usual type obtains for boundary values of functions analytic in $|z| < 1$ and bounded on $|z| = 1$. Those sequences of operators which can be expressed simultaneously in terms of a single operator measure are characterized (generalizing a result of Nagy, Acta Hungaricae vol. 3 (1952) pp. 289–292) by the condition that if $c_0+\sum c_n e^{int}+\sum c_{-n}e^{-int}$ is a positive trigonometric polynomial, then $c_0A_0+\sum c_nA_n+\sum c_{-n}A_n^*$ is a positive operator, and related results on order and commutativity are obtained. (Received November 15, 1954.)

291. I. E. Segal: *Tensor algebras over Hilbert spaces.*

The algebra of all covariant tensors over a complex Hilbert space H is shown to contain no "completely characteristic" ideals that are maximal with respect to not containing all tensors from a certain rank onwards other than the symmetric and skew-symmetric ideals. The corresponding symmetric quotient algebra is shown to be unitarily equivalent to the space $L_2(H')$ of "square-integrable" functionals over a real Hilbert space H' of which H is the complexification, in a canonical fashion which clarifies the structure of various linear operators on the algebras. Among these is the Plancherel transform, which like some features of the theory of Lebesgue integration, is extended from the classical case when H' is finite-dimensional to the case of a space of arbitrary dimension. (Received November 3, 1954.)

292*t*. R. L. Shively: *Pseudo Laguerre polynomials.*

The classical Laguerre polynomials $L_n^{(\alpha)}(x)$ may be defined by the series $L_n^{(\alpha)}(x)=((\alpha+1)_n/n!)_1F_1(-n;\alpha+1;x)$. Most authors do not distinguish between the case in which α is independent of n, and the case in which α depends on n. For the purpose of this paper $L_n^{(\alpha)}(x)$ is defined to be a proper Laguerre polynomial if α is independent of n, and a pseudo Laguerre polynomial if α depends on n. The particular pseudo Laguerre polynomials $R_n^{(a)}(x)=((a+n)_n/n!)_1F_1(-n;\ a+n;\ x)$ are studied in order to learn something about the effect of changing a parameter $(\alpha+1)$ to $(a+n)$, a and α independent of n, in a hypergeometric polynomial. It is found that many of the important properties of the proper Laguerre polynomials are either ambiguous or totally meaningless when $(\alpha+1)$ is replaced by $(a+n)$, a and α independent of n. A generalization of $R_n^{(a)}(x)$, obtained by inserting an arbitrary number of numerator and denominator parameters all independent of n, is also studied in some detail. (Received November 12, 1954.)

293. C. J. Standish: *Inversion of a generalised Poisson transform.*

Let $f(x)=\pi^{-1}\int_{-\infty}^{\infty}[A(x-t)+B/(x-t)^2+C^2]g(t)dt$ where $g(t)$ is Lebesgue integrable on every finite interval. Convergence properties of the transform are investigated and real and complex inversion formulas are obtained. The real inversion formula in the special case of the Poisson transform $(A=0,\ B=C=1)$ was discovered by Pollard (*The Poisson transform*, Trans. Amer. Math. Soc., to appear). (Received September 15, 1954.)

294B. Warren Stenberg: *On sequences with divergent total variation.*

Let $a_1, a_2, \cdots$ be a sequence of positive terms tending to zero with $\sum_{n=1}^{\infty} a_n=\infty$. It is proved that there is a univalent sequence $k_1, k_2, \cdots$ of positive integers such that the sequence $b_1, b_2, \cdots$, where $b_n=a_{k_n}$ has $\sum_{n=1}^{\infty} b_n=\infty$ and has the following property. If $c_1, c_2, \cdots$ is a subsequence of $b_1, b_2, \cdots$ with $\sum_{n=1}^{\infty} c_n=\infty$, then also $\sum_{n=1}^{\infty}|c_n-c_{n+1}|=\infty$. Even further, the sequence $b_1, b_2, \cdots$ may be so constructed that if $c_1, c_2, \cdots$ is a subsequence of $b_1, b_2, \cdots$ then for each positive integer m, $\sum_{n=1}^{m-1}|c_n-c_{n+1}|\geq 2\sum_{n=1}^{m}c_n-5b^*$, where b^* is the greatest of the terms of $b_1, b_2, \cdots$. Clearly the constant 2 cannot be increased; b^* cannot be replaced by c^*, the largest of the terms $c_1, c_2, \cdots$, even if 5 is replaced by an arbitrarily large positive number and 2 by an arbitrarily small positive number. (Received November 13, 1954.)

295. Wilhelm Stoll: *The singularities of meromorphic modifications.*

Let G and H be $2n$-dimensional complex manifolds, let τ be an analytic homeomorphism of an open subset $A\subseteq G$ onto $B\subseteq H$, and let $v=\tau^{-1}$. $\mathfrak{M}=\mathfrak{M}(G, A, M, \tau/H, B, N, v)$ is said to be a modification provided $M=G-A$, $N=H-B$ are subsets of analytic sets of dimension smaller than $2n$. The modification $\mathfrak{M}^{-1}=\mathfrak{M}(H, B, N, v/G, A, M, \tau)$ is the inverse of $\mathfrak{M}$. $\mathfrak{M}$ is open provided $\tau(U\cap A)\cup N$ is open when U is an open neighborhood of M. $\mathfrak{M}$ is said to be meromorphic provided that for each 2-dimensional complex manifold $L\subset G$ with $L\cap M=\{P_0\}$ the set consisting of all points $Q=\lim_{n\to\infty}\tau(P^n)$, with $\lim_{n\to\infty}P^n=P_0$ for a proper sequence $P^n\in L\cap A$, contains at most one point. For $n=2$ one has the theorem: If $\mathfrak{M}$ is meromorphic, if $\mathfrak{M}$ and $\mathfrak{M}^{-1}$ are open, then for each singular point P_0 of τ there exist a 2-dimensional analytic set $C(P_0)\subseteq N$ and a set $D(P_0)\subset C(P_0)$ without an accumulation point such that v is regular on $C(P_0)-D(P_0)$ and $v^*(Q)=P_0$ for $Q\in C(P_0)-D(P_0)$, where v^* is the continuation of v. Therefore $P_0\rightarrow C(P_0)$ is a one-to-one map. An easy consequence is: If H has a countable base of open sets, then the singularities of τ are isolated. (Received November 8, 1954.)

296B. George Swift: *n-valued irregular Borel measures.*

A Borel measure is n-valued on a topological space X if it assumes exactly n different values on X, where n is a positive integer. In this paper, it is shown that every n-valued, totally finite, irregular Borel measure on a σ-compact Hausdorff space is outer irregular at some point of the space. Furthermore, an n-valued, totally finite, irregular Borel measure μ on a σ-compact or locally compact Hausdorff space X is outer irregular at at most m points where m is the greatest integer less than or equal to $\mu(X)/\inf\{\mu(B): \mu(B)>0, B \text{ is a Borel set}\}$. Necessary and sufficient conditions on a σ-compact Hausdorff space X for the existence of a totally finite n-valued irregular Borel measure μ are that X be uncountable and that X possess a point x_0 and a class $\mathcal{A}$ of Borel sets such that (1) $\{x_0\}\notin\mathcal{A}$ and $x_0\in A^-$ for all $A\in\mathcal{A}$, (2) $\{A_i\}_{i=1}^{\infty}\subset\mathcal{A}$ implies that there exists an $A\in\mathcal{A}$ such that $A\subset\bigcap_{i=1}^{\infty}A_i$, and (3) $A\in\mathcal{A}$ and $A=B_1\cup B_2$, where B_1, B_2 are disjoint Borel sets, imply that there exists an $A_0\in\mathcal{A}$ such that either $A_0\in B_1$ or $A_0\in B_2$. (Received November 12, 1954.)

297t. C. T. Taam: *Schlicht functions and linear differential equations of second order.*

It is known that an analytic function $f(z)$ is schlicht in a region R if and only if no solution of (A): $W''+Q(z)W=0$ has more than one zero in R, where $Q(z)$ is one-half of the Schwarzian derivative of $f(z)$ with respect to z. In this paper several sufficient criteria for the schlichtness of $f(z)$ are obtained for different regions R by proving that no solution of (A) has more than one zero in R. Two basic theorems are used to establish the results: one is an extension of a nonoscillation criterion of Wintner to the complex plane (A. Wintner, *On the non-existence of conjugate points*, Amer. J. Math. vol. 73 (1951) pp. 368–380) and the other is a comparison theorem (C. T. Taam, *On the solutions of second order linear differential equations*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 876–879). The second theorem yields results which are either the same as those obtained by Z. Nehari (Lecture Notes, Conference on Functions of a Complex Variable at University of Michigan, 1953, pp. 148–151) or slight generalizations of his. (Received November 15, 1954.)

298. C. J. Titus: *Linear vector spaces of interior mappings.*

Let D be a domain in the xy-plane. Let F be the family of continuously differentiable mappings of D into the plane which are nowhere sense-reversing. Let F_0 be the family of mappings in F with the property that the Jacobian of any mapping in F_0 is zero only at points where it is rank zero. All maximal real linear vector spaces in F_0 are characterized. If V is such a vector space, then V is maximal even in F. A space V is completely determined by a pair of mappings which are independent in a certain sense. The relationship to quasi-conformal mappings and to pseudo-analytic functions of Bers is discussed. (Received November 15, 1954.)

299. Albert Wilansky (p) and Karl Zeller: *Controlled summability matrices.*

Classical treatments of summation methods have begun by showing that a matrix A satisfies (*) $\left|\sum_{k=0}^{m} a_{rk}x_k\right| \leq M\sup_{0\leq n\leq r}\left|\sum_{k=0}^{n} a_{nk}x_k\right|$. (Cf. Hardy-Riesz, *Dirichlet's series*, 1915, p. 28, Lemma 7, and p. 29, lines 12–17.) We apply functional analysis to unify the methods and results, calling A *controlled* if (**) $\{\sum_{k=0}^{m} a_{nk}x_k\}$ is bounded for each $x\in c_A=\{x \mid \lim_n \sum a_{nk}x_k \text{ exists}\}$. This implies (*) for triangles ($a_{nn}\neq 0$, $a_{nk}=0$ for $k>n$). By p. 107 in Banach's treatise, (**) is equivalent to "with $\delta^n=(0, 0, \cdots, 0, 1, 0, \cdots)$, $\{\delta^i\}$ is a basis for its linear closure in c_A (one uses $\|x\|=\sup_n\left|\sum_k a_{nk}x_k\right|$), a maximal subspace of, or all of, c_A." Applications of the uniform boundedness principle, weak-star completeness, and other (F)-space techniques yield the classical results, often with strengthening and converses. Criteria occur in the general setting, e.g. (M. Riesz): $B=A^{-1}$, $b_{nk}\leq 0$ for $k<n$, $\sum_k b_{nk}\geq 0$ imply that A is controlled. Typical results: $x_n=o(a_{nn}^{-1})$ for all $x\in c_A$, hence, sometimes, A sums no divergent sequences, Corollary: Mercer's theorem. There are results on convergence factors, summability factors, comparison theorems, inversion theorems. Closely related concepts are PMI, AK used by the authors in earlier, separate, papers. (Received November 10, 1954.)

300. Y. K. Wong: *On projections in reflexive Banach spaces.*

E. R. Lorch proved that (Trans. Amer. Math. Soc. vol. 45, p. 202) if P, Q are permutable projections, then PQ and $P+Q-PQ$ are projections. In this paper, the converse is shown. If P, Q are projections, the preceding three properties are equivalent. Other equivalent conditions are stated in terms of the associated manifolds. Nagy gave an expression for $P_1+\cdots+P_n$ in Hilbert space, which can be extended to reflexive Banach spaces. Lorch's result is a special case of that of Nagy, in which $n=2$. When $n\geq 3$, the converse does not hold, in general, without further restriction. (Received November 12, 1954.)

301*t*. Y. K. Wong: *Some properties of the proper values of a matrix.*

In the paper on *Nonnegative-valued matrices* (Proc. Nat. Acad. Sci. U.S.A. (1954) pp. 121–123), the author showed that if (1) the column-sums $\sum_i a_{ij}$ are at most 1, then (2) the maximal proper value of $A=(a_{ij})$ is less than 1 if and only if (3) $\sum_{i=1}^{j} a_{ij}$ is less than 1 for all j. It is now shown that if property (1) is not assumed for nonnegative matrices, then (2) implies (3), but not conversely. However, if (1) is modified with absolute value for complex-valued matrices, then $\sum_{i=1}^{l} |a_{ij}| < 1$ is sufficient for the validity of (2), but not necessary. Without assuming condition (1), equivalent conditions for nonnegative matrices to have property (2) are given. The aim is to find some simple criteria for the validity of property (2) so as to facilitate certain numerical computations. (Received October 14, 1954.)

APPLIED MATHEMATICS

302*t*. Y. W. Chen: *On a problem of minimal surface.* II.

The present paper extends previous results (see Bull. Amer. Math. Soc. vol. 60, p. 338) to the case when the boundary is not a simple polygon but a simple closed convex curve with a finite number of corners. As before, the admission of corners on the boundary is significant. The convergence of the approximating polygon solution is based upon the following property of the minimizing integral J (which is not positive definite): If one boundary curve C is contained in the other, C', then $J(C) > J(C')$. This property is proved by an application of the subordination principle of conformal mapping. Uniqueness is obtained for the symmetrical curves, using results of Lavrentieff and Gilbarg. The discontinuous behavior of the solution $z(x, y)$ at the corner is not the same as that of a harmonic function; indeed z remains continuous in two sectors formed by the tangents and normals at the corner. The formulation of the results in terms of conjugate minimal surface is of interest. It shows that when a surface is unable to meet the boundary requirement, it may smooth out the boundary condition by itself. This explains certain well-known features in the soap films experiments. (Received November 15, 1954.)

303*t*. Jim Douglas, Jr.: *On the numerical integration of quasi-linear parabolic differential equations.*

The numerical solution of the differential equation $u_{xx} = F(x, t, u)u_t + G(x, t, u)$, $0 \leq x \leq 1$, $0 \leq t \leq T$, is treated by means of the backwards difference equation $\Delta_x^2 w_{i,n+1} = F(x_i, t_n, w_{in})(w_{i,n+1} - w_{in})/\Delta t + G(x_i, t_n, w_{in})$, where $\Delta_x^2 w_{i,n+1} = (w_{i+1,n+1} - 2w_{i,n+1} + w_{i-1,n+1})/(\Delta x)^2$. Let $\beta = \min(1, 2a)$. Then, under reasonable assumptions on F, G, the initial condition, and the boundary conditions, if $\Delta x/(\Delta t)^a$, $a > 0$, is held fixed $|u_{in} - w_{in}| = O((\Delta t)^\beta)$ as $\Delta t \to 0$, uniformly in $0 \leq x \leq 1$, $0 \leq t \leq T$. It is also shown that, in the sense of least work, the best choice of a is one-half. (Received November 9, 1954.)

304. T. C. Doyle: *Higher order invariants of stress or deformation tensors and their sysygies.*

As a preliminary investigation motivated by the admission of higher order derivatives of displacements into the elastic strain energy density, a fundamental system of *irrational* joint invariants of order p of a euclidean metric tensor $\mathbf{g}$ and a generic two index tensor $\mathbf{a}$ is derived. A *rational* system of 21 invariants of order 1 is also determined. On identifying $\mathbf{a}$ with a stress tensor $\mathbf{t}$ in the absence of body force or with an isochoric deformation tensor $\mathbf{c}$, certain syzgyies occur which are exhibited in a canonical form, i.e., as the vanishing of properly chosen base invariants. A repetition of this specialization of $\mathbf{a}$ in the second order irrational invariants introduces 6 compatibility syzygies for $\mathbf{a} = \mathbf{c}$ in addition to the once differentiated first order syzygies. (Received November 10, 1954.)

305. R. J. Duffin: *A minimax theory for overdamped networks.*

Linear dynamical systems with n degrees of freedom are considered. Such a system is characterized by three positive-definite quadratic forms: the kinetic energy form a, the dissipation form b, and the potential energy form c. If $b=0$, the system is conservative and the frequencies of the normal modes of vibration are determined by the stationary values of the Rayleigh quotient c/a. In this paper a system is termed "overdamped" if $d^2=b^2-4ac$ is positive for all virtual modes of vibration. It is shown that the $2n$ eigenvalues (decay constants) of an overdamped system are the stationary values of $(-b\pm d)/2a$. A minimax treatment of this functional is developed, in analogy to the treatment for the conservative case. It results that a constraint on the system tends to increase the eigenvalue of each normal mode. Among other things this property justifies the use of the Rayleigh-Ritz procedure for the approximate determination of the eigenvalues. (Received November 15, 1954.)

306. R. J. Duffin and Elsa Keitzer (p): *A trilinear formula relating network matrices.*

A positive real function $g(s)$ is a rational function with real coefficients whose real part is positive when the real part of s is positive. An important trilinear formula $T(f, g, s)=0$, due to P. I. Richards [Duke Math. J. vol. 14 (1947) pp. 777–786] defines a new positive real function f in terms of g. Following a suggestion of R. Bott it is shown that this formula has a generalization $T(f, g, h)=0$ where h is a positive real function whose poles are all on the imaginary axis. A positive real matrix is essentially a matrix whose quadratic form is a positive real function. The above trilinear formulae are generalized to matrices, and their properties are investigated. (Received November 15, 1954.)

307t. William H. Durfee: *Heat flow in a fluid with eddying flow.*

The differential equation for the flow of heat in a fluid flowing with eddying flow in a pipe as derived by Latzko (Zeitschrift für Angewandte Mathematik und Mechanik vol. 1 (1921) pp. 268–290) is solved by numerical methods with the aid of high speed computing machinery. The first nine eigenvalues and the tabulated values of the corresponding functions together with their first derivatives were computed. The results represent a considerable improvement both in completeness and accuracy over Latzko's computations. (Received November 12, 1954.)

308. David Gale: *The law of supply and demand.*

Let M be a market consisting of n *goods* G_j. A *price vector* $p=(\pi_1, \cdots, \pi_n)$ is a non-negative vector such that $\sum\pi_j=1$, where π_j is the unit price of G_j. A commodity bundle $x=(\xi_1, \cdots, \xi_n)$ is a vector in R_n where ξ_j, the amount of G_j in the bundle x, is positive or negative according as G_j is supplied or consumed. Let there be m economic units (individuals or firms) U_i. For each price vector p the unit U_i chooses a set of commodity bundles $S_i(p)$ subject to the condition (1) if $x_i\in S_i(p)$, then $p\cdot x_i\geq 0$ (this is the requirement that U_i be able to afford x_i at prices p). The function S_i is called the ith units *supply function*. A price vector p_0 is said to balance supply and demand if there exist bundles $x_1, \cdots, x_m$ where $x_i\in S_i(p_0)$ and $\sum x_i\geq 0$, the last condition being the requirement that the net amount of each G_j supplied be at least equal to the amount consumed. Theorem: Let the functions S_i satisfy: (a) $S_i(p)$ is compact and convex for all p; (b) the graph of each S_i is closed. Then there exists a price vector p_0 which balances supply and demand. The proof depends on a well-known lemma of elementary combinatorial topology. (Received November 5, 1954.)

309. W. T. Guy, Jr.: *A generalized Laplace transform.* II.

Two convolution and composition type results were presented for a generalized Laplace transform. The kernel of the integral transform used was a Meijer *G*-function which had been previously defined. (Received November 15, 1954.)

310B. W. C. Hoffman: *Solution of the wave equation for a randomly inhomogeneous medium.*

The Helmholtz equation $\nabla^2\psi+k^2\psi=0$, where $k=k_0\kappa^{1/2}$ is a random function of position and $|\nabla\kappa|\lambda\ll 1$, possesses plane wave solutions of the form $\psi(x, y, z)=\psi_0 \exp\{i(\omega t-k_0S)\}$, where S and ψ_0 are functions of position, the latter being slowly-varying. The differential equation $(\nabla S)^2+(i/k_0)\nabla^2 S=\kappa$ then results for S. Letting $S=\sum_0^\infty S_n r^n$ and $\kappa=\sum_0^\infty \kappa_n r^n$, $r\ll 1$, perturbation theory yields the following differential equations for S_0 and S_1: $(\nabla S_0)^2+(i/k_0)\nabla^2 S_0=\kappa_0=\text{const.}$; $2\nabla S_0\ \nabla S_1+(i/k_0)\nabla^2 S_1=\kappa_1$. The former is solved by separation of variables, and certain properties of ψ and its mean value that are implied by the form of S_0 are studied. The first perturbations of the phase and the refractive index, S_1 and κ_1 respectively, are next supposed to be stationary random processes and their respective spectral representations in terms of the orthogonal processes $\int_{\Delta\omega} s_1(\omega_1, \omega_2, \omega_3)\prod_1^3 d\omega_i$ and $\int_{\Delta\omega}\zeta_1(\omega_1, \omega_2, \omega_3)\sum_1^3 d\omega_i$ are introduced. This leads to the equation $s_1(\omega_1, \omega_2, \omega_3)=ik_0[\omega\cdot\omega-2k_0\nabla S_0\ \omega]^{-1}\zeta_1(\omega_1, \omega_2, \omega_3)$. The relation between the covariance functions S_1 and κ_1 is then determined by the *faltung* theorem, and it is shown that the covariance function of S is proportional to that of S_1. (Received November 10, 1954.)

311t. A. B. Lehman: *A note on modular lattices and series-parallel circuits.* Preliminary report.

Let $a, b, c, \cdots$ denote the elements of a series-parallel electric circuit configuration. Let $x'(a, b, c, \cdots)$ denote the polynomial representation of the two terminal circuit viewed from the nodes connecting element x. Theorem: In a modular lattice, $x'(a', b', c', \cdots)=x'(a, b, c, \cdots)$ as lattice polynomials. The proof is by substitution and induction. The necessity of the modular law may be shown by applying the theorem to a triangular circuit. More general lattice theorems on the substitution and shorting algebras of general circuits have been established. (Received December 14, 1954.)

312B. R. D. Luce: *k-stability of symmetric and of quota games.*

Let v be the 0, 1-reduced form of the characteristic function of an essential n-person game, $X=\|x_i\|$ an imputation of the game, and τ a proper partition of the players into coalitions. The pair (X, τ) is *k-stable*, where k is an integer, $0\leq k\leq n-2$, [Luce, R. D., Ann. of Math. vol. 59 (1954) pp. 357–366] if (i) for each subset S of players such that there exists a $T\in\tau$ with $|(S-T)\cup(T-S)|\leq k$, then $v(S)\leq\sum_{i\in S}x_i$ and if (ii) for each $i\in T$, where $T\in\tau$ and $|T|>1$, then $x_i>v(\{i\})=0$. The game is k-stable if it has a k-stable pair. Theorem 1. A symmetric game $[v(S)=v(|S|)]$ is k-stable if and only if $v(i)\leq i/n$ for $0\leq i\leq k+1$. Theorem 2. A quota game [Shapley, L. S., *Contributions to the theory of games*, II, Annals of Mathematics Studies, no. 28, Princeton, 1953, pp. 343–359] is 1-stable if and only if there is no weak player. Theorem 3. If (X, τ)

is k-stable in a quota game with quota $Q=\|q_i\|$, then $X=Q$ if n is odd or if n is even and $k\geqq 2$. If n is even and $k=1$, then either $X=Q$ or $|T|$ is even and $v(T)=\sum_{i\in T}x_i=\sum_{i\in T}q_i$ for $T\in\tau$. Theorem 4. A quota game without weak player is k-stable if (*) $v(S)\leqq\sum_{i\in S}q_i$ for all S with $|S|\leqq k+1$. If it is k-stable (*) holds for all S with $|S|\leqq k$; if it is k-stable and (*) is violated for some S with $|S|=k+1$ and if (X,τ) is k-stable pair, then $|T\cap S|=1$ and $|T|$ is even for all $T\in\tau$ such that $T\cap S\neq 0$. (Received November 8, 1954.)

313. Y. L. Luke: *Simple formulas for the evaluation of some higher transcendental functions.*

A certain class of transcendents can be simply computed by using the trapezoidal or modified trapezoidal rule on definite integrals defining these functions. Computation is simplified. Technique requires evaluation of elementary functions usually available as a subroutine on automatic computers. Let $I=\int_0^b f(t)dt$. Apply first and second Eulerian summation formulas. Correction series is usually divergent and involves $E_r=[f^{(2r-1)}(b)-f^{(2r-1)}(0)]$. If $E=E_r=0$ for all r, series is null. Highly accurate representations for I using either form of trapezoidal rule are obtained. If b is finite, f an even periodic function of period b, $E=0$. If b is infinite, f an even integral function, $E=0$. Error is conveniently studied by expressing it as an infinite sum of Fourier coefficients of f. Important example is Bessel function $J_n(z)$. Three integral representations are considered: Bessel's integral, Poisson's integral, and representation for $J_n(x^2-y^2)^{1/2}$; simple estimates of error are found. Seven terms of modified trapezoidal rule gives $e^{-z}J_0(z)$, $z=x+iy$ accurate to at least 9d for all $|z|\leqq 10$. Three kinds of complete elliptic integrals are also studied. Another example is modified Bessel function $K_\nu(z)$, Re $(z)>0$ ($\nu=1/3$ yields Airy's integral, essentially). Other examples are error function and various rocket functions. (Received November 12, 1954.)

314*t*. T. S. Motzkin: *Evaluation of rational functions.*

For an algorithm starting from constants and v variables and consisting of an unbounded number of additions and n other binary operations, the number p of parameters of the family F obtained is $\leqq 2n+1$. Even for $v=1$, $p=2n+1$ is obtained by n divisions, e.g., in the algorithms of partial and (getting all real rational functions from real constants) recurrent fractions. In these algorithms a result obtained is used only once; under this assumption, for m multiplications and $d=n-m$ divisions, the exact bound for k is $2n-[(m-1)/(d+1)]$, where [ ] means largest integer. (Received November 12, 1959.)

315B. T. S. Motzkin: *Evaluation of polynomials.*

Let A_m denote an algorithm consisting of $m>0$ multiplications and arbitrarily many additions, starting from real constants and one variable, and F the family of polynomials obtained. A_m can be chosen so that F consists of almost all (for $m\leqq 3$: all) polynomials of degree $2m$ with highest coefficient 1 and contains a given polynomial. To obtain almost all polynomials of degree n with unrestricted highest coefficient, the lowest m is $[n/2]+1$ except for $n=3$, 5, 7, 9 where it is $[n/2]+2$. Ostrowski has shown that to obtain all polynomials of degree n, $n\leqq 4$, m must be $\geqq n$; this holds also for $n=5$, but not for $n>5$. As an application of these improvements on the Newton-Horner synthetic division procedure it is possible, using Tchebyshev approximation to $1/x$, to halve the time for division on divisionless automatic computers. (Received November 12, 1954.)

316. J. A. Nohel (p) and W. K. Ergen: *Stability of solutions of a particular delay-differential equation.* Preliminary report.

In a circulating fuel nuclear reactor the power $x(t)$ satisfies the equation (1) $x'(t) = (-c/\theta)\int_0^\theta(\theta-k)[f(x(t-k))-1]dk$, $' = d/dt$, where c and θ are positive constants, and t is real. The function $f(x)$ satisfies the conditions: (a) $f(x)>0$ for all x, (b) $f'(x)>0$ for all x, (c) $f(0)=1$. Assumption (c) implies that $x=0$ is a solution of (1). The stability and asymptotic stability in the sense of Liapounov of the zero solution of (1) are investigated after these concepts are redefined so as to apply to equations of the type (1). Asymptotic stability, and in one case only stability, is established for the zero solution of the equation of first variation associated with (1). The method used is similar to that of F. G. Prohammer, *Note on the linear kinetics of the circulating-fuel nuclear reactor*, Y-F10-99 (Declassified), where an entirely different result is obtained. The general stability problem of the zero solution of the nonlinear equation (1) is not completed. For the case of periodic solutions of (1) which are known to exist (F. H. Brownell and W. K. Ergen, A *theorem on rearrangements and its application to certain delay differential equations*, Journal of Rational Mechanics vol. 3 (1954)) the stability problem is formulated, but remains unsolved. (Received November 10, 1954.)

317. L. E. Payne: *Proof of a conjecture of A. Weinstein.*

The following conjecture is due to A. Weinstein. *The first eigenvalue in the buckling problem for a clamped plate is not less than the second eigenvalue of the membrane of the same shape which is fixed on the boundary.* The author proves this conjecture by using the first partial derivatives of the first eigenfunction for the clamped plate in the variational principle for the membrane eigenvalues. It is also shown that the two eigenvalues are equal if and only if the plate is circular. This last statement follows from some well known properties of nodal lines of a vibrating membrane. The research of this author was supported by the United States Air Force through the Office of Scientific Research. (Received October 7, 1954.)

318. George Seifert: *On solutions of pendulum-type equations.*

Let $\ddot{\theta}+f(\theta)\dot{\theta}=g(\theta)$, where f and g have continuous first derivatives everywhere, have period 2π in θ, and let $\int_0^{2\pi}f(\theta)d\theta>0$. If $z=\dot{\theta}$, then $\dot{z}=g(\theta)-f(\theta)z$, and this system may be described in terms of a cylindrical phase space, the θ and z being the usual cylindrical coordinates on, say, $x^2+y^2=1$. It is shown that for each solution $\theta(t)$, the corresponding phase trajectory (or characteristic) has a finite positive limiting set. If $g(\theta)\neq 0$ for each θ, then the limiting set of each trajectory is some closed curve surrounding the phase cylinder; i.e., a limit cycle of the second kind. If $g(\theta)$ has only simple zeros, the system will have elementary critical points and the limiting sets may also include these points, ordinary limit cycles, or closed separatrices. (Received November 12, 1954.)

319. Domina E. Spencer: *Bessel wave functions.*

Bessel functions have been thoroughly studied but the *Bessel wave functions*, which occur when the Helmholtz equation is separated in parabolic coordinates, have hardly been considered. The Bessel wave equation is the Bôcher equation designated as $\{24\}$, while its degenerate case, the Bessel equation, is classified as $\{23\}$. The solutions are called *Bessel wave functions* $J_p(k, qz)$ and $Y_p(k, qz)$. The paper develops series expansions, asymptotic expansions, recursion formulas and considers orthogonality and integral representation. (Received November 8, 1954.)

320t. R. L. Sternberg: *Successive approximation and expansion methods in the numerical design of microwave dielectric lenses.*

Two essentially complete formal solutions suitable for computation are given for the problem of designing either zoned or unzoned microwave dielectric lenses of revolution having two finite and two infinite focal points pairwise symmetrically displaced about the axis of symmetry and perfect with respect to meridional rays when the refractive index of the lens, the diameter of the lens, the normal distance to the finite focal points, the off axis angle to the infinite focal points, and the wave length of the incident radiation are given. The first of these formal solutions is a characteristic value and successive approximation method while the second is a series expansion technique similar to the Cauchy-Kovalevsky method in the solution of analytic differential equations. The results are particularly useful for designing extremely short focal length and wide aperture lenses for scanning antenna applications. (Received November 15, 1954.)

321. H. F. Weinberger: *An error estimate for "box normalization" in the one-dimensional Schroedinger equation.*

An error estimate is given for the approximation of the negative eigenvalues λ_k of (1) $u''+(\lambda-q)u=0$ on $0<x<\infty$ with (2) $ru'(0)+su(0)=0$, where $q(x)$ is piecewise continuous and bounded, with $\lim_{x\to\infty} q=0$. The eigenvalues are approximated by the eigenvalues $\lambda_k(c, a)$ of (1), (2), and (3) $u'(a)+cu(a)=0$ on the interval $0\leqq x\leqq a$. It is known that $\lim_{a\to\infty} \lambda_k(c, a)=\lambda_k$, but it is here shown how to find an a such that the error is less than a prescribed ϵ. It is further known that $\lambda_k(0, a)\leqq\lambda_k\leqq\lambda_k(\infty, a)$ and that $\lambda_k(c, a)$ increases with c. Thus, for $c>0$, it is sufficient to make $\lambda_k(\infty, a)-\lambda_k(0, a)<\epsilon$. It is shown that if (4) $q_1\leqq q(x)\leqq q_2$ for $x\geqq b$, where q_1 and q_2 are constants, then for all k such that $\lambda_k(\infty, a)\leqq q_1-\beta^2$ one has (6) $\lambda_k(\infty, a)-\lambda_k(0, a)\leqq q_2-q_1+(q_1-q_m)(\exp(2\pi K/(4-K^2)^{1/2})-1)$, where $q_m=\min q$ and (7) $K=2[\exp\{2(a-b)\beta\}-1-2(a-b)\beta]^{-1}$. Since $q(x)\to 0$, b and then a may be chosen to make the right-hand side of (6) arbitrarily small. The inequality (6) is first proved when q is constant for $x\geqq b$ by explicit computation. Use is then made of the fact that the eigenvalues decrease if q is replaced by q_1 and increase if q is replaced by q_2 for $x\geqq b$, and that the eigenvalues of the modified problems differ by less than q_2-q_1. This work was supported by the U.S.A.F. through the Office of Scientific Research. (Received November 15, 1954.)

322. R. A. Willoughby: *A stability consideration for matrix inversion.* Preliminary report.

The statement, C is an approximate inverse of A, has many possible interpretations. Usually it means that for some definition of the norm of a matrix $N(B)\leqq\epsilon$ (B being one of the matrices $CA-I$, $AC-I$, $(AC+CA)/2-I$). For example: $N(B)=\max|b_{ij}|$ is a norm which is easy to calculate. Two factors omitted in the preceding definition are: (1) Does C possess properties A^{-1} is known to have (e.g. symmetry)? Are the number of significant digits in the various c_{ij} adequate? These questions arose in connection with the inversion of a 41×41 complex symmetric matrix on a high speed digital computer. The process involved replacing each complex number $a+ib$ by its matrix equivalent ([illegible]) and inverting the resulting 82×82 real matrix partially by partitioning and partially by elimination. An approximate inverse of a 12×12 principal sub-matrix did adequately represent a 6×6 complex symmetric matrix, but such was not the case for a 28×28 even though the ϵ seemed as small as could reasonably be expected in such a problem considering the accuracy of the input data. (Received November 8, 1954.)

GEOMETRY

323. Shreeram Abhyankar: *The theorem of local uniformization on algebraic surfaces over modular ground fields.*

Let K be an algebraic function field of two independent variables over an algebraically closed ground field k of characteristic p. Let v be a valuation of K/k. One says that v can be uniformized if a projective model S of K/k can be found on which the center of v is at a simple subvariety of S whose dimension is equal to the dimension of v. The theorem of local uniformization states that all the valuations of K/k can be uniformized. Zariski proved this theorem for $p=0$ (Ann. of Math. vol. 40 (1939) pp. 639–689). In this paper the theorem is proved for $p\neq 0$. Assuming only the validity of the local uniformization theorem, Zariski has established: (1) the existence of a nonsingular projective model of K/k, (2) the existence of the arithmetic genus of K/k, and (3) the Hilbert-Zariski theorem for K/k, which states that if R is a finite integrally closed integral domain over k whose quotient field contains K, then $R\cap K$ is finite over k (unpublished). These three results, hitherto completely proved only for $p=0$, may be therefore now regarded as completely demonstrated for $p\neq 0$ as well. Further, let T be a rational map of a normal V^{*r} onto a nonsingular V^r without fundamental points on either variety. Then the branch locus B of T on V is pure $r-1$ dimensional; if $r=2$ and if Q is a simple point or normal crossing of B, and if P is a corresponding point on V^*, the local Galois group is characterized. (Received November 15, 1954.)

324. E. F. Allen: *The Feuerbach configuration applied to central conics.*

Tangents at three distinct points t_1, t_2, t_3, of a central conic A, form a triangle T, whose vertices are x_1, x_2, x_3, in which the conic is either inscribed or escribed. The author proved that the conic A is tangent to the nine-point conic D of the triangle T (E. F. Allen, *On a triangle inscribed in a rectangular hyperbola*, Amer. Math. Monthly vol. 48, p. 675), and thus extends Feuerbach's theorem to central conics as well as circles. The conic D is the homothetic of the conic A, with center $K=S_3\bar{S}_1/S_1$, having the homothetic ratio $1/(S_3\bar{S}_1-1)$. S_1, S_2, S_3 are symmetric functions of t_1, t_2, and t_3. Certain conics of the Feuerbach configuration are inverse with respect to the conic A. The author defines the inverse C''' of a conic C' with respect to a conic C'', where C' and C'' are conics of the same kind, having parallel axes, and shows how to obtain the equation of C'''. Thus if B is the nine-point conic of the triangle t_1, t_2, t_3 and C the conic circumscribing triangle T, it is shown that B and C are inverse conics with respect to the conic A. The method of proof is analytic using the notation of the extended inversive geometry (E. F. Allen *An extended inversive geometry*, Amer. Math. Monthly vol. 60, p. 233). (Received October 14, 1954.)

325t. P. O. Bell: *A projective curvature tensor for hypersurfaces.*

Let $x(u^1, u^2, \cdots, u^{n-1})$, $x'(u^1, u^2, \cdots, u^{n-1})$ denote corresponding points of two hypersurfaces V, V' of n-dimensional projective space. In the present note a projective curvature tensor $R_{\alpha\beta}$ of V at x with respect to V' at x' is defined which is shown to be a generalization of the well known Ricci curvature tensor of a hypersurface V in

Euclidean space. Among the results obtained is the following geometric characterization of the invariant form $\Phi = R_{\alpha\beta}du^{\alpha}du^{\beta}$. Let V'_{μ} denote the hypersurface generated by a point $y_{\mu} = x' - \mu x$ collinear with x, x'. Let X, Y_{μ} denote the points of V, V'_{μ} which correspond to the parameter values $u^{\alpha} + du^{\alpha}$ $(\alpha = 1, 2, \cdots, n-1)$ and let p, P denote $(n-1)$-planes tangent to V at x, X, respectively. Let z_{μ}, Z_{μ} denote the points of intersection of the line (y_{μ}, Y_{μ}) and the $(n-1)$-planes p, P, respectively. The principal part of the cross-ratio $(y_{\mu}, z_{\mu}, Y_{\mu}, Z_{\mu})$ is the quadratic differential form denoted by ϕ_{μ}. Let μ_{α} $(\alpha = 1, 2, \cdots, n-1)$ denote the functions which correspond to the foci of the line (x, x') (of the hypercongruence generated by (x, x')). The invariant Φ is given by $\Phi = (n-2)\phi_0 - \sum \phi_{\mu_{\alpha}}$, in which ϕ_0 is the function ϕ_{μ} for $\mu = 0$ and the sum is extended over the range $\alpha = 1, 2, \cdots, n-1$. If the general coordinates of the point x are homogeneous rectangular cartesian coordinates $(1, x^1, x^2, \cdots, x^n)$ and x' is the point "at infinity" on the normal to V at x, Φ becomes the quadratic form whose coefficients form the Ricci curvature tensor of V. (Received November 16, 1954.)

326. W. M. Boothby: *Bochner's lemma for Hermitian manifolds.*

In a recent paper (Amer. J. Math. vol. 76 (1954) pp. 509–534) the author derived a form of Bochner's lemma for Hermitian manifolds. However, the form of the lemma obtained in this instance was unnecessarily complicated by the presence of a term involving the torsion, and here a better proof shows that in the Hermitian case the statement is formally the same as in the Kähler case. Although results relating curvature and Betti numbers do not appear to be any more accessible in consequence, two theorems of Bochner can be generalized to Hermitian manifolds, namely: 1. If $T_{i j^*}$ is positive definite on a compact, Hermitian manifold M, then there are no analytic differential forms on M, and 2. If $T_{i j^*}$ is negative definite on a compact, Hermitian manifold M, then M does not admit a one parameter group of analytic homeomorphisms. In both cases $T_{i j^*} = R_{i j^* k k^*}$ is one of the three "Ricci tensors" of the Hermitian curvature. (Received November 12, 1954.)

327B. Leo Branovan: *Umbilics on hyperellipsoids and related closed surfaces in four dimensions.* Preliminary report.

The subject of umbilics on hypersurfaces in S_4 has rarely been considered in mathematical literature. The only previous treatment of this subject is a paper by Sommer (*Focaleigenschaften quadratischer Mannigfaltigkeiten im vierdimensionalen Raum*, Math. Ann. vol. 53 (1900) pp. 113–160) in which he considered a confocal manifold in S_4. He proved synthetically that the confocal manifold cannot have real umbilics. However, he did not obtain the coordinates of the umbilics under any conditions. In this paper, the author investigates the conditions under which umbilics on hyperellipsoids and related closed surfaces in S_4 exist; then he actually obtains the coordinates of the real umbilics on these surfaces. Furthermore, he compares umbilics on ordinary surfaces in S_3 with those on hypersurfaces in S_4 and draws interesting conclusions. A differential geometry approach is used to solve the problem. (Received November 4, 1954.)

328. H. S. M. Coxeter: *On Laves' graph of girth ten.*

Consider, in Euclidean 3-space, the infinite sets of points whose coordinates are (0, 0, 0), (1, 2, 3), (2, 3, 1), (3, 1, 2), (2, 2, 2), (3, 0, 1), (0, 1, 3), (1, 3, 0) (mod 4). Certain pairs are distant $2^{1/2}$, and by joining all such pairs one obtains an infinite graph of degree 3 and girth 10: the smallest circuit is a skew decagon. Spheres of diameter $2^{1/2}$, with these points as centres, form a "thin packing" which was discussed by F. Laves (Z. Kristallogr. vol. 82 (1932) p. 10). When the infinite space is reduced to a three-dimensional torus by identifying points whose coordinates differ by multiples of $4n$, the resulting finite graph has $8n^3$ nodes and $12n^3$ branches. The girth is still 10 when $n \geqq 3$, but only 8 when $n=2$. (Received October 25, 1954.)

329. S. I. Goldberg: *Note on metric manifolds with torsion.*

Let V_n be a compact n-dimensional manifold on which there is given a positive definite metric $ds^2 = g_{jk}dx^j dx^k$ and a metric connection $\Gamma^i_{;jk}$ so that the covariant derivative $g_{jk;l}$ of g_{jk} taken with respect to $\Gamma^i_{;jk}$ vanishes. $\Gamma^i_{;jk}$ is the most general nonsymmetric connection that complies with the condition that $g_{jk;l}=0$. Denote the torsion tensor by $S_{;jk}{}^{i} = (\Gamma^i_{;jk}) - \Gamma^i_{;kj})/2$ and assume that $S_{ri}^{\ i} = 0$. Consider a pseudo-harmonic vector $\xi^i(\xi_{i;j} = \xi_{j;i}$ and $\xi^i_{;i} = 0)$ and form the vector $G_{ij}\xi^{i;k}\xi^j$ where G_{ij} is an arbitrary symmetric tensor. Using Green's theorem and then putting $G_{ij} = \rho g_{ij}$ where ρ is a nonzero scalar one obtains $\int_{V_n}\rho[(E_{jk}+E_{kj}-(\Delta\rho/\rho)g_{jk})\xi^j\xi^k - 2(S_{kmj}+S_{kjm})\xi^k\xi^{j;m} + (g_{rl}g_{sm}+g_{rm}g_{sl})\xi^{r;s}\xi^{l;m}]dv = 0$ where E_{jk} is the contracted curvature tensor referred to Γ^i_{jk} and $\Delta\rho$ is the Laplacean of ρ. This gives a generalization of Theorem 7.9 (cf. *Curvature and Betti numbers*, K. Yano and S. Bochner) in case V_n is orientable. In a similar manner one can generalize Theorem 7.14 (loc. cit.) by considering pseudo-Killing vector. These results can easily be extended to cover pseudo-harmonic and pseudo-Killing tensors. Furthermore, ρ can be adjusted to yield some interesting formulae. (Received November 15, 1954.)

330. Frank Harary: *On the notion of balance in signed graphs.*

The concepts and results of this note were motivated by some problems in social psychology. A signed graph (s-graph) results when one takes an ordinary linear graph and regards some of its lines as positive and the remaining lines as negative. A cycle of an s-graph is positive if it contains an even number of negative lines. An s-graph is balanced if all its cycles are positive. It is shown that a necessary and sufficient condition for an s-graph to be balanced is that its point set can be divided into two disjoint subsets such that each positive line joins two points of the same subset and each negative line joins two points of different subsets. (Sponsored by a grant from the Rockefeller Foundation to the Research Center for Group Dynamics.) (Received November 12, 1954.)

331. C. C. Hsiung: *An integral formula for a hypersurface in a Riemannian space.*

Let V^n be an orientable hypersurface twice differentiably imbedded in a Riemannian space R^{n+1} of $n+1 \geqq 3$ dimensions; $\kappa_1, \cdots, \kappa_n$ the n principal curvatures at a point P of V^n; and M_1 the first mean curvature of V^n at P defined by $M_1 = \kappa_1 + \cdots + \kappa_n$. Let dA be the area element of V^n at P, and p the scalar product of the unit normal vector of V^n at P and the position vector of P with respect to any orthogonal frame in R^{n+1}. The purpose of this paper is to extend a recent result of the author [Bull. Amer. Math. Soc. Abstract 60-4-441] by showing that for an orientable hypersurface V^n with a closed boundary V^{n-1} of dimension $n-1$ the integral $\int_{V^n}(M_1p+n)dA$ can be expressed as an integral over the boundary V^{n-1} and hence that the integral vanishes for a closed orientable V^n. (Received November 9, 1954.)

332. Wilhelm Klingenberg: *Configurations in euclidean planes.*

Removing a line L_∞ and the points on it from a projective plane π one gets a euclidean plane π^*. It is well known (M. Hall, Trans. Amer. Math. Soc. vol. 54) that by choosing a fundamental quadrilateral, which has two vertices on L_∞, π^* may be described by means of the elements of a ternary ring T. T is a field (or a commutative field) if and only if the Desargues Theorem (D) holds in π^* (or the Pappos Theorem (P)). The author denotes by (R) ("Reidemeisterfigur") or by (T) ("Thomsenfigur") the configuration which corresponds to the associative law or the commutative law of multiplication in a ternary ring T. Then the following theorem holds: (R) is equivalent to (D) and (T) is equivalent to (P). This means: If for every ternary ring T, defined by π^*, the associative law of multiplication (or the commutative law of multiplication) holds, then T is a field (or a commutative field) and vice versa. It is well known that (R) is a consequence of (T). Thus this theorem includes another proof of the theorem of Hessenberg, i.e. that (D) in a consequence of (P). (Received November 5, 1954.)

333. L. F. Markus: *The universe and Mr. Euler.*

It is known that a (C^∞) differentiable n-manifold M admits a continuous line element field if and only if M admits a differentiable, covariant, symmetric, second order tensor field with signature of one at each point, that is, a Lorentz metric on M. It is shown here that a compact differentiable n-manifold M admits a continuous line element field, and thus a Lorentz metric, if and only if the Euler characteristic is zero. A cosmological interpretation of this result for a bounded space of general relativity is that the characteristic of the universe is zero. In particular, the universe is not the product of two pretzels. (Received November 15, 1954.)

334t. Albert Nijenhuis: *On local and infinitesimal holonomy groups.*

The methods and results of the author's paper, which dealt with the connections in linear bundles only [K. Nederl. Akad. Wetensch. A vol. 56 (1953) pp. 233–249; A vol. 57 (1954) pp. 17–25] can be extended to the general case of bundles with Lie groups. Some results along that line were obtained independently by Ambrose and Singer [Trans. Amer. Math. Soc. vol. 75 (1953) pp. 428–443]. The local holonomy group $\mathbf{h}^*(p)$ at $p \in M$ is the intersection of all restricted holonomy groups $\mathbf{h}^0(U, p)$ at the base point p, where U ranges over the open neighborhoods of p; the infinitesimal holonomy group $\mathbf{h}'(p)$ is determined by explicit expressions in terms of the curvature and its derivatives at p. Some of the results are: (1) $\mathbf{h}'(p) \subset \mathbf{h}^*(p)$; (2) If dim $\mathbf{h}^*(x)$ is constant for all $x \in M$, then $\mathbf{h}^0(M, p) = \mathbf{h}^*(p)$; (3) If $\mathbf{h}'(x)$ and $\mathbf{h}^*(x)$ are equal for all $x \in M$, then $\mathbf{h}^0(M, x)$ is equal to these; (4) If the bundle and the connection are analytic, then $\mathbf{h}^0(M, p) = \mathbf{h}^*(p) = \mathbf{h}'(p)$; (5) If Ω^* is the set of $x \in M$ where dim $\mathbf{h}^*(x)$ is discontinuous, and if Ω' is the same for $\mathbf{h}'(x)$, then $\Omega^* \subset \Omega'$; both are closed in M and have no interior points. (Received November 16, 1954.)

335. T. K. Pan: *Angular spread and indicatric torsion.*

Different forms of angular spread, analogous to those of geodesic curvature due to Beltrami, Bonnet and others, are derived; and a modified expression of Gauss-Bonnet theorem is obtained with respect to a vector field in a surface. The arc-rate of turning along a space curve of the plane, which is determined by a vector of a field in space and its derived vector along the curve, is defined as the torsion of the field along the curve. This leads to a generalization of the Frenet formulas in the theory of space curves. When a vector field is in a surface, a torsion, called the indicatric torsion of the field along the curve, is defined to include the geodesic torsion of a curve as a special case. It is found that the indicatric torsion of the asymptotic field of a surface vector field along the curve of the field is numerically equal to the principal curvature of the field and that the indicatric torsion of a vector field along a curve is zero if and only if the curve is the line of curvature of the field, the first part being a generalization of the theorem of Enneper. (Received November 12, 1954.)

336. Valdemars Punga: *The derivation of O. Varga's affine connection by H. Friesecke's method.*

O. Varga in *Über affinzusammenhängende Mannigfaltigkeiten von Linienelementen insbesondere deren Äquivalenz*, Publicationes Mathematicae, Tomus 1, Debrecen, 1949, introduces the parallel displacement of a vector ξ^i in the space of line elements (x^i, x'^i) by $D\xi^i = d\xi^i + C^i_{kj}\xi^k\pi^j(d) + \Gamma^{*i}_{kj}\xi^k dx^j$. In the present paper the author shows that applying to the space of line elements the same arguments as H. Friesecke in his *Vektorubertragung, Richtungsubertragung, Metrik*, Math. Ann. vol. 94 (1925) one will naturally arrive at O. Varga's affine connection defined by C^i_{kj} and Γ^{*i}_{kj} (Received November 5, 1954.)

337*t*. P. V. Reichelderfer: *On the product of two essentially absolutely continuous transformations.*

Given bounded domains D' and D'' in euclidean n-space let T' be a continuous transformation from D' into D'' and let T'' be a continuous transformation from D'' into a bounded portion of n-space. Assume that both T' and T'' are essentially absolutely continuous. (T. Radó and P. V. Reichelderfer, *On n-dimensional concepts of bounded variation, absolute continuity and generalized jacobian*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 678–681). Then a necessary and sufficient condition in order that $T''T'$ be essentially absolutely continuous is that $T''T'$ be essentially of bounded variation. If $j_e(u', T')$ and $j_e(u'', T'')$ are the essential jacobians for T' and T'' respectively, then a necessary and sufficient condition in order that $T''T'$ be essentially absolutely continuous is that $j_e(T'u', T'')\,j_e(u', T')$ be summable in D'. If $T''T'$ is essentially absolutely continuous then its essential jacobian $j_e(u', T''T')$ is equal to $j_e(T'u', T'')\,j_e(u', T')$ almost everywhere in D'. (Received November 8, 1954.)

338. H. W. E. Schwerdtfeger: *An approach to non-Euclidean geometry.*

The plane D of operation is the whole or a part of the complete plane of the complex numbers which serve as points in the geometries. The basic ideas are prescribed by F. Klein's Erlanger Program. Euclidean geometry is thus generalized by: 1. Assuming a group G of transformations of D into itself as motions; 2. Defining congruence with respect to G; 3. Defining straight lines, circles and cycles by its one-parameter subgroups; 4. Introducing distance as a "two-point invariant." The group G will be restricted by the following assumptions: A. G is simply, not twofold transitive. B. For every element $g \in G$ ($g \neq$ unit element) the powers g^s (s real) form a continuous one-parametric subgroup of G. C. Every stability subgroup is cyclic. D. One of the stability subgroups is similar to a rotation group. *Theorem*: There is a unique two-point invariant $f(z_1, z_2)$ of the group G. E. A transformation $h_z \in G$ for which $h_z(z) = 0$ has the property $|h_z(0)| = |z|$. F. The transformations of G are conformal. Under these restrictions on the motion group G the domain D is either the inner of a circle about 0 (hyperbolic geometry), or the plane without infinity, or the completed plane (spherical geometry). (Received November 12, 1954.)

339. Saly Ruth Struik: *Affine geometry*. II.

Enriques (Encyklopädie d. Math. Wiss. III, 1, 1) gives an axiomatics of area based on Euclid's axioms of congruence utilizing equality of decomposition for two equal triangles and utilizing the square for measure. For three dimensions Enriques only mentions that according to Dehn's proof the analogous definition of equality of volume for tetrahedrons fails. No alternative is proposed. Hilbert adds to this fact Dehn's assertion that a foundation for three-space will have to utilize another principle, e.g. the Cavalieri principle. In *Affine geometry*, I this has been done, by giving an axiomatics of affine geometry and basing equality of triangle and tetrahedron on the notion of affine two-space reflection and affine three-space reflection respectively. In the present paper two theorems of Euclidean geometry are added based on purely affine findings of the first paper. (1) Two triangles are of equal area if and only if they can be placed into a position where they are each other's two-space affine reflection. (2) Two tetrahedrons are of equal volume if and only if they can be brought into a position where they are both the affine three-space reflection of a third one. Proof is by construction. (Received November 8, 1954.)

340. N. Z. Wolfsohn: *On differentiable maps of Euclidean n-space into Euclidean m-space*. Preliminary report.

Let R be a subset of E^n and let $f:R\rightarrow E^m$ where $f\in C^1$. Then with $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_m)$ coordinate systems in E^n and E^m respectively, $\partial f(P)/\partial x_i$ is regarded as a vector in E^m. Using an extension of methods of Whitney (*The general type of singularity of a set of* $2n-1$ *smooth functions of* n *variables*, Duke Math. J. vol. 10 (1943) pp. 161–172) it can be shown that given f, there exists f^*, approximating f as closely as desired and such that the rank of f^* at every point P is at least min $(m, n)-1$ if $2(m+2)/3<n\leqq m$ or if $m<n$ and $m<(n+4)/2$. If $n\geqq m$, if the rank of f is $m-1$ for all P, and if for some P, $\partial f(P)/\partial x_{m+i}=0$ $(i=0, \cdots, n-m)$ while $\partial f(P)/\partial x_1, \cdots, \partial f(P)/\partial x_{m-1}$ span an $m-1$ dimensional space parallel to that spanned by $(y_1, \cdots, y_{m-1})$ the quadratic rank and signature of f at P is defined to be equal to the rank and signature of the quadratic form $\sum_{i,j=0}^{n-m} (\partial^2 y_m(P)/\partial x_{m+i}\partial x_{m+j})w_iw_j$. F may be approximated by f^* so that the quadratic rank of f^* at any singular point is $\geqq n-m+1-p$ where p is the greatest integer such that $m\geqq(p^2+p+2)/2$. If $n\leqq m$, if the rank of f at every singular point is $n-1$, and if $\partial f(P)/\partial x_n=0$ at a singular point P, then f^* may be found, approximating f such that a certain matrix has rank m. In certain of these cases, local coordinate systems may be found which give the map a particularly simple form. (Received November 10, 1954.)

LOGIC AND FOUNDATIONS

341. J. W. Addison: *Analogies in the Borel, Lusin, and Kleene hierarchies*. II. Preliminary report.

The effective Lusin hierarchy (ω_1 levels) on N (natural numbers) and on N^N is defined (using images of recursive transforms from N^N, complements, and recursively enumerable unions). Recursive sieves and recursive operations (A) are applied in the theory. On N and N^N the finite effective Lusin (effective projective) hierarchy is equivalent to the Kleene function-quantifier ($\mathcal{J}$) hierarchy. Parallel theorems about the projective hierarchy on N^N and $\mathcal{J}$ hierarchies on N and N^N are demonstrated by essentially identical proofs. A set of recursive predicates which are 'notations' for 'effective ordinals' is defined; it is an $\forall^1$- but not an $\exists^1$-set of N^N, i.e. it is expressible in $\widehat{\phi}(\psi)(Ey)R(\phi, \psi, y)$ but not in $\widehat{\phi}(E\psi)(y)R(\phi, \psi, y)$ form, where $\phi, \psi \in N^N$, $y \in N$, and R is recursive. Using this, any two disjoint $\exists^1$-sets of N are separable by an effective Borel set (and hence by an $\exists^1 \cap \forall^1$-set). Kleene's theorem (any $\exists^1 \cap \forall^1$-set is an effective Borel set) follows as a special case. There exist two disjoint $\forall^1$-sets of N not separable by any $\exists^1 \cap \forall^1$-set. The analogy of the projective with the $\mathfrak{J}$ hierarchy thus lacks the imperfections of the earlier analogy with the Kleene (number-quantifier) hierarchy (cf. Kleene, Neder. Akad. Wetensch. vol. 53 (1950) pp. 800–802). (Received November 15, 1954.)

342*t*. Solomon Feferman: *Sum operations on relational systems.*

The arithmetic theory of a given relational system $\mathfrak{A} = \langle A, R_0, \cdots, R_{n-1} \rangle$ is here denoted by $Th(\mathfrak{A})$; the fact that a sentence ϕ of $Th(\mathfrak{A})$ is true in $\mathfrak{A}$ is expressed by $\vdash_{\mathfrak{A}} \phi$ (cf. Mostowski, Journal of Symbolic Logic vol. 17 (1952) pp. 2–3). The relation $\equiv$ of arithmetical equivalence between systems is defined by Tarski (Proc. Int. Cong. Math. 1950, vol. 1, p. 712). For any two systems $\mathfrak{A}$ and $\mathfrak{B}$ with $A \cap B = \Lambda$ it is possible to define a new system $\mathfrak{A} + (\phi)\mathfrak{B}$ (where ϕ is any finite sequence of formulas out of a certain theory $Th([\mathfrak{A}, \mathfrak{B}])$ comprehending $Th(\mathfrak{A})$, $Th(\mathfrak{B})$) having the following properties: I^+. $+(\phi)$ preserves $\cong$. II^+. If θ is any sentence of $Th(\mathfrak{A} + (\phi)\mathfrak{B})$, there exist effectively constructible sentences $\psi_0, \cdots, \psi_k$ of $Th(\mathfrak{A})$ and $\chi_0, \cdots, \chi_k$ of $Th(\mathfrak{B})$ such that $\vdash_{\mathfrak{A}+(\phi)\mathfrak{B}} \theta$ iff for some $j \leqq k$, $\vdash_{\mathfrak{A}} \psi_j$ and $\vdash_{\mathfrak{B}} \chi_j$. III^+. $+(\phi)$ preserves $\equiv$. IV^+. If $Th(\mathfrak{A})$, $Th(\mathfrak{B})$ are both decidable, so also is $Th(\mathfrak{A} + (\phi)\mathfrak{B})$. By suitable specifications of ϕ, $\mathfrak{A}$ and $\mathfrak{B}$, the operation $+(\phi)$ may be made to agree with the cardinal sum or the ordinal sum. Thus IV^+ generalizes a result of Beth (Collection de Logiques Mathématique Série A (Paris-Louvain 1954) fasc. V pp. 29–35) concerning the ordinal sum of simply ordered systems. (Received November 12, 1954.)

343*t*. Solomon Feferman: *Product operations on relational systems.*

For notation see preceeding abstract. By selecting a certain class P out of the class of formulas of $Th([\mathfrak{A}, \mathfrak{B}])$, it is possible to define a new system $\mathfrak{A} \times (\phi)\mathfrak{B}$ for any systems $\mathfrak{A}$ and $\mathfrak{B}$ and any finite sequence ϕ of formulas of P. The properties I^+-IV^+ of the operation $+(\phi)$ hold as well for the operation $\times(\phi)$ (under the restrictions to P given above), giving new theorems $\mathrm{I}^\times$–$\mathrm{IV}^\times$. By suitable specification of ϕ, $\mathfrak{A}$ and $\mathfrak{B}$, the operation $\times(\phi)$ may be made to agree with the cardinal product or the ordinal product or with a mixture of these operations. Using this last fact leads to an interesting application of $\mathrm{IV}^\times$: For each integer n, let W_n be the set of ordinals less than ω^n, $<_n$ the well-ordering relation in W_n, and S_n and N_n the ternary relations in W_n corresponding respectively to ordinary and natural addition of ordinals. $\langle W_n, N_n, <_n \rangle$ may be expressed as a product $\times(\phi)$ of n identical factors $\langle W_1, S_1, <_1 \rangle$ in which S_1 is cardinally multiplied, $<_1$ ordinally multiplied. Since $Th(\langle W_1, S_1, <_1 \rangle)$ is decidable by a result of Presburger (Comptes-rendus du I Congres des Mathematiciens des Pays Slaves (Warszawa 1929) pp. 92–101 and 395), and since S_n is elementarily definable in terms of N_n and $<_n$, it follows that $Th(\langle W_n, S_n \rangle)$ is decidable for each integer n. The general problem (originating with Tarski), whether the theory of addition of arbitrary ordinals is decidable, remains open. (Received November 12, 1954.)

344B. Richard Montague: *Non-finitizable and essentially non-finitizable theories.*

For terminology consult Tarski, Mostowski, Robinson, *Undecidable theories*. A theory is ***non-finitizable*** if it is not finitely axiomatizable, ***essentially non-finitizable*** if no consistent extension with the same constants is finitely axiomatizable. *Theorem* 1. *Zermelo-Fraenkel set-theory is essentially non-finitizable.* A theory T is *strongly arithmetical* if, intuitively speaking, Peano's arithmetic (with $+$, $\cdot$, and 0) can be interpreted in T in such a way that induction and definition by recursion (involving, respectively, arbitrary formulas of T and arbitrary terms definable in T) are permissible. *Theorem* 2. *If T is strongly arithmetical and Φ either is a universal sentence interpretable in T or follows logically from such a sentence, then the consistency of Φ can be proved in T.* With the aid of Theorem 2, Theorems 3 and 4 are proved. *Theorem* 3. *The following theories are essentially non-finitizable*: (i) *Peano's arithmetic*, (ii) *Zermelo's set-theory without the axiom of infinity but with a form of the axiom of constructibility. Theorem* 4. *If Zermelo's set-theory without the axiom of infinity is consistent, then it is non-finitizable.* Theorems 2 and 3(i) are obtained by generalizing a method of Mostowski (Fund. Math. vol. 39 (1952) pp. 133–158). (Received November 13, 1954.)

345B. R. M. Robinson: *Arithmetical representation of recursively enumerable sets.*

A set S of natural numbers is called recursively enumerable if there is a general recursive function $F(x, y)$ such that $y \in S \leftrightarrow (\bigvee x)[F(x, y) = 0]$. Martin Davis has shown [J. Symbolic Logic vol. 18 (1953) pp. 33–41] that every such set S can be represented in the form $y \in S \leftrightarrow (\bigvee b)(\bigwedge w)[w \leq b \rightarrow (\bigvee x_1, \cdots, x_\lambda)\ P(y, b, w, x_1, \cdots, x_\lambda) = 0]$, where P is a polynomial with integer coefficients, but no estimate for λ was given. In the present paper, it is shown that it is possible to take $\lambda = 4$. (On the other hand, it is easily seen that one cannot always take $\lambda = 0$, so that the smallest possible λ is 1, 2, 3, or 4.) The proof is based on the fact that the set S, if non-empty, is the range of a primitive recursive function of one variable. Now all such functions can be obtained from $x+1$ and Kx by repeated use of the formulas $Fx = J(Ax, Bx)$, $Fx = BAx$, and $Fx = B^x 0$ [Bull. Amer. Math. Soc. Abstract 61-1-162]. Applying these three formulas in a systematic way, we obtain a sequence $F_n x$ which enumerates all primitive recursive functions. The condition $(\bigvee x)[y = F_n x]$ is then expressed arithmetically. Various devices are used to reduce the number of quantifiers needed. (Received October 25, 1954.)

346. J. R. Shoenfield: *The independence of the Axiom of Choice.*

The Axiom of Choice is not provable from axioms A–C of Godel's *The consistency of the continuum hypothesis*. The proof uses an extension of techniques developed by Fraenkel and Mostowski. (Received November 12, 1954.)

347B. R. L. Vaught: *On the arithmetical equivalence of free algebras.*

Let $\mathcal{K}$ be a class of algebras $\mathfrak{A}$ formed by a set and a binary operation over the set, and suppose that $\mathcal{K}$ is closed under taking of isomorphic images, subalgebras, and arbitrary direct products. Then the free algebra $\mathfrak{C}$ over $\mathcal{K}$ with any given cardinal number of generators (as defined in Birkhoff, *Lattice theory*, p. viii, with the added condition that $\mathfrak{C} \in \mathcal{K}$) exists and is unique up to isomorphism. *Theorem*: Any two free algebras $\mathfrak{A}$ and $\mathfrak{B}$ over $\mathcal{K}$, each with infinitely many generators, are arithmetically equivalent (i.e., the same sentences of a language L with the equality symbol, a binary operation symbol, sentential connectives, and quantifiers, hold in $\mathfrak{A}$ and $\mathfrak{B}$). Indeed, if, moreover, $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, then any formula of L which is satisfied in $\mathfrak{A}$ by some elements is satisfied in $\mathfrak{B}$ by the same elements. (Thus $\mathfrak{B}$ is an arithmetical extension of $\mathfrak{A}$, an unpublished notion due to Tarski.) The theorem may be extended to algebras formed by a set and arbitrarily many finitary operations. These investigations arose from a still unresolved conjecture of Tarski's that any two free groups with at least *two* generators are arithmetically equivalent. (Received October 25, 1954.)

STATISTICS AND PROBABILITY

348. D. G. Austin: *Some differentiation properties of Markoff transition probability functions.*

Let $p_{ij}(t)$ denote the transition probability functions of a denumerable Markoff chain satisfying $\lim_{t\to 0} p_{ij}(t) = \delta_{ij}$. It is known that $p_{ii}(t)$ has a derivative, finite or infinite, at $t=0$ which will be denoted by $-q_i$. It is shown that if either q_i or q_j is finite, then $p_{ij}(t)$ has an ordinary derivative almost everywhere and has a difference quotient bounded above by q_i and below by $\max(-q_i, -q_j)$. If q_i is finite each of the series $\sum_j D^* p_{ij}(t)$ and $\sum_j D_* p_{ij}(t)$ converge absolutely to numbers between q_i and $-q_i$. In the special case where the q_i are bounded, the $p_{ij}(t)$ have a Taylor series expansion and the higher order derivatives satisfy a generalized Kolmogoroff differential equation $D^{(m+n)}p_{ij}(t_1+t_2) = \sum_k D^{(m)}p_{ik}(t_1)D^{(n)}p_{kj}(t_2)$. (Received November 10, 1954.)

349t. H. W. Becker: *Lattice interpolation between rhyme schemes and factorial words.*

If $\mathcal{R}_n$, W_n, and $n! = F_n$ enumerate *RS*, 2-limit *FW* (Bull. Amer. Math. Soc. vol. 57 (1951) p. 297), and *FW*, then $e^{t\mathcal{R}} = \exp(e^t - 1)$, $e^{tW} = \sec^2 t + \sec t \tan t$, and $e^{tF} = 1/(1-t)$, with $\mathcal{R}_n < W_n < F_n$, $n \geqq 4$. But $\mathcal{R}_n \not\subset W_n$, $n > 2$, *RS* with letters rhymed more than once not being members of *W*. $\mathcal{R}_n \subset I_n \subset F_n$ is satisfied by I_n, the n-letter factorial words in which r is both the variety of letters and the highest of them. The subset of F_{n+1} with s different letters, r highest, is $F_{n+1,r;s} = sF_{n,r;s} + (r) - s + 1)F_{n,r;s-1} + F_{n,r-1);s-1}$, the) denoting summation from 1 through $r-1$. A valuable check is the table of $F_{n+1,r} = rF_{n,r} + F_{n,r-1)}$. In particular, $F_{n,r);2} = 2^{n-r+1} - 1$, $F_{n,n,s} = F_{n-1;s-1}$ (a crosscheck, congruences due to Carlitz and Riordan, Duke Math. J. vol. 20 (1953) p. 341), and $F_{n,r);r} = I_{n,r} = I_{n,n-r+1}$. Independent formulation of I_n seems impracticable, so its recurrence and generating functions are unknown, beyond $I_{n,1} = 1$, $I_{n,2} = 2^{n-1} - 1$, $I_{n,3} = 1 + 3(I_{n-1,3} + I_{n-2,2})$. Breakdowns of I_n by terminal letter, number of singletons, and a's, and position of last singleton, and a, are tabulated. The latter is curious, in that the table is symmetrical but montone *decreasing* to midtable. $0 \leqq n \leqq 10$, $I_n = 1$, 1, 2, 5, 16, 63, 294, 1585, 9692, 66275, 501106. $W_n = I_n = S_n$, $n < 5$; $W_n < I_n < S_n$, $n \geqq 5$: where S_{n+1} = superfactorial $n = (F+1)^n = 1 + nS_n \sim n!e$. By Erdélyi et al., *Higher transcendental functions*, 1953, 1.18(2), 1.20(9), 1.21(8, 14), $W_n \sim 2^{n+7/2} n^{n+1/2}(n+1)/e^n \cdot \pi^{n+3/2}$, so $W_n/W_{n-1} \sim (n+1)2/\pi$. Empirically $I_n \sim 1.01(W_n^2 \cdot S_n)^{1/3}$, a cubic mean, so $I_n/I_{n-1} \sim n(2/\pi)^{2/3} = 0.74n$. (Received November 12, 1954.)

350t. Volodymyr Bohun-Chudyniv: *On a new method of constructing canonical sets of Latin $2^k \times 2^k$ squares $(k \geqq 2)$.*

In the works of W. L. Stevens and R. S. Bose and K. R. Nair is given a method of construction, by using some properties of Galois fields, for each value of power k, of only one canonical set of Latin squares that can be orthogonized. In the author's paper (see Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, pp. 281–282) was given with the help of author's schemes of K-nions, a general method of determining for each value of K not only one canonical set of Latin

$2^k \times 2^k$ squares, but all possible Latin squares that can be orthogonized, and it was shown that canonical sets of this type, determined by Bose and Stevens, can be obtained from the author's schemes as their separate cases. In the present paper the author determines another method of constructing canonical sets of Latin $2^k \times 2^k$ squares ($k \geq 2$), simpler than the aforementioned methods, by using pairs of mutually conjugate solutions in positive integers from the following expressions: (I) $1, X, Y$; $1, 1+2^k-Y, X+2^k-Y$; $1, Y-X+1, 1+2^k-X$ which satisfy the following conditions: (1) $X < Y$, and all values of elements (I) in the second and third triples of both solutions must be different; (2) in the Latin square pertaining to each of the solutions, for each pair $A_{\beta,\alpha}; A_{\gamma,\alpha}$ belonging to any given column, exists a reciprocal pair of these elements in another column of the same row. Illustrative examples for $K=3, 4, 5, 6$, and 7 have been worked out by the author. (Received November 15, 1954.)

351*t*. Volodymyr Bohun-Chudyniv: *On determining all possible orthogonalizable sets of Latin $2^3 \times 2^3$ squares and pertaining to them completely orthogonal Latin squares.*

Using the known Bose and Nair's method, pointed out in paper *On complex sets of Latin squares* (Sankhya vol. 5, p. 34), one can obtain for every value of K only one set with two mutually conjugate Latin $2^k \times 2^k$ squares, called a *canonical set*, which can be orthogonalized. In Ronald Fisher's *The design of experiments* (Oliver and Boyd, Edinburgh, 4th ed., 1947, p. 81) is pointed out an example of orthogonalizable Latin $2^3 \times 2^3$ square, which cannot be obtained by the method mentioned above. By the author's methods pointed out in the paper, *On a general method for constructing completely orthogonal $2^k \times 2^k$ square by using closed orthogonal systems of K-nions* (see Proceedings of the International Congress of Mathematicians, 1954, pp. 281–282), one can determine for every k all possible orthogonalizable sets of Latin $2^k \times 2^k$ squares and pertaining to them completely orthogonal Latin $2^k \times 2^k$ squares. Bose and Stevens' *canonical sets* and Fisher's example mentioned above can be obtained from the author's schemes as separate cases. The aims of this paper are: (1) To determine the number of all possible sets with two mutually conjugate Latin $2^3 \times 2^3$ squares that can be orthogonalized, and to construct all of them; (2) To prove the existence of not only one completely orthogonal Latin $2^3 \times 2^3$ square for each orthogonalizable Latin $2^3 \times 2^3$ square, but 8. (3) To give a method for constructing all completely orthogonal Latin $2^3 \times 2^3$ squares for each type of orthogonalizable Latin $2^3 \times 2^3$ squares and to construct them. (Received November 17, 1954.)

352. Maria Castellani: *Fantappiè's analytical indicators and Levy's characteristic functions.*

This is a preliminary investigation on relations among Fantappiè's "analytica functionals" and Levy's characteristic functions. Given on R_n the distribution function $F(x_1, x_2, \cdots, x_n)$ of n random variables, the linear operators to be considered are: $E[g(z_1, z_2, \cdots, z_n; x_1\ x_2 \cdots x_n] = \int_R g\,dF = f(z_1, z_2, \cdots, z_n)$. F.A. indicators are given as follows: $u(z_1, z_2, \cdots, z_n) = E(\prod 1/(z_i - x_i))$ (antisymmetric indicators); $v(z_1, z_2, \cdots, z_n) = E(\prod 1/(1 - z_i x_i))$ (symmetric indicators); $w(z_1, z_2, \cdots, z_n) = E(1/(1+\sum z_i x_i))$ (projective indicators). Relations among indicators are given by using convenient operators. Relations among F.A. indicators and Levy's characteristic functions are easely obtained when the n random variables are independent. The operator to be used is a product of nR_1 transforms, viz: $\phi(it) = E(e^{itx}) = e^{itx} w(it) = e^{itx} w(t)$. For random variables not independent the projective indicator is advisable. When F.A. indicators are regular at the origin the problem of moment could be solved by known procedures. (Received November 10, 1954.)

353. K. L. Chung: *Continuous parameter Markov chains*. Preliminary report.

In part one only the case of stable states is considered, namely all $q_i = -p'_{ii}(0+) < \infty$. A complete description of the sample functions of such a chain is given. From this we can define the successive intervals of sojourn and the various random variables connected therewith. Recurrence and ergodicity properties follow, in the same pattern as in the discrete parameter case. Many known results in the latter case, including a number of them in my *Contributions to the theory of Markov chains* (Journal of Research, Bureau of Standards vol. 50 (1953) pp. 203–208; Trans. Amer. Math. Soc. vol. 76 (1954) pp. 397–419) are extended to the continuous parameter case. Comparisons with discrete skeletons are given. The main limit theorem is an extension of Doeblin's ratio theorem, from which the average limit theorem follows. An ergodic theorem in the ratio form is proved in any recurrent class. This is preceded by a discussion of certain stochastic integrals. Some of these results have been extended to instantaneous states (with $q_i = \infty$) and will be given in part two. (Received September 20, 1954.)

354. Joseph Gillis: *Random walk with correlation.*

Random walk on a d-dimensional lattice is investigated such that, at any stage, the probabilities of the step being in the various possible directions depend upon the direction of the previous step. The motion may be characterized by a generating function which is here derived. The generating function is then used to obtain some general properties of the walk. Certain special cases are considered in greater detail. The existence of recurrent points is investigated in particular, and it is shown that Pólya's result (Math. Ann. vol. 84 (1921) pp. 149–160) (that every point is a recurrent point for $d=1$, 2 but that there are no such points for $d \geq 3$) is unaffected by the correlation between adjacent steps. (Received November 1, 1954.)

355*t*. H. S. Konijn: *Properties of two classes of bivariate distribution functions.*

Let F denote the joint distribution of two independent random variables Y and Z. The paper investigates properties of (1) the joint distribution F_λ of the linearly transformed random variables Y_λ and Z_λ, of (2) the distributions (previously examined by Fréchet) F_+ [and F_-] of random variables which have the same marginal distributions as Y and Z but which are nondecreasing [nonincreasing] functions of each other, and (3) mixtures F_κ^+ [F_κ^-] of F and F_+ [F_-] with coefficient κ. (Received November 12, 1954.)

356*t*. H. S. Konijn: *On the power of certain tests for independence in bivariate populations.*

Let $\mathfrak{T}_0$ be the Spearman rank correlation test, $\mathfrak{T}_1$ the difference sign correlation test, $\mathfrak{T}_2$ the unbiased grade correlation test (which is asymptotically equivalent with $\mathfrak{T}_0$), $\mathfrak{T}_3$ the medial correlation test, and $\mathfrak{R}$ the ordinary (parametric) correlation test. (Whenever discussing $\mathfrak{R}$, existence of fourth moments is assumed.) Let the joint distribution of two independent random variables be a continuous function F. Properties of the power of the above-mentioned tests are found for the alternatives F_λ, F_κ^+ and F_κ (defined in the preceding abstract), particularly for alternatives "close" to the hypothesis of independence and for large samples. Against alternatives F_λ the efficiency of $\mathfrak{T}_3$ is found to depend strongly on local properties of the densities, which should invite caution, and the efficiency of $\mathfrak{T}_1$ with respect to $\mathfrak{T}_3$ is often unity. Against alternatives F_κ^+ and F_κ^- these tests are shown to be unbiased and equally efficient with efficiency independent of F, and the population parameters corresponding to the nonparametric test statistics are simply related to κ. Incidentally, Pitman's result on efficiency is extended in several directions. (Received November 12, 1954.)

357B. R. B. Leipnik: *Cycling a stationary process.*

If X is a stionary (wide sense) stochastic process, a the autocorrelation function of X, $a(s)=E[X_tX_{t+s}]$, and $E[X_t]=0$, and $\sum_{n=0}^{\infty} a(ns)$ converges for each $s>0$, then for each $T>0$, the processes $Y_{tn}^{(T)}=(1/n)^{1/2}(X_t+X_{t+T}+\cdots+X_{t+(n-1)T})$ converge in distribution to a stationary process $Y^{(T)}$ on $[0, T)$ with an autocorrelation function $a^{(T)}$ such that $a^{(T)}(s)=a^{(T)}(T-s)$ and $\lim_{T\to\infty} a^{(T)}(s)=a(s)$. The autocorrelation matrix $(a^{(T)}(t_i, t_j)$ is simpler in several respects than $(a(t_i, t_j))$ which is important for the problem of estimation in stationary processes. (Received November 12, 1954.)

358B. Emanuel Parzen: *On the convergence of distribution functions at points of discontinuity.* I.

Let $F_n(\mathbf{x})$ for $n=1, 2, \cdots$ and $F(\mathbf{x})$ be distribution functions on Euclidean N-space E_N, with lower limit 0 and upper limit 1. For simplicity, results are stated only for $N=2$, although they hold for general N. F_n is defined to converge to F at the point $\mathbf{x}$ if $\Delta_n(\mathbf{x})=|F_n(x_1+0,\ x_2+0)-F(x_1+0,\ x_2+0)|+|F_n(x_1+0,\ x_2-0)-F(x_1+0,\ x_2-0)|+|F_n(x_1-0,\ x_2+0)-F(x_1-0,\ x_2+0)|+|F_n(x_1-0,\ x_2-0)-F(x_1-0,\ x_2-0)|\to 0$ as $n\to\infty$. Usually considered in the literature is "complete" convergence: $F_n\rightarrow^{c}F$ if $\Delta_n(\mathbf{x})\to 0$ at every continuity point of F. "Total" convergence is defined: $F_n\rightarrow^{t}F$ if $\Delta_n(\mathbf{x})\to 0$ at every point $\mathbf{x}$. A condition involving convergence of jumps is also introduced, denoted $F_n\rightarrow^{J}F$, which has the basic property that: $F_n\rightarrow^{t}F$ if, and only if, $F_n\rightarrow^{c}F$ and $F_n\rightarrow^{J}F$. Necessary and sufficient conditions are given for these modes of convergence to hold in terms of the conditions stated in the following abstract (which lead to sufficient conditions in terms of characteristic functions). In the notation of the following abstract, total convergence is equivalent to (IV), complete convergence to (V), and jump convergence to (III). The results for jump convergence lead to conditions for convergence of conditional distribution functions in certain cases. (Received October 26, 1954.)

359t. Emanuel Parzen: *On the convergence of distribution functions at points of discontinuity.* II.

Given a distribution function F on Euclidean 2-space E_2, define $D(F;\ \mathbf{a},\ h)$ for $\mathbf{a}$ in E_2 and $h=(h_1,\ h_2)>0$ by $h_1h_2D(F;\ \mathbf{a},\ h)=\int_0^{h_1}\int_0^{h_2} ds_1ds_2\{F(a_1+s_1,\ a_2+s_2)-F(a_1+s_1,\ a_2-s_2)-F(a_1-s_1,\ a_2+s_2)+F(a_1-s_1,\ a_2-s_2)\}$. Given a sequence of distribution functions F_n, similarly define $D(F_n;\ \mathbf{a},\ h)$. Probabilistic significance is given to the following possible hypotheses. Suppose, for every $\mathbf{a}$ in E_2, $D(F_n;\ \mathbf{a},\ h)\to D(F;\ \mathbf{a},\ h)$ as (I) $n\to\infty$, $h_1\to 0$, $h_2\to 0$; (II) $n\to\infty$, $h_1\to 0$, h_2 fixed; (III) $n\to\infty$, $h_1\to 0$, uniformly in $h_2>0$; (IV) $n\to\infty$, uniformly in h_1, $h_2>0$; (V) $n\to\infty$, h_1, h_2 fixed. The foregoing conditions lead to conditions in terms of the characteristic functions f and f_n of the distribution functions. In particular, (IV) holds if $|f_n(t)-f(t)|\to 0$ as $n\to\infty$ uniformly in

$t \in E_1$, and (IV) holds if $|f_n(t) - f(t)| \to \infty$ as $n \to \infty$ and $\|t\| \to \infty$. (Received October 26, 1954.)

360. J. M. Shapiro: *Error estimates for certain probability limit theorems.*

Consider a system of random variables (x_{nk}) with distribution functions F_{nk}, $k = 1, 2, \cdots, k_n$; $n = 1, 2, \cdots$, such that for each n, $x_{n1}, \cdots, x_{nk_n}$ are independent. Let $S_n = x_{n1} + \cdots + x_{nk_n}$ and let $F_n(x)$ be the distribution function of S_n. Under the assumption that the x_{nk} have two moments, estimates are obtained on $M_n = \sup_{-\infty < x < \infty} |F_n(x) - F(x)|$ where $F(x)$ is any infinitely divisible distribution with mean μ and finite second moment such that $dF(x)/dx$ is bounded. Let $G(u)$ be the nondecreasing function given by Kolmogorov's formula for the representation of the characteristic function of $F(x)$. The estimates involve $\left| \sum_{k=1}^{k_n} \int_{-\infty}^{x_i} \mu^2 dF_{nk}(\mu + E[x_{nk}]) - G(x_i) \right|$ evaluated at a finite number of points x_i, the variances of the x_{nk}, and $|E[S_n] - \mu|$. It is shown that under necessary and sufficient conditions for $F_n(x)$ to approach $F(x)$, the bounds on M_n obtained approach zero as n becomes infinite. In the special case of the classical Poisson theorem (i.e. where the x_{nk} are determined by $P\{x_{nk} = 0\} = 1 - \lambda/n$, $P\{x_{nk} = 1\} = \lambda/n$, $k = 1, \cdots, n$) it is shown that $M_n \leqq c[4\lambda^2/n - 3\lambda^3/n^2 + \lambda^4/n^3]$ where $F(x)$ is the Poisson distribution with parameter λ and c is a constant. (Received October 28, 1954.)

361t. F. L. Spitzer: *On interval recurrent sums of independent random variables.*

The partial sums S_n of independent identically distributed random variables X_i, $i = 1, \cdots, n$, $n = 1, 2, \cdots$, are known to be interval recurrent if the X_i have a density $f(x)$ in $L^p(-\infty, \infty)$ for some $p > 1$, and if the distribution of $n^{-1/\alpha} S_n$ converges to the symmetric stable distribution of index α for some $1 \leqq \alpha \leqq 2$. The following limit theorems show how the value of α determines the recurrence behavior of S_n satisfying the above conditions. For every a and every $x > 0$, $\lim_{n \to \infty} \Pr [\min_{1 \leqq k \leqq n} |S_k - a| > xC(n, \alpha)] = E_{1-1/\alpha}(-x)$, where $C(n, \alpha) = (\alpha/2) n^{1/\alpha - 1} \sin (\pi/\alpha)$ if $1 < \alpha \leqq 2$, and $(1/2)\pi(\log n)^{-1}$ if $\alpha = 1$, and where $E_\gamma(z)$ is the Mittag-Leffler function. Further, if $\{a_n\}$ is any monotone positive sequence, then the probability that $|S_n - a| \leqq n^{1/\alpha} a_n$ for infinitely many values of n is zero or one according as the series $\sum_{n=1}^{\infty} a_n$ converges or diverges. (Received November 17, 1954.)

TOPOLOGY

362. R. D. Anderson: *Zero-dimensional infinite compact groups of homeomorphisms.*

The purpose of this paper is to contribute to the knowledge of the ways in which zero-dimensional infinite compact groups of homeomorphisms can operate effectively on locally connected metric continua. A group G of homeomorphisms is said to operate on a closed point set X in a strongly effective way if for any element g of G and any point x of X, $g(x) = x$ only if g is the identity. The principal result is to show that the dyadic group (all of whose elements except the identity are of infinite order) can operate in a strongly effective way on the (one-dimensional) universal curve. It is also noted that, rather trivially, the compact infinite-dimensional torus also admits the dyadic group operating in a strongly effective way. Such an example apparently is not generally known. Standard known examples of p-adic groups operating effectively on locally connected continua all involve a type of "branching" clearly not present if the group operates in a strongly effective way. The examples cited may also be used to exhibit regular almost periodic homeomorphisms of locally connected spaces with no periodic points. (Received November 15, 1954.)

363. R. D. Anderson and Mary-Elizabeth Hamstrom (p): *On spaces filled up by continuous collections of arcs.*

The principal result of this paper is the theorem that if G is a nondegenerate continuous collection of arcs filling up the compact metric continuum M, then no connected open subset of M is separated by any closed and totally disconnected point set. It follows immediately that M must be at least 2-dimensional at each of its points. A slight extension of the methods establishes that if in addition to the hypotheses above G is a Cantor manifold of dimension greater than 1, then M is not separated by any rational curve. (Received November 15, 1954.)

364t. R. M. Baer: *A characterization theorem for lattices with Hausdorff interval topology.*

A characterization theorem for Boolean algebras with Hausdorff interval topology has been given by Katetov [Colloq. Math. vol. 4, pp. 229–235] and Northam [Proc. Amer. Math. Soc. vol. 4 (1953) pp. 824–827]. The general lattice is the subject of the following theorem. First, write (with respect to some underlying partially ordered set) $\dot{a}$ for the set $[x\colon x\leqq a]$, and $\breve{a}$ for the set $[x\colon a\leqq x]$, and define, for nonempty *subsets* X and Y in the space, $X<Y$ to mean: if x in X, y in Y, then either $x<y$ or x and y incomparable (or $X\leqq Y$, in which case $x\leqq y$ or x and y incomparable). Theorem: a necessary and sufficient condition that the interval topology of a lattice L be Hausdorff is that, for every pair of elements a, b in L with $a<b$, there exist finite, nonempty subsets A and B in L such that both the following (i) and (ii) are satisfied. (i) $a<A\leqq b$, $a\leqq B<b$; (ii) $(\breve{x})$, $x\in A$, $(\dot{y})$, $y\in B$ is a covering of L. (Received September 23, 1954.)

365. Lida K. Barrett: *Regular curves and regular points of finite order.*

J. R. Kline has raised the question whether for any positive integer n greater than two, there exists a continuous curve such that between each pair of its points there are exactly n simple arcs mutually exclusive except for end points. J. H. Kusner has settled this question for n equal to three or four (*On continuous curves with cyclic connection of higher order*, Comptes Rendus de Varsovie vol. 25 (1932) Classe III, pp. 71–92). In this paper it is shown that for any integer n greater than two no such curve exists. The final part of this paper answers two questions raised by W. L. Ayres (*On the regular points of a continuum*, Trans. Amer. Math. Soc. (1931)) concerning continuous curves containing points of only two orders. Ayres conjectured that if $m>n>2$ then (1) the points of order m must be countable and (2) for some integer k, $m=k(n-1)$. Examples are given to show that neither of these conjectures is true. (Received November 10, 1954.)

366. Raoul Bott: *On the Sturm intersection theory.*

The complete set $X_\lambda(t)$ of fundamental solutions of a regular self-adjoint system of second order differential equations $Ly=\lambda y$ gives rise to a map $X^L\colon (t, \lambda)\rightarrow X_\lambda(t)$ of the t, λ plane into a certain Lie group H. It is shown that every self-adjoint boundary condition B defines a cycle $\phi(B)$ in H, such that the intersections of the curve $\lambda \rightarrow X_\lambda(a)$ with $\phi(B)$ (in the sense of algebraic topology) correspond precisely to the spectrum of the problem $Ly=\lambda y$ subject to B at $t=0$ and $t=a$. This point of view permits the generalisation of the classical oscillation and comparison theorems of Sturm (for the case $n=2$) to higher dimensional systems (as was done by Morse in *The Calculus of variation in the large*, Amer. Math. Soc. Colloquium Publications, vol. 18) and to general boundary conditions. They are all consequences of the fact that as X^L is defined on all of the (t, x) plane, the image of an oriented curve F in that plane under X^L will be homologous to zero and hence have total algebraic intersection zero with $\phi(B)$. These methods are applied to refine comparison theorems announced by M. Morse and E. Pitcher (*On certain invariants of closed extremals*, Nat. Acad. Sci. vol. 20 (1934) pp. 282–287) concerning periodic operators. (Received November 12, 1954.)

367. Eldon Dyer: *Certain transformations which reduce dimension. II.*

In this paper it is shown that if G is a continuous collection of mutually exclusive dendrons filling up a compact metric n-dimensional space, then G with respect to its elements as points is an $(n-1)$-dimensional space. The above statement is true if "dimension" means either Brouwer-Menger-Urysohn dimension or homology dimension over the integers modulo k, $k \geq 2$. A corollary of this theorem is that if $\mathfrak{A}$ is either the additive group of real numbers modulo 1 or that of the integers modulo k, $k \geq 2$, and L is a compact metric n-dimensional space (homology dimension with coefficient group $\mathfrak{A}$) having a nontrivial nth-homology group over $\mathfrak{A}$, then L is not filled up by a continuous collection of mutually exclusive dendrons. (Received November 12, 1954.)

368. W. M. Faucett: *Topological semi-groups and trees.*

A mob S is a Hausdorff space that admits a continuous associative multiplication. A compact connected Hausdorff space S is a tree if for every pair of points in S there exists a third point separating them. A preliminary result states that if there exists $k \in K$, the minimal two-sided ideal of the compact connected mob S, such that $K-k$ is the union of two separated sets, then either (i) every element in K is a left zero for S, or (ii) every element in K is a right zero for S. Let S be a metric tree and N the set of end points of S. It is shown that if S is a mob and if N is a group, then S has a zero. Further, if N is an abelian group, then S is abelian. Another result states that if S has a unit, then S is abelian if and only if N is contained in the center of S. All of these results have been verified for the compact Hausdorff case. (Received November 15, 1954.)

369B. H. A. Forrester: *Acyclic models and de Rham's theorem.*

Let $\mathcal{A}$ be a category, $\mathcal{G}_\Lambda$ the category of left Λ-modules, Λ a ring with unit. $F(\mathcal{A}, \mathcal{G}_\Lambda)$ denotes the category of covariant functors $T: \mathcal{A} \rightarrow \mathcal{G}$. A direct (inverse) model theory (R, Γ) is formed by a covariant functor $R: F(\mathcal{A}, \mathcal{G}_\Lambda) \rightarrow F(\mathcal{A}, \mathcal{G}_\Lambda)$ and a transformation $\Gamma: R \rightarrow I$ (resp. $\Gamma: I \rightarrow R$), where I is the identity functor. This is a natural version of the Eilenberg-MacLane concept of models (S. Eilenberg and S. MacLane, *Acyclic models*, Amer. J. Math. vol. 75 (1953) pp. 189–199). A representative Ψ of a functor $T: \mathcal{A} \rightarrow \mathcal{G}_\Lambda$ is a transformation $\Psi: T \rightarrow R(T)$ such that $\Gamma(T)\Psi$ is the identity transformation (for inverse model theories the order must be reversed);

if $\Psi(A):T(A)\to R(T)(A)$ for each $A\in\mathcal{A}$, but no naturality conditions are required, Ψ is called a semi-representation. The fundamental theorems of Eilenberg-MacLane can be established. Further, *if (R, Γ) is an inverse model theory, K, L are cochain functors, $R(K)$ and $R(L)$ are acyclic, and K, L are representable, and if $f:K\to L$ is a cochain map which is a homotopy equivalence in low dimensions, then $f_*:H(K)\to H(L)$ is an isomorphism.* If ϕ is the de Rham functor of differential forms, C is the real singular cochain functor, and $f:\phi\to C$ is the Stokes' map, it follows that $f_*:H(\phi)\approx H(C)$. It is easy to see then that f_* preserves products. (Received October 25, 1954.)

370*t*. F. B. Jones: *A note on homogeneous metric continua.*

Every compact, decomposable, metric continuum has a nondegenerate, aposyndetic decomposition such that (1) the decomposition (with respect to its elements as points) is a compact, homogeneous, metric continuum and (2) each element of the decomposition is a homogeneous continuum. Roughly speaking, the elements of the decomposition are atomic. (Received November 15, 1954.)

371B. V. L. Klee, Jr.: *Separation properties of convex cones.*

Except in rather special cases, the known separation theorems for convex sets require that the sets involved have no common point, and thus fail to cover the interesting case of two convex cones whose intersection and common vertex is the origin ϕ. This case is discussed in the present note, the principal results extending theorems of Aronszajn and of Tagamlitzki. A *ϕ-cone* in a topological linear space is a closed convex cone with vertex ϕ. For a ϕ-cone A, $A'=A\cap -A$. Theorem. *Suppose E is a locally convex topological linear space, A and B are ϕ-cones in E such that $A\cap B=\{\phi\}$, and A is locally compact. Then E admits a continuous linear functional f such that $f<0$ on $A\backslash A'$, $f=0$ on $A'\cup B'$, and $f\geqq 0$ on $B\backslash B'$. If, in addition, either B is locally compact or E is normable and B separable, then E admits f such that $f<0$ on $A\backslash A'$, $f=0$ on $A'\cup B'$, and $f>0$ on $B\backslash B'$.* Examples show that these results cannot be substantially improved. In particular, Hilbert space contains a disjoint pair of linearly bounded closed convex sets which cannot be separated by any hyperplanes and hence contains a pair of ϕ-cones A and B which cannot be separated even though $A'=B'=A\cap B=\{\phi\}$. (Received October 18, 1954.)

372. Harold W. Kuhn: *A combinatorial lemma equivalent to the Brouwer fixed-point theorem.*

The following combinatorial lemma is equivalent to the Brouwer fixed-point theorem: Denote by I_n the cube in E_n formed by all points $x=(x_k)$ with $|x_k|\leqq 1$ for $k=1,\cdots,n$ and let K be any subdivision of I_n by hyperplanes orthogonal to the coordinate directions. Suppose further that each vertex x of the subdivision is labeled with $\lambda(x)=(\lambda_k(x))$, where $\lambda_k(x)=\pm 1$ subject only to $\lambda_k(x)=x_k$ if $x_k=\pm 1$ for $k=1,\cdots,n$. Then there is a rectangular n-cell of the subdivision with both labels in each coordinate direction appearing among its vertices. (Received November 12, 1954.)

373. E. B. Leach: *Cohomology theory in the space of n complex variables.*

Let R be an exact (schlicht) existence domain for an analytic function. Let Ω^p be the faisceau of pure differential forms of degree p, with complex analytic coefficients, defined near points of R. Dolbeault [C. R. Acad. Sci. Paris vol. 236 (1953) pp. 175–

177] has given an analytic characterization of $H^q(\Omega^p)$, the group of cohomology with complex-analytic p-form coefficients. In the case $q=1$, it has been shown that $H^1(\Omega^p)$ is trivial, i.e., it contains only the identity element. A proof of this is given by H. Cartan [Bull. Soc. Math. France vol. 78 (1950) pp. 29–64]. In this paper, it is shown that the groups $H^q(\Omega^p)$ are trivial for all $q \geqq 1$. By Dolbeault's theorem, this is accomplished by showing that certain systems of partial differential equations which possess local solution possess solution throughout R. This is done by a method similar to that of Cartan. (Received October 13, 1954.)

374. L. F. McAuley: *A note on generalized upper semi-continuous collections.*

With the knowledge that an aposyndetic connected topological space is a Hausdorff topological space, the following generalization of upper semi-continuous collections is useful in a study of aposyndeteic decompositions of continua. Suppose that G is a collection of point sets. A subcollection R of G is said to be a region in G provided that there exists no point set P in G^*-R^* such that (1) for g in G, $g \cdot P$ has no limit point in R^* and (2) R^* contains a limit point of P. An element g of G is said to be a limit element of a subcollection H of G provided that every region R in G which contains g also contains an element of $(H+g)-g$. A collection G of point sets is said to be upper semi-continuous provided that if (1) H is a subcollection of G and (2) g and h are two limit elements of H, then there exists a subcollection K of H such that (a) either g or h is a limit element of K and (b) the limiting set of K either (i) contains no point of $h-h\cdot g$ or (ii) contains no point of $g-g\cdot h$. (Received November 15, 1954.)

375*t*. P. J. McCarthy: *Non-archimedean uniform structures.*

A uniform structure on a set E is called non-archimedean if it has a fundamental system of entourages consisting entirely of entourages V satisfying $V \circ V=V$. A uniform space is called non-archimedean if its uniform structure is non-archimedean. These uniform spaces were defined and studied by A. F. Monna (Proc. Kon. Nederl. Akad. Wetensch. vol. 53, pp. 470–481, 625–637). It is shown that if E is a 0-dimensional space (Hausdorff or not), then there is a non-archimedean uniform structure on E compatible with the topology of E. Furthermore, if E is Hausdorff, it is precompact with respect to this uniform structure. Using this result it is proved that every 0-dimensional Hausdorff space is homeomorphic to a dense subset of a compact 0-dimensional space. (Received November 9, 1954.)

376. F. P. Palermo: *The cohomology rings of product spaces.*

Let K_1, K_2 be finite simplicial complexes. It may be shown by examples that $H(K_1\times K_2)$, the integral cohomology ring of $K_1\times K_2$, is not determined by $H(K_i)$, the integral cohomology rings of K_i, $i=1, 2$. In this paper it is shown that $H(K_1\times K_2)$ is completely determined by the following: $H(K_i)$ and $H(K_i, n)$, the cohomology ring of K_i with coefficients in the ring of integers mod n, for all integers n, together with the group homomorphisms $\Delta_n: H(K_i, n)\rightarrow H(K_i)$ and $h_{m,n}: H(K_i, n)\rightarrow H(K_i, m)$ for all integers m, n $(i=1, 2)$. Here Δ_n is the Bockstein coboundary homomorphism and $h_{m,n}$ is induced by the homomorphism which takes a cocycle x mod n into $(m/(m, n))\cdot x$ which is a cocycle mod m (J. H. C. Whitehead, Comment. Math. Helv. vol. 22 (1949) pp. 48–92). A ring $R(K_1, K_2)$ which is naturally isomorphic to $H(K_1\times K_2)$ is constructed from the rings $H(K_i)$ and $H(K_i, n)$ and the homomorphisms Δ_n and $h_{m,n}$ for all integers m, n. This construction gives a functor in the sense of Eilenberg and MacLane (Trans. Amer. Math. Soc. vol. 58 (1945) pp. 231–294). The construction can be extended to infinite simplicial complexes. This completes and extends the earlier work of M. Bockstein (C. R. (Doklady) vol. 40 (1943) pp. 339–342). (Received October 25, 1954.)

377*t*. J. P. Roth: *An application of algebraic topology of numerical analysis: Kron's method.* I.

It is shown that the process of solving an electrical network in terms of itself is formulable in terms of a "twisted" isomorphism of the homology and cohomology sequence of the 1-dimensional complex defined by the network. Specifically, the space of mesh currents coincides with the first homology group $H^1(K)$ (complex coefficient field), the space of branch (coil) currents with the group $C^1(K)$ of 1-chains; the equations (Kron, *Tensor analysis of networks*, New York, Wiley, 1939); $i = Ci'$ an expression of the natural homomorphism of $H'(K)$ into $C'(K)$. The space of node currents corresponds to the group of bounding zero chains, the map $I' = A_t I$ being an expression of the boundary operator (also $A_t i = 0$). The space of node potentials is the group of 0-cochains, $C_1(K)$ the space of coil voltages, $E = AE'$ the coboundary operator, $H_1(K)$ the space of mesh emf's and $e' = C_t e$ the natural homomorphism of $C_1(K)$ on $H_1(K)$. The mapping Z and its inverse Y are isomorphisms defined by the "hardware" of $C^1(K)$ and $C_1(K)$. It is shown that $Z' = C_t Z C$ has an inverse if Z is "power-definite —a generalization of Hermann Weyl's result (Revista Math. Hispano-Americana vol. 5 (1923))—as well as $A_t Y A = Y'$. (Received November 15, 1954.)

378. J. P. Roth: *An application of algebraic topology to numerical analysis: Kron's method.* II.

By suitable interpretation of Kron's method of solving a network by means of "tearing" in terms of homomorphisms of homology and cohomology sequences induced by simplicial mappings of 1 dimensional complexes, the author obtains a proof of the validity of this method. Specifically let K, $\tilde{K}$ be 1 complexes with the same number of branches, ϕ a simplicial map of $\tilde{K}$ on K. Let N be a matrix defining the isomorphism of $C_1(K)$ on $C_1(\tilde{K})$, bases for each space being chosen. Then if $\tilde{Y}$ is the inverse of $\tilde{Z}$, the inverse of Z is $N_t \tilde{Y} N$, computed exclusively in terms of the inverse $\tilde{Y}$ of $\tilde{Z}$ and the matrix N of the map ϕ having entries of 0, 1, -1. The application to numerical analysis arises from the fact that finite difference equations can be represented by "electrical" networks (Kron, Journal of Applied Physics vol. 24 (1953)). (Received November 15, 1954.)

379. Mary E. Rudin: *A property of normal spaces related to the existence of a Souslin space.*

A set S is called a Souslin space if (a) S is linearly ordered, (b) S is not separable, (c) there are at most a countable number of mutually exclusive segments in S. The existence of Souslin spaces is an open problem. It is proved that if a Souslin space exists, then there exists a normal Hausdorff space X with the following property: there is a sequence $M_1, M_2, M_3, \cdots$ of closed sets in X such that $M_1 \cdot M_2 \cdot M_3 \cdot \cdots = 0$ but $N_1 \cdot N_2 \cdot N_3 \cdot \cdots \neq 0$ for every sequence $N_1, N_2, N_3, \cdots$ of open sets such that $M_i \subset N_i$. (Received November 8, 1954.)

380. Annette Sinclair: *Pseudo locally compact spaces.*

The author defines a topological space $(\mathcal{R}, \mathfrak{R})$ to be *$\mathfrak{R}'$-pseudo locally compact* if any neighborhood by the $\mathfrak{R}$-topology of an arbitrary point p of $\mathcal{R}$ contains some neighborhood $N_p(\mathcal{R}, \mathfrak{R})$ by the $\mathfrak{R}$-topology such that the point set $\overline{N}_p(\mathcal{R}, \mathfrak{R})$ is compact by the $\mathfrak{R}'$-topology. Examples are given of a set of functions analytic in a region with a topology such that the function space so defined is not locally compact but has a property close to local compactness. If $\mathcal{R}$ is the set of functions analytic in an open region R and $\mathfrak{R}$ is the topology induced by the metric $d(f, g) = \sup_{z \in R} |f(z) - g(z)|$, then $(\mathcal{R}, \mathfrak{R})$ is not locally compact. Let $\mathfrak{R}'$ be the topology induced by the metric $\rho(f, g) = \sum_{n=1}^{\infty} (1/2^n) d_n(f, g)/(1 + d_n(f, g))$, where $d_n(f, g) = \max_{z \in R_n} |f(z) - g(z)|$ for $\{R_n\}$ a certain monotone increasing sequence of R-covering sets. Then $(\mathcal{R}, \mathfrak{R}')$ is also not locally compact. However, the space $(\mathcal{R}, \mathfrak{R})$ is $\mathfrak{R}'$-pseudo locally compact. (Received November 8, 1954.)

381. Fred Supnick: *On the Hauptvermutung for polyhedral 3-manifolds with boundary.*

E. E. Moise proved the Hauptvermutung for 3-manifolds (Ann. of Math. (1952) pp. 96–114); R. H. Bing extended this to 3-manifolds with boundary (Ann. of Math. (1954) p. 150). In their work the 3-simplexes of the triangulations and their subdivisions are not necessarily *rectilinear* (i.e. straight-edged, flat-faced (open) tetrahedrons in E^3). In the present work this *rectilinear* restriction upon the simplexes of the triangulations and their subdivisions is made. (Two n-complexes are *isomorphic* if there is a one to one correspondence between the i-simplexes of one, and those of the other, such that incidence relations are preserved.) The following theorem is proved: *Let K_1 and K_2 be locally finite rectilinear simplicial 3-complexes in E^3, which are homeomorphic 3-manifolds with boundaries B_1 and B_2 respectively, B_1 and B_2 being locally finite rectilinear simplicial 2-complexes. Then K_1 and K_2 possess isomorphic rectilinear simplicial subdivisions. In particular, if K_1 and K_2 are finite (in number of simplexes), then K_1 and K_2 possess isomorphic rectilinear simplicial finite subdivisions.* Theorem 3 of Bing (ibid. pp. 148–149) plays an essential role in the proof. (Received November 12, 1954.)

382. Hidehiko Yamabe: *Compact transformation groups on spheres.*

Let G be a compact group of differential homeomorphisms on d dimensional sphere S^d, admitting one fixed point p_0. Assume there is a Riemannian structure invariant under G. Then eigenspaces of the Laplace Bertrami operator Δ give representation spaces of G, which are decomposed into irreducible spaces under G. On the other hand tangent space at p_0 turns out to be another representation space. Hence by applying approximation theorem and Schur's Lemma, one has d eigenfunctions of Δ which form a coordinate system around p_0. By applying a fixed point theorem, another point p_0' is obtained where all these d functions take the value 0. If the Riemannian structure is analytic, and if G is connected, this p_0' can be proved to be another fixed point of G by considering the intersection of zero points of these d functions. (Received October 4, 1954.)

R. D. SCHAFER,
Associate Secretary
J. W. GREEN,
Associate Secretary

BOOK REVIEWS

Calculus, a modern approach. By K. Menger. 2d ed. Illinois Institute of Technology Bookstore, 1953.

This book is the outcome of a long program of research by Professor Menger to simplify the ideas and notation of calculus. The most striking feature of it is the treatment of variables.

The classical variable, for which the notations of calculus were developed, was vaguely a generic symbol x which stands for the elements of a set X. This is akin to the algebraist's idea of an indeterminate. In the classical variable terminology, the symbol $f(x)$ represents a variable which is a function of the variable x. The Weierstrassian variable x was a symbol which stands for a "fixed" ("arbitrary") but unspecified element of a set X. Ordinarily a proposition involving a Weierstrassian variable is preceded by quantification. In terms of Weierstrassian variables the symbol $f(x)$ is regarded as representing the value of the function f at x.

The confusion between the symbol which represents the function and that which represents its evaluation remains a serious defect in the language of calculus to the present day, for although a writer may set out to employ a pure Weierstrassian point of view, and though all goes well with the basic topological theory, he gets into trouble with the machinery of calculus when he comes to speak of the function x^2+1, or $\sin(n\theta+\phi)$. The trouble grows deeper when he begins to cope with substitution theory, with multiple integrals and partial derivatives.

Menger retains the Weierstrassian variable with more or less standard notation but he makes a sharp break with the classical variable x whose domain is the set X, replacing it by the identity function I on X to X. The result is not only a conceptual clarification and simplification of the ideas of calculus, but, when this program is worked out in detail in an elementary text, there appear some illuminating perspectives on the rôles of the two languages. The Weierstrassian language works well for the function theoretic foundations, but once we are in possession of the properties of continuous functions on compact intervals, the calculus proper is better served by Menger's modern substitute for the classical variable language. Essentially what happens is that the introduction of the identity function in place of a variable carries the subject upstairs into the realm of linear operators in the algebra $C(x)$ of continuous functions.

This is the proper framework for calculus; that is, for the representational and calculational aspects of the theory.

In the opinion of the reviewer it would be a mistake to underestimate the importance of Menger's contribution by regarding it as merely a notational change. Moreover, in this pioneer work it is essential to the author's purposes to adopt a new system of symbols in order to emphasize the distinctions. However, for the purposes of this review it may be helpful to consider the notational effects of making the opposite choices to the ones made by Menger. Since a third meaning of the symbol x has always been the identity function; let us adopt this meaning for x and distinguish the other variable ideas by different devices. For the Weierstrass variable in the domain of x a time-honored device is to replace x by x_1 or x_2. A second device is to replace x by one of the earlier letters of the alphabet, as in $f(b)-f(a)$. A third device could be used for emphasis, square brackets, $f[a]$ for "f at a"; thus saving parentheses for composition as in $\sin(2x+1)$. Then $f(x)$ becomes identical with $f\colon X\to R$ and has the added virtue that $f(x)$ indicates more about the domain than f does. Then $\int_E f(x)dx$ means the integral of f over some set E in its domain, X, with respect to the identity function $x\colon X\to X$. And $D_x(2x^2+1)$ has a proper meaning which can be achieved only by involved circumlocution in terms of Weierstrass variables. Menger writes this $\mathcal{D}(2I^2+1)$.

Actually a second basic function is needed to make the scheme elegant if not to make it conceptually correct. This is the one-function $\mathbf{1}\colon R\to R$ defined for every a in R by $\mathbf{1}[a]=1$. With it, the polynomial ring $P[x, \mathbf{1}]$ becomes algebraically isomorphic to the polynomial ring $P[x, 1]$ over R in which x is an indeterminate. But now $P[x, \mathbf{1}]$ is a ring of polynomials *as functions* $p(x)\colon R\to R$. Menger uses the notations c and $\underline{c}$ for constant functions which is rather inconsistent with the idea of generating functions by ring operations in $C(x)$.

A second outstanding feature of this book is the author's painstaking approach to the mathematical representation of physical quantities. Here he virtually abandons his identity function as a variable in favor of a scheme of pairing the states of a physical observable with numbers in such a way that the "physical variables" which emerge are apparently old fashioned scalar variables. In view of the good start towards the representation of physical quantities by linear operators which is inherent in the author's theoretical treatment of variables and functions, and in view of the now well established physical doctrine, "to every physical observable there corresponds a linear operator," why did the author set up a separate idea of a physical variable?

The reviewer pondered this question at some length and offers the following conjecture. There are several closely related geometries in the Cartesian plane, the confusion of which can be as troublesome as the confusion of different concepts of variable, *especially in the mathematical representation of physical quantities*, or "dimensional analysis" in the physicist's sense. The principal geometry of the Cartesian plane for the purposes of measurement is that of the Cartesian product of the affine line and affine line $A_1 \times A_1$ in which independent transformations, $x' = ax + b$, $y' = cy + d$, $a \neq 0$, $c \neq 0$, are admitted. Physical laws must have statements invariant in this geometry. The subject of dimensional analysis lives in this geometry. However, if in apparently innocuous indulgence, the more familiar Euclidean plane geometry E_2 is adopted instead, the subject of dimensional analysis vanishes, leaving behind only ghosts whose goings on are to be recorded by fiat. The author is not explicit about his choice of geometry but the reviewer's conjecture is that his point of view about "physical variables" is that of a man who is thinking in E_2.

In the opinion of the reviewer a consistent operator approach to the representation of physical quantities in the elementary sense is still needed to prepare the way for this point of view which forces itself upon us of necessity in more advanced situations. The author of this book has laid the groundwork for it by introducing the identity function in lieu of the scalar variable, so that his linear operators $\mathfrak{D}$ and $\mathfrak{D}^{-1}$ work properly. The natural extension of this same idea should do much to simplify dimensional analysis.

Another interesting aspect[1] is that the variable used by Menger is much closer to the idea of a random variable than the conventional one, in fact, *is* a random variable with uniform distribution.

WILLIAM L. DUREN, JR.

Elements of number theory. By I. M. Vinogradov. Trans. from the 5th rev. ed. by S. Kravetz. New York, Dover, 1954. 8+227 pp. $1.75 paper; $3.00 cloth.

This elementary book was originally published in 1936 with a little under 100 pages and subsequent editions appeared every few years until the sixth in 1952 of almost double the size. This work, as most other Russian books, has not been easy to obtain in the past and its present appearance is gratefully noted. It is a very welcome addition to the books on number theory.

As has been observed by other reviewers, the main body of the text, containing a mere 77 pages, is largely of a standard nature. The remainder of the work, aside from tables listing indices, primes, and primitive roots, is devoted to the statement and solution of problems. These are almost 100 in number with many that have several parts; their statement occupies about 55 pages and their solution 84 pages. In addition, there are about 40 numerical exercises, their answers being given at the end of the book.

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[1] Pointed out by R. N. Bradt.

The problems, therefore, constitute the chief novelty of the work. As might be expected, these are very heavily slanted in the direction of the author's outstanding researches on exponential sums and their applications. Although the methods used in the solutions are elementary, many of the problems are too difficult for beginning students. On the whole, the book may be considered as an excellent introduction to the author's monograph *The method of trigonometrical sums in the theory of numbers* now also translated into English. As can be seen from the discussion below, the number of topics is limited and even such a classical topic as Diophantine equations only barely makes its appearance in the form of a few problems.

Chapter I, *Divisibility theory*, treats such material as the greatest common divisor, Euclid's algorithm and its relation to simple continued fractions, fundamental properties of these fractions, and the unique factorization theorem. Among the problems, one finds the Diophantine equations $x^2+y^2=z^2$, $x^4+y^4=z^4$, and the basic properties of Farey series.

Chapter II, *Important number-theoretic functions*, deals with the factorization of $n!$, with the functions $[x]$, $x-[x]$, $\sigma_s(n)$, $\mu(n)$, $\phi(n)$, and with the general properties of multiplicative functions. The problems are concerned with various identities connecting number-theoretic functions, problems on lattice points, the almost indispensable Euler-Maclaurin sum formula (here ascribed to Sonin), estimates and asymptotic formulas for various sums, and the Möbius inversion formula in sum and product form.

Chapter III, *Congruences*, deals with basic properties, complete and reduced residue systems, and the theorems of Fermat and Euler. A number of the problems are concerned with sums of the form $\sum_x \{f(x)-[f(x)]\}$ where x ranges over an interval, while others treat the related question of the number of lattice points in certain regions. Other problems are concerned with exponential sums.

Chapter IV, *Congruences in one unknown*, sets forth the continued fraction solution of the linear congruence, the usual theorems on congruences with composite and prime power moduli, and Wilson's theorem. The problems deal with the number of solutions of congruences, the Kloosterman sums and with various aspects of congruences.

Chapter V, *Congruences of the second degree*, contains the standard material on the Legendre and Jacobi symbols as well as the solution of the congruence $x^2 \equiv a \pmod{m}$. Among the unusual problems, one finds many on the sum of various types of Legendre symbols, the Gaussian sum $\sum_{x=1}^{m} e^{2\pi i a x^2/m}$ as well as other exponential sums.

Chapter VI, *Primitive roots and indices*, is concerned with the determination of all moduli having primitive roots and with the corresponding theory of indices. For such moduli, the congruence $x^n \equiv a \pmod{m}$ is treated. The modulus 2^α, which has no primitive root if $\alpha > 2$, is considered and the essentially unique representation $a \equiv (-1)^{\gamma} 5^{\gamma_0} \pmod{2^\alpha}$ is obtained when $2 \nmid a$. A large number of the problems deal with characters, character·sums, and exponential sums including the Kloosterman sums.

The translation reads smoothly except for a number of exceptions, and the typography is an improvement over the original even though the right-hand margins of the pages have not been rectified. Although the errors mentioned in an insert to the sixth edition have been corrected, a considerable number have been introduced and some others have not been caught. For example, there are a number of places where the numeral 1 has been used in place of the letter l (pp. 35, 125, 133). In other places, lower case letters have been used in place of upper case letters (p. 87) or vice versa (p. 34). In at least three places (pp. 128, 128, 214), the sign $=$ is used in place of the correct sign $\leq$. There are other errors as well. Also, unfortunately, the chapter and paragraph titles have been omitted from the tops of pages. It is to be hoped that the publisher will correct these deficiencies in a subsequent printing.

LOWELL SCHOENFELD

Enzyklopädie der mathematischen Wissenschaften. Vol. I, *Algebra und Zahlentheorie*. Part 1: A. *Grundlagen*; B. *Algebra*. 2d ed. Vol. I_1, no. 3, part I: I_1, 6, *Lineare Algebra*; I_1, 7, *Normalformen von Matrizen*. By G. Pickert. Leipzig, Teubner, 1953. 72 pp. 7.50 DM.

The new edition of the *Enzyklopädie*, which was long delayed by the war, seems now to be getting well under way, at least for the first parts, covering foundations, algebra and number theory. While retaining essentially as their goal the same one as the first edition, namely to give as complete as possible a description of the mathematics of their day, the editors have wisely dropped all the historical material which can be found in the first edition (and would have been needlessly duplicated); nor do they insist on complete coverage of the literature, and this is also a sensible move if it enables them to leave the main results in a better perspective. Finally, the general plan of the work has been deeply and drastically altered, to fit better the changes in outlook which have occured in mathematics during the last half-century; whether the editors have entirely succeeded in this respect is a question to which we shall return a little later.

At any rate nowhere is this change more apparent than in the first half of the present volume, which gives due emphasis to modern linear and multilinear algebra, a topic barely mentioned in the old *Enzyklopädie*. In 43 dense pages, it gives a luminous and thorough description of the concepts and results of this theory. The author has here followed rather closely the arrangement, notations and terminology of Bourbaki's corresponding chapters; modules, vector spaces, duality, linear equations are treated in this order; matrices duly come *after* linear transformations. The notion of tensor product is next taken up, in relation to multilinear mappings, and is followed by exterior algebra, which naturally leads to determinants, and their relations to duality in exterior algebra. In addition, much valuable material, not to be found in Bourbaki's treatise, is included here, such as a rather complete discussion of the abstract notion of linear dependence, various results on bases of modules over rings satisfying some kind of chain condition, determinants over sfields, the Kronecker identities for minors of a symmetric determinant (where Lepage's proof, which is only mentioned in a footnote, would have deserved a more explicit discussion), and several other interesting identities taken from Grassmann's works. But the author manages to bring in all this detail, and to give an almost exhaustive bibliography, without losing track of the main lines of thought at any moment; the result is very pleasant and readable, and constitutes a valuable addition to the literature on the subject.

The second half of the volume makes such a contrast with the first one as to come almost as a shock to an unprepared reader. No better anticlimax could have been imagined to the up to date account of linear algebra given in the first part, than this report on "normal forms of matrices," which, with the few additions needed to include the later literature, might have been lifted bodily from a book written twenty years ago. It seems that here the author chose to follow chiefly MacDuffee's *Ergebnisse* volume, so that he naturally suffers from the defects of his model. Such a book may have had its use in 1933; but, after all the efforts spent in the meantime by the proponents of the conceptual approach to linear algebra on the clarification of the theory, how is one to judge this miscellany of totally unrelated results, brought together for no other reason than their being expressed in terms of matrices? Thus, the notions of eigenvector, eigenvalue, and characteristic determinant are *followed* by results on divisibility of matrices, and this in turn by equivalence and similarity of matrices! Reversing completely the point of view of the first part, all these notions are first given for matrices, and only afterwards linked with intrinsic concepts (the translation of the theorem on elementary divisors of a matrix, in terms of the decomposition of a module in a direct sum of cyclic summands, is given only after more than a page of results expressed in pure matrix language). But if already this type of exposition completely blurs out the essential features of the theory, things become even worse when it comes to the so-called "congruence" of matrices, which occupies the last 10 pages of the book. From the general plan of "Band I" (Algebra and number theory) published by the editors, it appears that these paragraphs are intended as a treatment of the algebraic aspects of quadratic and hermitian forms (a further volume is planned for the "arithmetical" theory of these forms). If such is the case, one can only deplore that the author completely failed to carry out this plan. Quadratic or hermitian forms are not even mentioned, until, after 7 pages, the author, as a kind of remorse, introduces, in an "alternate" proof of a theorem of Toeplitz, the notion of *positive definite* hermitian form! The fundamental relationship between bilinear forms and linear (or semilinear) mappings of a space into its dual is never mentioned at all; nor, of course, any of the fundamental geometric notions of the theory (such as orthogonality, isotropic vectors, Witt's theorem etc.), which maybe are reserved for the sections of the *Enzyklopädie* on "Geometry"? The Clifford algebra (without its name) is given the most cursory treatment in 7 lines; after which we meet a paragraph which begins with the statement that the eigenvalues of a hermitian matrix are real, and in the next sentence deals with Sylvester's law of inertia, as if the two questions belonged to the same theory! One could add many more instances of the absence of any serious attempt to organize the material here presented, but enough has been said to warn the reader that if he looks for clarification of his ideas on the topic, he had better try some other book. Except for their references, this section is, in the opinion of the reviewer, of very little value, and it is to be hoped that the editors of the series will realize their mistake and give the whole subject the treatment it deserves in a later edition.

J. DIEUDONNÉ

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HIERARCHIES OF NUMBER-THEORETIC PREDICATES

S. C. KLEENE

The existence of hierarchies of point sets in analysis has long been familiar from the work of Borel and Lusin. The study of the hierarchies in number theory which we consider here began with a theorem presented to the Society in 1940 and published in [12]. These hierarchies have applications in foundational investigations, but we shall be concerned here with the exploration of their structure (using classical logic). We shall survey the previous results from the beginning, and conclude with a few new ones. We have endeavored to make the exposition complete enough so that the layman in this field can get the gist of the arguments without consulting the references.

1. **Recursive functions and predicates.** By a *number-theoretic function* (*predicate*) we mean a function, of independent variables ranging over the natural numbers 0, 1, 2, $\cdots$, x, $x+1$, $\cdots$, taking natural numbers (propositions, true or false) as values.

By *general recursive* functions (predicates) we mean ones whose values can be computed (decided) by ideal computing machines not limited in their space for storing information. A theory of such machines was given by Turing [30] and in less detail by Post [24] (also cf. [16, Chapter XIII]). The general recursive functions can also be described as those whose values can be expressed by equations derivable formally from "recursion equations" defining the functions, in a sense first formulated precisely by Gödel [9] who built on a suggestion of Herbrand's (also cf. Church's [3], and our [10; 12; 16, Chapter XI]). A *general recursive* predicate $P(x_1, \cdots, x_n)$ is then one whose *representing function* $\phi(x_1, \cdots, x_n)$ ($=0$ or 1 according as $P(x_1, \cdots, x_n)$ is true or false) is general recursive.

The computation of a value of a general recursive function may involve a search through the natural numbers for the least one y with a given property without a bound for such a y having already been computed. By allowing such searches also when they may not terminate, we obtain an extension of the class of the general recursive functions to the *partial recursive* functions, which need not be defined for all sets of arguments [11; 12; 16, Chapter XII]. By disallowing such

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searches altogether, and requiring the functions to be generated, starting with the variables, 0 and $x+1$, by "primitive recursions", in which the function value for an argument $y+1$ is given in terms of that for y, and substitutions, we obtain a specialization of the general recursive functions to the *primitive recursive* functions [8; 10; 12; 16, Chapter IX]. Predicates are *partial* (*primitive*) *recursive*, if their representing functions are such.

Since a particular partial recursive function is defined by a machine or by a system of equations, which is a finite object, the class of the partial (including the general) recursive functions is countable. It is in fact possible to assign numbers many-one to the partial recursive functions so that, given a number of a function and a set of arguments for it, we can compute the value (if defined). We state this result (with additional details) as a theorem.

I. *For each $n \geqq 0$: There is a fixed partial recursive function $\Phi_n(z, x_1, \cdots, x_n)$ such that, to any partial recursive function $\phi(x_1, \cdots, x_n)$ there is a number e* (called a *Gödel number of ϕ*) *for which $\phi(x_1, \cdots, x_n) \simeq \Phi_n(e, x_1, \cdots, x_n)$*. In fact: $\Phi_n(z, x_1, \cdots, x_n) \simeq U(\mu y T_n(z, x_1, \cdots, x_n, y))$ *where U is a primitive recursive function, T_n is a primitive recursive predicate, and μy means "the least y such that"*. (The normal form theorem.)

We write $\simeq$ instead of $=$ in equations between partial recursive functions to indicate that they can hold by both sides being undefined (e.g. $\phi(x) = \phi(x)+1$ is absurd, but $\phi(x) \simeq \phi(x)+1$ merely implies that $\phi(x)$ is undefined). The numbers e we call "Gödel numbers", because we obtained them in [10; 11; 12; 16, pp. 288, 330, 340] by applying essentially Gödel's method of numbering [8] to the systems of equations.

II. *For each m, $n \geqq 0$: There is a primitive recursive function $S^m_n(z, y_1, \cdots, y_m)$ such that, if e is a Gödel number of $\phi(y_1, \cdots, y_m, x_1, \cdots, x_n)$ as a function of all $m+n$ variables, then for each m fixed numbers $y_1, \cdots, y_m$, $S^m_n(e, y_1, \cdots, y_m)$ is a Gödel number of $\phi(y_1, \cdots, y_m, x_1, \cdots, x_n)$ as a function of the n remaining variables $x_1, \cdots, x_n$.*

We trust this will appear plausible. (For details, cf. [11; 16, p. 342 (erratum, 1952 printing, l. 7 from below, replace "$d * \cdots$" by "$d * [2 \exp \cdots]$")].)

III. *Given any partial recursive function $\chi(z, x_1, \cdots, x_n)$, there is a Gödel number e of $\chi(e, x_1, \cdots, x_n)$.* (The recursion theorem.)

Let f be a Gödel number of $\chi(S^1_n(y, y), x_1, \cdots, x_n)$, and let $e = S^1_n(f, f)$.

By III, we can introduce a partial recursive function ϕ whose value

$\phi(x_1, \cdots, x_n)$ is expressed partial recursively in terms of a Gödel number e of ϕ itself and the arguments $x_1, \cdots, x_n$; i.e. we can put $\phi(x_1, \cdots, x_n) \simeq \chi(e, x_1, \cdots, x_n)$ for χ a given partial recursive function and e a Gödel number of ϕ. Sometimes the form of χ will then enable us to infer that ϕ is general or even primitive recursive. [11; 16, pp. 352 ff.].

2. **The arithmetical hierarchy.** Most predicates used in elementary number theory can be expressed by starting with general recursive predicates and applying the logical operations of the restricted predicate calculus, i.e. the *propositional connectives* & (and), $\vee$ (or), $\overline{}$ (not) and $\rightarrow$ (implies), and the *quantifiers*, *generality* (x) (for all x) and *existential* (Ex) (there exists an x such that). We call these predicates *arithmetical*, following Gödel who introduced them a little differently [8; 9; 16, pp. 239, 285].

Let p_i be the ith prime number, counting 2 as the 0th. Let $(a)_i = \{$the exponent of p_i in a as a product of powers of distinct primes, if $a \neq 0$; 0, if $a = 0\}$ [16, p. 230]. Then $(x_0) \cdots (x_m)A(x_0, \cdots, x_m) \equiv (x)A((x)_0, \cdots, (x)_m)$, and similarly with existential quantifiers.

IV. *Every arithmetical predicate $P(a)$ is expressible in one of the forms*

$$\text{(a)}\quad R(a)\quad \begin{array}{llll} (Ex)R(a, x) & (x)(Ey)R(a, x, y) & (Ex)(y)(Ez)R(a, x, y, z) & \cdots \\ (x)R(a, x) & (Ex)(y)R(a, x, y) & (x)(Ey)(z)R(a, x, y, z) & \cdots \end{array}$$

where the R for each form is general recursive.

One of these forms will be obtained, starting with an expression for $P(a)$ as an arithmetical predicate, by first advancing the quantifiers to the front, and then contracting consecutive like quantifiers as just indicated [12; 16, p. 285]. For brevity we are stating IV–VIII for one variable a, but they hold for $x_1, \cdots, x_n$ instead ($n \geqq 0$ for IV, V, VIII, $n > 0$ for VI, VII).

V. *Given any general recursive predicate $R(a, x)$, there is a number f such that $(Ex)R(a, x) \equiv (Ex)T_1(f, a, x)$. Similarly for the other quantified forms of* (a). (The enumeration theorem.)

Let f be a Gödel number of $\mu x R(a, x)$. This function is defined for given a, by definition if and only if $(Ex)R(a, x)$; by f being its Gödel number if and only if $(Ex)T_1(f, a, x)$. Similarly $(Ex)(y)R(a, x, y) \equiv (Ex)(y)T_2(g, a, x, y)$ when g is a Gödel number of $\mu y\overline{R}(a, x, y)$. [12; 16, p. 281]. Incidentally, V shows that the same predicates are expressible in a given quantified form of (a) with a primitive as with a general recursive R.

VI. *To each of the quantified forms of* (a) *there is a respective predicate*

$$(Ex)T_1(a, a, x) \quad (x)(Ey)T_2(a, a, x, y) \quad (Ex)(y)(Ez)T_3(a, a, x, y, z) \quad \cdots$$
$$(x)\overline{T}_1(a, a, x) \quad (Ex)(y)\overline{T}_2(a, a, x, y) \quad (x)(Ey)(z)\overline{T}_3(a, a, x, y, z) \quad \cdots$$

of that form which is not expressible in the dual form, a fortiori not in any of the forms with fewer quantifiers. (The hierarchy theorem.)

Given a recursive $R(a, x)$, the f of V is a value of a for which $(x)\overline{T}_1(a, a, x) \not\equiv (Ex)R(a, x)$ [12; 16, p. 283].

By a *complete* 1-place predicate of a certain form (or belonging to a certain class) we mean a predicate $C(a)$ of the form and such that to each predicate $P(a)$ of the form there is a recursive function $\theta(a)$ (actually primitive recursive in all our examples) for which $P(a) \equiv C(\theta(a))$.

VII. *The predicate* $(Ex)T_1(a, a, x)$ *of* VI *is a complete predicate of the respective form* $(Ex)R(a, x)$ *of* (a). *Similarly for the other quantified forms of* (a).

Let e be a Gödel number of $\mu x R(a, x)$ as a function of a, t (in Church's λ-notation [16, p. 34], of the function $\lambda a t \mu x R(a, x)$). Then $S_1^1(e, a)$ is one of $\lambda t \mu x R(a, x)$, which for any t is defined if and only if $(Ex)R(a, x)$. So $(Ex)R(a, x) \equiv (Ex)T_1(S_1^1(e, a), t, x)$, whence substituting $S_1^1(e, a)$ for t, $(Ex)R(a, x) \equiv C(\theta(a))$ for $C(a) \equiv (Ex)T_1(a, a, x)$ and $\theta(a) = S_1^1(e, a)$. ([16, p. 343]. A different $C(a)$ was used by Post [25], and by the author in lectures in 1941; cf. [19, Footnote 8].)

VIII. *The general recursive predicates* $R(a)$ *are exactly those expressible in both the* 1-*quantifier forms of* (a). (Analog of Souslin's theorem.)

For $R(a) \equiv (Ex)R(a) \equiv (x)R(a)$. Conversely, if $R(a) \equiv (Ex)R(a, x) \equiv (x)S(a, x)$ with R and S recursive, then $\overline{R}(a) \equiv (Ex)\overline{S}(a, x)$, and by first calculating $\mu x(R(a, x) \vee \overline{S}(a, x))$ we can decide $R(a)$. ([16, p. 284]. The theorem was given by the author [12], by Post [25], and by Mostowski [22] who noted that it is analogous to a theorem of Souslin on Borel and analytic sets.)

3. Relative recursiveness. Instead of a machine for computing a function $\phi(x_1, \cdots, x_n)$ outright, we can imagine one which would compute ϕ, if there were supplied to the machine values of a function $\psi(x)$ on demand, i.e. for arguments arising in the computation. (In this paper the ψ shall be a completely defined function.) This idea leads us to functions and predicates *general* (*partial, primitive*) *recursive in* ψ, and predicates *arithmetical in* ψ. ([16, pp. 275, 326, 224, 291]. In terms of machines, the idea appeared in a special setting in Turing [31, §4]; in terms of general recursiveness, in [12].)

We can obtain relativized versions of the above theorems simply by modifying the proof of I to suppose values of ψ available in the computation of ϕ. However it is useful to observe that at any stage of the computation only a finite number of values of ψ can have been used. So the computation can be arranged to involve a search in which at the yth stage only the first y values $\psi(0), \cdots, \psi(y-1)$ of ψ have been made available. These y values can be represented by either of the two numbers

$$\bar{\psi}(y) = \prod_{s<y} p_s^{\psi(s)+1}, \quad \tilde{\psi}(y) = \prod_{s<y} p_s^{\psi(s)}.$$

Note that $y = \mathrm{lh}(\bar{\psi}(y))$ with lh primitive recursive [16, p. 230].

IX. *Theorems* I–VIII *hold when in place of the notions 'recursive', 'Φ_n', 'Gödel number', 'T_n', 'S_n^m', 'arithmetical', '1-quantifier forms' we use 'recursive in ψ', 'Φ_n^ψ', 'Gödel number from ψ', 'T_n^ψ', '$S_n^{m,1}$', 'arithmetical in ψ', '1-quantifier forms relative to ψ', respectively, except that U, $S_1^{m,1}$ are still primitive recursive* (call them then I*–VIII*). *Moreover* $T_n^\psi(z, x_1, \cdots, x_n, y) \equiv T_n^1(\bar{\psi}(y), z, x_1, \cdots, x_n)$ *where T_n^1 is primitive recursive.*

We have stated IX for one (completely defined) function ψ of one variable (hence the superscript "ψ" or sometimes "1" for the number of its variables). We could instead have a (completely defined) predicate Q, where ψ is its representing function; or we could use any finite list Ψ of (completely defined) functions and predicates [16, p. 292].

X. *There is a number g such that, if $A(a)$ is recursive in $B(a)$ with Gödel number e, then $(Ex)T_1^A(a, a, x)$ is recursive in $(Ex)T_1^B(a, a, x)$ with Gödel number $S_1^{1,1}(g, e)$.*

For any 1-place predicate B and number e, let B_e be the 1-place function partial recursive in B with Gödel number e. By the method of proof of VII*, there is a primitive recursive function θ, independent of B, such that $(Ex)T_1^{B_e}(a, a, x) \equiv (Ex)T_1^B(\theta(e, a), \theta(e, a), x)$ for values of B, e, a which make $\lambda x T_1^{B_e}(a, a, x)$ completely defined. Let g be a ***uniform*** Gödel number from P (i.e. one independent of P) of $\lambda ea P(\theta(e, a))$. [19, beginning 4.1]. A X* holds, e.g. with "Q, $B(a)$", "$T_1^{Q,A}$", "$T_1^{Q,B}$" in place of "$B(a)$", "T_1^A", "T_1^B".

The relation 'P is general recursive in Q' is reflexive and transitive [16, p. 275]. Hence 'P is general recursive in Q and vice versa' is reflexive, symmetric and transitive. Post in [25; 26] associates with each predicate P a *degree* so that P and Q possess the *same* degree exactly if each is general recursive in the other. The degree of P is *less than* that of Q, if P is recursive in Q but not vice versa. A complete predicate of a given form (cf. VII, VII*) is of maximal degree for predicates of that form. Applying X in two directions to predicates Q_1, Q_2 of the same degree, the degree of $(Ex)T_1^{Q}(a, a, x)$ depends only on the degree of Q. The structure of the system of the degrees is discussed in detail in [19].

The predicates of VI of the two $k+1$-quantifier forms are negations of each other, and hence are of the same degree. But in XII we shall see that they are of higher degree than any predicate expressible with fewer quantifiers. For this we need the following generalization of VIII due to Post [26].

XI. *For each* $k \geq 0$: *The predicates general recursive in predicates expressible in the k-quantifier forms of* (a) *are exactly the predicates expressible in both the* $k+1$-*quantifier forms.* (Post's theorem.)

That a predicate expressible in both $k+1$-quantifier forms is recursive in k-quantifier predicates is proved for $k>0$ essentially as before for $k=0$. The proof of the converse is a little too detailed to be sketched here; it can be based on the special form of V* in terms of $\bar{\psi}(y)$ or $\tilde{\psi}(y)$ [16, p. 293]. An XI* referring similarly to IV* is obtained by reading "forms relative to Q" for "forms".

Let $N_0(a) \equiv L_0(a) \equiv a = a$. Let the predicates in the upper row of VI be abbreviated $N_1(a)$, $N_2(a)$, $N_3(a)$, $\cdots$. Let $L_{k+1}(a) \equiv (Ex) T_1^{L_k}(a, a, x)$.

XII. *For each* $k \geq 0$: N_{k+1} *is expressible in a* $k+1$-*quantifier form of* (a), *is of maximal degree for predicates so expressible, and is of higher degree than* N_k. *Similarly with 'L' in place of 'N'.* (The strengthened hierarchy theorem.)

The proof with 'N' is immediate from VII, XI and VI; with 'L' it follows thence, using induction on k and XI, VII*. A XII* with 'N^Q' and 'L^Q' is based similarly on VI*, VII*, XI* ($N_0^Q(a) \equiv L_0^Q(a) \equiv Q(a)$; N_1^Q, N_2^Q, N_3^Q, $\cdots$ is the upper row of VI*; $L_{k+1}^Q(a) \equiv (Ex) T_1^{L_k^Q}(a, a, x)$).

4. **The extended arithmetical hierarchy.** Kalmár first noticed that nonarithmetical predicates can be defined by induction using quantifiers within the induction step. (Cf. Skolem [27]; we gave a simple example in [12; 16, p. 287].) Let $N(a, k) \equiv N_k(a)$, $L(a, k) \equiv L_k(a)$. Then N and L are of degree exceeding that of every arithmetical predicate. For each arithmetical predicate is recursive in one of N_0, N_1, N_2, $\cdots$, each of which is recursive in N, and the degrees of which are ascending; and similarly with 'L'. (N and L are of the same degree, by [19, Footnote 29].)

The predicate $L(a, k)$ can be contracted to a 1-place predicate $L(a) \equiv L((a)_0, (a)_1)$ of the same degree. Then by using XII* with $Q(a) \equiv L(a)$ we get a sequence of predicates of degrees ascending from that of L. This process can be repeated, so that we obtain predicates correlated to various transfinite ordinal numbers, beginning with L correlated to ω.

However for this to make sense we must be careful in our handling of the ordinals. We might correlate to ω also $L(a, \phi(k))$ where $\phi(k)$ is any other increasing function than $\phi(k) \equiv k$. But then by the next theorem, various choices of ϕ would give predicates correlated to ω of arbitrarily high degrees. (To any function ψ, $\phi(k) = \sum_{x<k}\psi(x)+1$ is an increasing function of the same degree.)

XIII. *For any function* $\phi(k)$: *The function* $\phi(k)$ *is general recursive in the predicate* $N(a, \phi(k))$, *and in the predicate* $L(a, \phi(k))$.

Proof. For 'N', using the actual definition of $T_n(z, x_1, \cdots, x_n, y)$ [16, pp. 278, 281], it can be seen that $\mu a N_k(a)$ for $k=0$ is 0, and for $k>0$ is the Gödel number ($\neq 0$) of a system of equations consisting of a single equation $\mathrm{f}(\mathrm{r}_1, \cdots, \mathrm{r}_k)=0$, whence k itself can be read. Thus k (as a function of 0 variables) is partial recursive in N_k, *uniformly* in the sense that the indicated method of calculating k from N_k is applicable independently of what predicate N_k is. On replacing k by $\phi(k)$, this gives the theorem for 'N' (cf. [16, p. 344]). For 'L', using the actual definition of $T_1^A(z, a, x)$ [16, pp. 290, 291], $L_0(0)$ is true, but $L_k(a)$ for $k>0$ is only true when a is the Gödel number of a system of equations and hence is $\neq 0$. Also, using VII*, a predicate $A(a)$ $(\equiv (Ex)A(a))$ is recursive uniformly in $(Ex)T_1^A(a, a, x)$. So from $L_k(a)$ we can find k, by asking whether $L_k(0)$ is true, if not whether $L_{k-1}(0)$ is true, etc., and counting the questions asked until the answer is affirmative. Thus (using III* for $n=0$) k is partial recursive uniformly in L_k. A XIII* is obtained by reading "N^Q", "L^Q" in place of "N", "L", for the case of any $Q(a)$ such that $Q(0)$ is true.

What we do is to handle the ordinals on the basis of recursiveness. Such a theory of constructive ordinals was set up in a series of papers by Church and the author [5; 4; 11; 13; 18].

In the version we shall use (the system S_3 of [11; 13; 18]), ordinals are represented by natural numbers. Let O be the class of the natural numbers used, to each $y \in O$ let $|y|$ be the ordinal represented, and let $u <_O y$ be the partial ordering relation which holds when u enters into the generation of y as a member of O by the following principles. $1 \in O$ and $|1|=0$. If $y \in O$, then $2^y \in O$, $|2^y| = |y|+1$ and $y <_O 2^y$. (Thus in particular the finite ordinals $0, 1, 2, \cdots$ are represented by the numbers $1, 2^1, 2^{2^1}, \cdots$, which we also write $0_O, 1_O, 2_O, \cdots$. When $y \in O$, we often write 2^y as y^*.) If $y_0, y_1, y_2, \cdots \in O$, and $y_0 <_O y_1 <_O y_2 <_O \cdots$, and for each n, y_n is the value for n_O of the partial recursive function with Gödel number y (so $y_n = \Phi_1(y, n_O)$), then $3\cdot 5^y \in O$, $|3\cdot 5^y| = \lim_n |y_n|$, and for each n, $y_n <_O 3\cdot 5^y$. These principles, together with the transitivity of $<_O$, constitute a simultaneous inductive definition of O and $<_O$ and a definition of $|y|$ by induction over O.

If $u <_O y$, then $|u| < |y|$. (If $u <_O y$, then $u, y \in O$ [13, §12; 18, §20].) The ordinals $|y|$ for $y \in O$ constitute a proper segment of Cantor's first and second number classes determined by "the least nonconstructive ordinal" ω_1. The predicate $<_O$ only partially orders O, since for each limit ordinal $\alpha < \omega_1$, different ascending sequences with $\lim_n |y_n| = \alpha$, different partial recursive functions ϕ with $\phi(n_O) = y_n$, and different Gödel numbers y of ϕ, can be chosen. However the natural numbers $u <_O$ any fixed natural number y are linearly (and well-) ordered by $<_O$ (e.g. [18, §20]).

Now we define a predicate $H_y(a)$ for each $y \in O$, so that H_{0_O}, H_{1_O}, H_{2_O}, $\cdots$ will be $L_0, L_1, L_2, \cdots$, and H_w will be L when w is a suitable notation for ω (namely, when $w = 3\cdot 5^y$ and $\Phi_1(y, n_O) = n_O$).

We simply put $H_1(a) \equiv a = a$; $H_{2^y}(a) \equiv (Ex)T_1^{H_y}(a, a, x)$ if $y \in O$; and $H_{3\cdot 5^y}(a) \equiv H_{y_{(a)_1}}((a)_0)$ if $3\ 5^y \in O$ and $y_n = \Phi_1(y, n_O)$.

XIV. *To each $y \in O$, the predicate H_y is defined; and for $u <_O y$, H_u is of lower degree than H_y. Moreover, there is a primitive recursive function $\rho(u, y)$ such that, if $y \in O$ and $u \leqq_O y$, then $\rho(u, y)$ is a Gödel number of H_u from H_y.*

The first sentence is clear by the reasoning applied above to $L_0, L_1, L_2, \cdots$, and L. So in particular, for $u \leqq_O y$, H_u is recursive in H_y; examining this recursiveness in detail, we are able using III to obtain a Gödel number for it by a primitive recursive function $\rho(u, y)$ [17, 6.3].

Sums $u +_O y$ of ordinal notations $u, y \in O$ can be defined by induction so that $+_O$ is primitive recursive and $|u +_O y| = |u| + |y|$ [18, §22]. In fact $u \leqq_O u +_O y$ but not always $y \leqq_O u +_O y$ [18, §22 and Footnote 30]. Finite sums

$$\sum_{i<n}{}_O\, \psi(i)$$

where $(i)[\psi(i) \in O]$ can then be defined primitive recursively in ψ [18, §23].

XV. *There is a primitive recursive function $\sigma(u, y)$ such that, if $u, y \in O$, then H_y is general recursive in $H_{u +_O y}$ with Gödel number $\sigma(u, y)$.*

This can be proved similarly to XIV, using X [17, 6.3].

Extensions of the arithmetical hierarchy have been discussed by Davis [6; 7] and Mostowski [23], and were known independently to Post and to the author. Davis' extension is as here based on

$L_0, L_1, L_2, \cdots$; Mostowski's on $N_0, N_1, N_2, \cdots$, besides which he proceeds differently at limit ordinals. After adjusting for this difference, the degrees of the predicates correlated to a given $y \in O$ are the same [17, §9]. Davis [6] proves that the predicates correlated to different notations for the same ordinal $|y|$ are of the same degree, for $|y| < \omega^2$. In [28] Spector will extend this to all $|y| < \omega_1$.

5. **The analytic hierarchy.** When a function ϕ or predicate P is (general, partial, primitive) recursive in ψ and the relationship is uniform (i.e. the method of computing ϕ, or deciding P, from ψ is independent of what number-theoretic function ψ is), we can regard ϕ as a recursive function, and P as a recursive predicate, of ψ as an independent function variable besides of the number variables [15; 17]. More generally there may be several function variables; we shall usually write them $\alpha, \beta, \gamma, \alpha_1, \cdots, \alpha_n$, etc. We take them to be variables for 1-place functions, which suffices for present purposes.

If we apply to general recursive predicates of number and function variables the operations of the predicate calculus with quantification only of number variables, we get as before *arithmetical* predicates except that now they may have function variables. If we apply the operations of the predicate calculus with quantification of function variables also allowed, we get *analytic* predicates. If their only (free) variables are number variables, they come under the subject of this paper.

The analytic predicates were studied in [17], where we began by seeing to what extent the theory could be developed analogously to [12].

First, by exploring the possible permutations and contractions of number and function quantifiers together, we get an analog of IV.

XVI. *Every analytic predicate $P(a)$ is expressible in one of the forms*

$$(\mathrm{b}_1)\quad A(a)\quad \begin{array}{lll} (\alpha)(Ex)R(a, \alpha, x) & (E\alpha)(\beta)(Ex)R(a, \alpha, \beta, x) & \cdots \\ (E\alpha)(x)R(a, \alpha, x) & (\alpha)(E\beta)(x)R(a, \alpha, \beta, x) & \cdots \end{array}$$

where A is arithmetical and each R is general recursive. Equivalently these forms can be written respectively

$$(\mathrm{b}_2)\quad A(a)\quad \begin{array}{lll} (\alpha)A(a, \alpha) & (E\alpha)(\beta)A(a, \alpha, \beta) & \cdots \\ (E\alpha)A(a, \alpha) & (\alpha)(E\beta)A(a, \alpha, \beta) & \cdots \end{array}$$

where each A is arithmetical.

The first and third of the next theorems follow from V^* and VII^* by quantification of function variables; the second from the first as VI from V.

XVII. *Given any general recursive predicate* $R(a, \alpha, x)$, *there is a number* f *such that* $(\alpha)(Ex)R(a, \alpha, x) \equiv (\alpha)(Ex)T_1^{\alpha}(f, a, x) \equiv (\alpha)(Ex)T_1^{1}(\bar{\alpha}(x), f, a)$. *Similarly for the other quantified forms of* (b).

XVIII. *To each of the quantified forms of* (b) *there is a respective predicate*

$$(\alpha)(Ex)T_1^{\alpha}(a, a, x) \qquad (E\alpha)(\beta)(Ex)T_1^{\alpha,\beta}(a, a, x) \qquad \cdots$$

$$(E\alpha)(x)\overline{T}_1^{\alpha}(a, a, x) \qquad (\alpha)(E\beta)(x)\overline{T}_1^{\alpha,\beta}(a, a, x) \qquad \cdots$$

of that form which is not expressible in the dual form, a fortiori not in any of the forms with fewer quantifiers.

XIX. *The predicate* $(\alpha)(Ex)T_1^{\alpha}(a, a, x)$ *of* XVIII *is a complete predicate of the respective form* $(\alpha)(Ex)R(a, \alpha, x)$ *of* (b). *Similarly for the other quantified forms of* (b).

The hierarchy (b_1) is formally similar to (a). However consider the analog of VIII under this comparison. Half of it holds (by XX for $k=0$), but not the converse part (by XXI, since H_y for $|y| \geqq \omega$ is nonarithmetical).

XX. *For each* $k \geqq 0$: *Each predicate arithmetical in predicates expressible in both the* $k+1$-*function-quantifier forms of* (b) (*a fortiori, each arithmetical predicate*) *is expressible in both the* $k+1$-*function-quantifier forms.*

This is proved similarly to XVI (details in [17, 5.2]).

XXI. *Each predicate* H_y *for* $y \in O$ *is expressible in both the* 1-*function quantifier forms of* (b). *In fact, there is a primitive recursive function* $\tau(y)$ *such that, for each* $y \in O$,

$$H_y(a) \equiv (\alpha)(Ex)T_1^{\alpha}((\tau(y))_0, a, x) \equiv (E\alpha)(x)\overline{T}_1^{\alpha}((\tau(y))_1, a, x).$$

The second sentence of the theorem gives the idea of the proof (details in [17, 8.3]). Predicates definable from recursive predicates by quantification of function variables can also be so defined by quantification of set variables and vice versa [17, 3.8]. That every predicate H_y is definable by such quantification was shown by Davis in [6], for $|y| < \omega^2$; and was stated by Mostowski in [23], for $|y| < \omega_1$. That one such quantifier suffices is new with this theorem.

6. The predicates expressible in both 1-function-quantifier forms. Since the analogy of (b) to (a) breaks down on VIII, the problem arises of characterizing, independently of the hierarchy (b), the class of the predicates expressible in both the 1-function-quantifier forms of that hierarchy. By XXI this class contains H_y for every $y \in O$, and then by XX every predicate recursive (and hence arithmetical) in some H_y is in the class. So in [17] we conjectured that the predicates each recursive in H_y for some $y \in O$ are the ones expressible in both 1-function-quantifier forms. The principal contribution of the present paper is a proof of this conjecture (cf. XXIV).

By XVII, $(\alpha)(Ex)R(a, \alpha, x)$ with R general recursive can be rewritten in the form $(\alpha)(Ex)R(a, \bar{\alpha}(x))$, where $R(a, w)$ is primitive recursive, and where in fact (by the definition of $T_1^1(w, z, a)$ [16, p. 291] with [17, Footnote 2]), for each a and α, $R(a, \bar{\alpha}(x))$ is true for at most one x.

The number $\bar{\alpha}(x)$ $(= \prod_{i<x} p_i^{\alpha(i)+1})$ represents the sequence $\alpha(0), \cdots, \alpha(x-1)$ of x numbers. Such a number $\bar{\alpha}(x)$ (for any α, and any $x \geqq 0$) we call a ***sequence number***. The property Seq(w) of being a sequence number is primitive recursive; in fact, $\text{Seq}(w) \equiv w \neq 0$ & $(i)_{i<\text{lh}(w)}[(w)_i \neq 0]$. The sequence numbers include 1 which represents the empty sequence, and for each i the number 2^{i+1} which represents the unit sequence i. The sequence $\alpha(0), \cdots, \alpha(x-1), \beta(0), \cdots, \beta(y-1)$ is represented by $\bar{\alpha}(x) * \bar{\beta}(y)$ where $*$ is primitive recursive [16, p. 230].

We use $\succ$ to denote the linear ordering of the sequence numbers which is established by ordering the finite sequences represented by them lexicographically, with shorter sequences coming above longer ones, and using the infinite descending alphabet $\cdots, 2, 1, 0$. In this ordering, 1 which represents the empty sequence is the highest element. (If $\overline{\text{Seq}}(u)$ or $\overline{\text{Seq}}(v)$, $u \succ v$ shall be false. The predicate $u \succ v$ is primitive recursive.)

Now take any predicate $R(a, w)$ and number a. We shall say that (with respect to R and a) a sequence number $w = \bar{\alpha}(x)$ is ***secured*** if $(Et)_{t \leq x} R(a, \bar{\alpha}(t))$, ***past secured*** if $(Et)_{t<x} R(a, \bar{\alpha}(t))$, ***immediately secured*** if w is secured but not past secured, ***securable*** if w is secured or $(\beta)(Et)R(a, \bar{\alpha}(x) * \bar{\beta}(t))$ (i.e. if $(Et)R(a, \bar{\alpha}(t))$ no matter how the values of $\alpha(t)$ for $t \geqq x$ are chosen). ([18, §24]. If $\overline{\text{Seq}}(w)$, w shall be unsecured and unsecurable. The terms "secured" and "securable" are adapted from Brouwer [1; 2].)

Now we define a subset $S_w^{R,a}$ of the sequence numbers, for a given sequence number w. If w is not past secured, $S_w^{R,a}$ shall be the set of all the numbers $w * \bar{\beta}(t)$ which are not past secured. If w is past secured, $S_w^{R,a}$ shall be the unit set $\{w\}$. (If $\overline{\text{Seq}}(w)$, $S_w^{R,a}$ shall be the empty set. The predicate $u \in S_w^{R,a}$ is primitive recursive uniformly in R.)

XXII. *Let* Seq(w). *The set* $S_w^{R,a}$ *is well-ordered by* $\succ$, *if and only if* w *is securable with respect to* R *and* a. *So in particular*, $S_1^{R,a}$ *is well-ordered by* $\succ$, *if and only if* $(\alpha)(Ex)R(a, \bar{\alpha}(x))$.

If w is unsecurable, then by definition w is unsecured and $\overline{(\beta)}(Et)R(a, w * \bar{\beta}(t))$, whence $(E\beta)(t)\overline{R}(a, w * \bar{\beta}(t))$. For this β, $w * \bar{\beta}(t)$ $(t=0, 1, 2, \cdots)$ is an infinite descending sequence within $S_w^{R,a}$. Conversely, given any infinite descending sequence within $S_w^{R,a}$, w is unsecured, and by considering that the alphabet $\cdots, 2, 1, 0$ is well-ordered, we are led to a β such that $(t)\overline{R}(a, w * \bar{\beta}(t))$. [18, §26, especially (H).]

XXIII. *Let* $R(a, w)$ *be primitive recursive, and let* Seq(w). *There is a primitive recursive function* $\xi(a, w)$ *such that, if* $S_w^{R,a}$ *is well-ordered, then* $\xi(a, w) \in O$ and $|\xi(a, w)|+1$ *is the ordinal number of* $S_w^{R,a}$.

If $S_w^{R,a}$ is not $\{w\}$, then

$$S_w^{R,a} = \Bigl(\sum_{t=0,1,2,\cdots} S_{w*2^{t+1}}^{R,a}\Bigr) + \{w\}.$$

The sum operation here on the sets is paralleled by a summing of ordinal notations in the construction of ξ. [18, §25, §26 (F).]

XXIV. *If* $P(a) \equiv (\alpha)(Ex)R(a, \alpha, x) \equiv (E\alpha)(x)S(a, \alpha, x)$ *with* R *and* S *recursive, then for some* $y \in O$, $P(a)$ *is general recursive in* H_y.

PROOF. By XVII as remarked above, and since $S \equiv \overline{\overline{S}}$ where $\overline{S}$ is also recursive, we can write

$$(1) \qquad P(a) \equiv (\alpha)(Ex)R_0(a, \bar{\alpha}(x)) \equiv (E\alpha)(x)\overline{R}_1(a, \bar{\alpha}(x))$$

where R_j $(j=0, 1)$ is primitive recursive, and for a given a and α, $R_j(a, \bar{\alpha}(x))$ is true for at most one x.

If α ranges over all 1-place number-theoretic functions, and we put $\alpha^j(x) = (\alpha(x))_j$, then (α^0, α^1) will range over all pairs (α_0, α_1) of 1-place number-theoretic functions. Let $x \dot{-} y = x - y$ if $x \geqq y$, $=0$ if $x < y$. For the present proof, let

$$w_j = \prod_{i<\mathrm{lh}(w)} p_i^{((w)_i \dot{-} 1)_j + 1}.$$

Then w_j is a primitive recursive function of w, and $\bar{\alpha}(x)_j = \overline{\alpha^j}(x)$. Let $R(a, w) \equiv R_0(a, w_0) \vee R_1(a, w_1)$. Let $w_{s,t} = w * 2^{2^s \cdot 3^t + 1}$.

By the law of the excluded middle, $(a)[P(a) \vee \overline{P}(a)]$. But

$$\begin{aligned}
&(a)[P(a) \vee \overline{P}(a)] \\
&\quad \equiv (a)[(\alpha)(Ex)R_0(a, \bar{\alpha}(x)) \vee \overline{(E\alpha)}(x)\overline{R}_1(a, \bar{\alpha}(x))] \\
&\quad \equiv (a)[(\alpha)(Ex)R_0(a, \bar{\alpha}(x)) \vee (\alpha)(Ex)R_1(a, \bar{\alpha}(x))] \\
&\quad \equiv (a)(\alpha_0)(\alpha_1)(Ex)[R_0(a, \overline{\alpha_0}(x)) \vee R_1(a, \overline{\alpha_1}(x))] \\
&\quad \equiv (a)(\alpha)(Ex)[R_0(a, \bar{\alpha}(x)_0) \vee R_1(a, \bar{\alpha}(x)_1)] \equiv (a)(\alpha)(Ex)R(a, \bar{\alpha}(x)).
\end{aligned}$$

Thus

$$(2) \qquad (a)(\alpha)(Ex)R(a, \bar{\alpha}(x)).$$

So by XXII, for every a, the set $S_1^{R,a}$ is well-ordered by $\succ$. (The y for the theorem will be a notation for the ordinal number of the sum of the sets $S_1^{R,a}$ for $a=0, 1, 2, \cdots$.)

If $w \in S_1^{R,a}$, then either (Case 1) w is immediately secured (i.e. $R(a, w)$), or else (Case 2) w is unsecured and the numbers $w * 2^{i+1}$ ($i=0, 1, 2, \cdots$) and hence in particular the numbers w_{st} ($s, t = 0, 1, 2, \cdots$) belong to $S_1^{R,a}$ and come below w in the ordering $\succ$.

Hence we can define predicates $P_j(a, w)$ ($j=0, 1$) for all a and for $w \in S_1^{R,a}$ by transfinite induction over $S_1^{R,a}$ as follows.

$$P_0(a, w) \equiv \begin{cases} R_0(a, w_0) & \text{if } R(a, w), \\ (s)(Et)P_0(a, w_{st}) & \text{if } \overline{R}(a, w), \end{cases}$$

$$P_1(a, w) \equiv \begin{cases} R_1(a, w_1) & \text{if } R(a, w), \\ (t)(Es)P_1(a, w_{st}) & \text{if } \overline{R}(a, w). \end{cases}$$

Now we prove that, for each a,

$$(3) \qquad P_0(a, 1) \vee P_1(a, 1).$$

This will follow by setting $w=1$, if we can prove by induction over $S_1^{R,a}$ that, for $w \in S_1^{R,a}$, $P_0(a, w) \vee P_1(a, w)$. CASE 1: $R(a, w)$. By definition, then either $R_0(a, w_0)$, in which case $P_0(a, w)$, or $R_1(a, w_1)$, in which case $P_1(a, w)$. CASE 2: $\overline{R}(a, w)$. Then for each s and t, $w_{st} \in S_1^{R,a}$; and by the hypothesis of the induction, $(s)(t)[P_0(a, w_{st}) \vee P_1(a, w_{st})]$. If $(Es)(t)\overline{P}_0(a, w_{st}) \& (Et)(s)\overline{P}_1(a, w_{st})$, then for the s given by the first and the t by the second we would have $\overline{P}_0(a, w_{st}) \& \overline{P}_1(a, w_{st})$, contradicting the hypothesis of the induction. So $(\overline{Es})(t)\overline{P}_0(a, w_{st}) \vee (\overline{Et})(s)\overline{P}_1(a, w_{st})$, i.e. $(s)(Et)P_0(a, w_{st}) \vee (t)(Es) P_1(a, w_{st})$, whence by the definitions of P_0 and P_1, $P_0(a, w) \vee P_1(a, w)$.

Next we prove that, for each a,

$$(4) \qquad (E\alpha)(x)\overline{R}_j(a, \bar{\alpha}(x)) \rightarrow \overline{P}_j(a, 1) \qquad (j = 0, 1).$$

Take e.g. $j=0$. Suppose given a and α such that $(x)\overline{R}_0(a, \bar{\alpha}(x))$. For this α, a sequence number w with $w_0 = \bar{\alpha}(\mathrm{lh}(w))$ we call an *α-number*. Now by (1), $(\alpha)(Ex)R_1(a, \bar{\alpha}(x))$. So by XXII, $S_1^{R_1,a}$ is well-ordered by $\succ$. We prove by transfinite induction over $S_1^{R_1,a}$ that for $v \in S_1^{R_1,a}$, if w is an α-number such that $w_1 = v$, then $\overline{P}_0(a, w)$. Since w is an α-number, $\overline{R}_0(a, w_0)$. CASE 1: $R_1(a, v)$. Then by $w_1 = v$ and the definition of R, $R(a, w)$. So by definition, $P_0(a, w) \equiv R_0(a, w_0)$. So by $\overline{R}_0(a, w_0)$, $\overline{P}_0(a, w)$. CASE 2: $\overline{R}_1(a, v)$. Then for every t, $v * 2^{t+1} \in S_1^{R_1,a}$. Let $s = \alpha(\mathrm{lh}(w))$. Then for every t, w_{st} is an α-number with $[w_{st}]_1 = v * 2^{t+1}$. So by the hypothesis of the induction, $(t)\overline{P}_0(a, w_{st})$. Thus $(Es)(t)$

$\overline{P}_0(a, w_{st})$, i.e. $\overline{(s)}(Et)P_0(a, w_{st})$. By $\overline{R}_1(a, v)$ with $w_1 = v$ and $\overline{R}_0(a, w_0)$, $\overline{R}(a, w)$. So by the definition of P_0, $\overline{P}_0(a, w)$. This completes the induction. But 1 is an α-number w with $w_1 = 1$, and $1 \in S_1^{R,a}$. So $\overline{P}_0(a, 1)$.

By (4) contraposed and (1),

$$(5) \qquad P_0(a, 1) \rightarrow P(a), \qquad P_1(a, 1) \rightarrow \overline{P}(a),$$

whence by (3),

$$(6) \qquad P(a) \equiv P_0(a, 1) \equiv \overline{P}_1(a, 1).$$

Now we shall obtain a primitive recursive function $v(a, w)$ such that, for each a and each $w \in S_1^{R,a}$, $P_0(a, w)$ as a function of 0 variables (i.e. $\lambda P_0(a, w)$) is recursive in $H_{\xi(a,w)^{**}}$ (cf. XXIII) with Gödel number $v(a, w)$. The property of $v(a, w)$ is to be established by transfinite induction over $S_1^{R,a}$. However, instead of defining v first, we shall begin by considering the two cases of the induction, in each case deriving a property of v sufficient to treat the case. Then in conclusion we shall verify that v can be defined so as to possess these properties. Case 1: $R(a, w)$. Then $P_0(a, w) \equiv R_0(a, w_0)$. So $\lambda awP_0(a, w)$ is recursive, a fortiori recursive in $H_{\xi(a,w)^{**}}$, say with Gödel number h_1. So by II* it will suffice to take $v(a, w) = S_0^{2,1}(h_1, a, w)$. Case 2: $\overline{R}(a, w)$. Then by the hypothesis of the induction, $\lambda P_0(a, w_{st})$ is recursive in $H_{\xi(a,w_{st})^{**}}$ with Gödel number $v(a, w_{st})$, call this number A. By the definition of ξ in [18, §25], $\xi(a, w) = 3 \cdot 5^v$ where (writing $v_n = \Phi_1(v, n_O)$)

$$v_n = \sum_{t<n} {}_O\, \xi(a, w * 2^{t+1})^*.$$

To simplify notation, we write st for $2^s \cdot 3^t$ in subscripts of v. By the definition of $\sum_O$ in [18, §23], $v_{st+1} = v_{st} +_O \xi(a, w_{st})^*$; so by XV, $H_{\xi(a,w_{st})^*}$ is recursive in $H_{v_{st+1}}$ with Gödel number $\sigma(v_{st}, \xi(a, w_{st})^*)$, call it B$_1$. Thence by X, $H_{\xi(a,w_{st})^{**}}$ is recursive in $H_{v_{st+1}^*}$ with Gödel number $S_1^{1,1}(g, \mathrm{B}_1)$, call it B. But $v_{st+1} <_O v_{st+2}$, so $v_{st+1}^* \leqq_O v_{st+2} <_O 3 \cdot 5^v = \xi(a, w)$; so by XIV, $H_{v_{st+1}^*}$ is recursive in $H_{\xi(a,w)}$ with Gödel number $\rho(v_{st+1}^*, \xi(a, w))$, call it C. Combining the recursivities with respective Gödel numbers A, B and C, we can write $P_0(a, w_{st}) \equiv G_2^{H_{\xi(a,w)}}(u, a, w, s, t)$, where u is a Gödel number of v, and $G_2^A(u, a, w, s, t)$ is partial recursive uniformly in A. Thence by two successive applications of the method of proof of VII* (noting that $\lambda st G_2^A(u, a, w, s, t)$ is completely defined for the A, u, a, w considered), $(s)(Et)G_2^{H_{\xi(a,w)}}(u, a, w, s, t) \equiv \overline{H}_{\xi(a,w)^{**}}(\theta(u, a, w))$ with a primitive recursive θ. So if h_2 is a uniform Gödel number from A of $\lambda uaw \overline{A}(\theta(u, a, w))$, then $S_0^{3,1}(h_2, u, a, w)$ is one from $H_{\xi(a,w)^{**}}$ of $\lambda(s)(Et)P_0(a, w_{st})$, i.e. of $\lambda P_0(a, w)$. So it suffices to take $v(a, w) = S_0^{3,1}(h_2, u, a, w)$ for u a Gödel number of v. Conclusion. Let

$$\chi(z, a, w) \simeq \begin{cases} S_0^{2,1}(k_1, a, w) & \text{if } R(a, w), \\ S_0^{3,1}(k_2, z, a, w) & \text{if } \overline{R}(a, w). \end{cases}$$

By the cases it suffices to take $v(a, w) = \chi(u, a, w)$ for u a Gödel number of $\chi(u, a, w)$. By III, such a u can be found.

Thence taking $w = 1$ and noting (6), $\lambda P(a)$ is recursive in $H_{\xi(a,1)^{**}}$ with Gödel number $v(a, 1)$, call it D.

We can choose z so that (writing $z_n = \Phi_1(z, n_O)$)

$$z_n = \sum_{i<n}{}_O\, \xi(i, 1)^*;$$

z is simply a Gödel number of

$$\sum_{i<\mathrm{nat}(t)}{}_O\ \xi(i, 1)^*$$

where $\mathrm{nat}(t) = \mu n_{n<t}\, t = n_O$. Let $y = 3 \cdot 5^z$. Then $y \in O$. Also $z_{a+1} = z_a +_O \xi(a, 1)^*$. So by XV, $H_{\xi(a,1)^*}$ is recursive in $H_{z_{a+1}}$ with Gödel number $\sigma(z_a, \xi(a, 1)^*)$, call it E_1. Thence by X, $H_{\xi(a,1)^{**}}$ is recursive in $H_{z_{a+1}^*}$ with Gödel number $S_1^{1,1}(g, \mathrm{E}_1)$, call it E. But $z_{a+1} <_O z_{a+2}$, so $z_{a+1}^* \leq_O z_{a+2} <_O y$; so by XIV, $H_{z_{a+1}^*}$ is recursive in H_y with Gödel number $\rho(z_{a+1}^*, y)$, call it F.

Combining the recursivities with Gödel numbers D, E and F, we can write $P(a) \simeq G^{H_y}(a)$ where G^A is partial recursive uniformly in A. Thus $P(a)$ is general recursive in H_y.

XXV. *The predicate $a \in O$ is expressible in the form $(\alpha)(Ex)R(a, \alpha, x)$ with R recursive, and to each recursive predicate $R(a, \alpha, x)$ there is a primitive recursive function $\xi(a)$ such that $(\alpha)(Ex)R(a, \alpha, x) \equiv \xi(a) \in O$.*

In brief, $a \in O$ is a complete predicate of the form. This is the main result of [18]. To summarize the proof, $a \in O$ can be expressed arithmetically, except for the stipulation that a certain recursive linear ordering is a well-ordering, which can be accomplished by using a universal function quantifier (α). For the second part, we put $\xi(a) = \xi(a, 1)$ after strengthening XXIII to assert also that $\xi(a, w) \in O$ only if $S_w^{R_a}$ is well-ordered (by managing $+_O$ so that $u +_O y \in O \rightarrow u, y \in O$).

In attempting to make the foundations of analysis more constructive, one may attempt in various situations to replace the uncountable infinity of the number-theoretic functions by a countable class of such functions. It is thus of interest to inquire how large a class may be necessary in a given situation. Questions of this kind have been considered by Kreisel [20; 21] and the author [15; 17]. Using terminology suggested by Kreisel, a class C of functions is a *basis* for a class D of predicates of a function variable, if for every predicate

$B(\alpha)$ of D, $(E\alpha)B(\alpha) \equiv (E\alpha)[\alpha \in C \& B(\alpha)]$.

Already in the case D consists of the predicates $(x)R(\alpha, x)$ with R recursive, a surprisingly large class C of α's is necessary. In [17, 5.5] we observed that the functions general recursive in 1-function-quantifier predicates are a basis, while for any fixed $y \in O$ those general recursive in H_y are not, but left open the possibility that the functions general recursive in predicates expressible in both 1-function-quantifier forms might be. Now we shall show, as an application of XXIV with XXV, that they are not.

XXVI. *There is a recursive predicate* $R(\alpha, x)$ *such that* $(E\alpha)(x)R(\alpha, x)$ *is true, but* $(x)R(\alpha, x)$ *is false for every function* α *general recursive in predicates* $P_1, \cdots, P_l$ *expressible in both the* 1-*function-quantifier forms of* (b).

PROOF. We can take the predicates $P_1, \cdots, P_l$ expressible in both 1-function-quantifier forms to be one 1-place predicate P; in fact, after contracting each P_i if necessary to a 1-place predicate P_i, by XX we can use $P(a) \equiv P_{\mathrm{rm}(a,l)+1}([a/l])$ where $a = [a/l]l + \mathrm{rm}(a, l)$ [16, p. 223].

The $R(\alpha, x)$ will be $\bar{T}_1^1(\bar{\alpha}(x), f, f)$ for a number f to be selected. Consider the predicate

$$(EP)(E\alpha)[\{P \text{ is expressible in both 1-function-quantifier forms}\}$$

$$\& \{\alpha \text{ is recursive in } P\} \& (x)\bar{T}_1^1(\bar{\alpha}(x), a, a)];$$

let it be abbreviated $(EP)(E\alpha)\mathsf{A}(P, \alpha, a)$. Obviously

$$(EP)(E\alpha)\mathsf{A}(P, \alpha, a) \rightarrow (E\alpha)(x)\bar{T}_1^1(\bar{\alpha}(x), a, a). \tag{1}$$

Assume $\mathsf{A}(P, \alpha, a)$. Then by XXIV, there is a y such that $y \in O$ and P is recursive in H_y. Then α being recursive in P is also recursive in H_y, say with Gödel number e. Then by I* (since α is completely defined, as is to be understood), $(i)(Et)T_1^{H_y}(e, i, t)$. Also $\bar{\alpha}(x) = \prod_{i<x} p_i \exp \Phi_1^{H_y}(e, i) + 1$, and hence $\bar{T}_1^1(\bar{\alpha}(x), a, a)$ can be rewritten $(Ew)[w = \prod_{i<x} p_i \exp \Phi_1^{H_y}(e, i) + 1 \& \bar{T}_1^1(w, a, a)]$, which is of the form $(Ew)R^{H_y}(x, e, a, w)$ where R^A is partial recursive uniformly in A. By the method of proof of VII*, $(Ew)R^{H_y}(x, e, a, w) \equiv H_{y'}(\theta(x, e, a))$, with θ primitive recursive, for values of y, x, e, a which make $\lambda w R^{H_y}(x, e, a, w)$ completely defined; but $y \in O$ and $(i)(Et)T_1^{H_y}(e, i, t)$ guarantee this completeness. Similarly $(Et)T_1^{H_y}(e, i, t) \equiv H_{y'}(\theta(e, i))$. Altogether, from $\mathsf{A}(P, \alpha, a)$, and hence from $(EP)(E\alpha)\mathsf{A}(P, \alpha, a)$, we thus infer

$$(Ey)(Ee)[y \in O \& (i)H_{y'}(\theta(e, i)) \& (x)H_{y'}(\theta(x, e, a))];$$

write this $(Ey)(Ee)B(y, e, a)$. Conversely, from $(Ey)(Ee)B(y, e, a)$ we infer $(EP)(E\alpha)\mathrm{A}(P, \alpha, a)$, by taking for P the predicate H_y (cf. XXI) and for α the function recursive in H_y with Gödel number e.

In $(Ey)(Ee)B(y, e, a)$, by XXV we can replace $y \in O$ by $(\alpha)(Ex) R_1(y, \alpha, x)$ with recursive R_1, and by XXI we can replace $H_{y^*}(c)$ by $(\alpha)(Et)T_1^{\alpha}((\tau(y^*))_0, c, t)$, after which replacements the resulting expression will reduce by the method of proof of XVI to the form $(\alpha)(Ex)R(a, \alpha, x)$ with recursive R.

So we have

$$(2) \qquad (EP)(E\alpha)\mathrm{A}(P, \alpha, a) \equiv (\alpha)(Ex)R(a, \alpha, x).$$

But by XVII, for some number f,

$$(3) \qquad (\alpha)(Ex)R(a, \alpha, x) \equiv (\alpha)(Ex)\overset{1}{T_1}(\bar{\alpha}(x), f, a).$$

Substituting f for a in (1), and in (2) with (3),

$$(4) \qquad (EP)(E\alpha)\mathrm{A}(P, \alpha, f) \rightarrow (E\alpha)(x)\overline{\overset{1}{T_1}}(\bar{\alpha}(x), f, f),$$

$$(5) \qquad (\alpha)(Ex)\overset{1}{T_1}(\bar{\alpha}(x), f, f) \equiv (EP)(E\alpha)\mathrm{A}(P, \alpha, f).$$

Assuming $(\alpha)(Ex)T_1^1(\bar{\alpha}(x), f, f)$, by (5) and (4) we could infer $(E\alpha)(x)\overline{T_1^1}(\bar{\alpha}(x), f, f)$, i.e. $\overline{(\alpha)}(Ex)T_1^1(\bar{\alpha}(x), f, f)$. So by reductio ad absurdum, $\overline{(\alpha)}(Ex)T_1^1(\bar{\alpha}(x), f, f)$, i.e. $(E\alpha)(x)\overline{T_1^1}(\bar{\alpha}(x), f, f)$.

Finally, from $\overline{(\alpha)}(Ex)T_1^1(\bar{\alpha}(x), f, f)$ by (5), we have $\overline{(EP)}(E\alpha)\mathrm{A}(P, \alpha, f)$, which can be rewritten

$$(P)(\alpha)\Big[\big\{\{P \text{ is expressible in both 1-function-quantifier forms}\} \\ \& \{\alpha \text{ is recursive in } P\}\big\} \rightarrow \overline{(x)}\overset{1}{T_1}(\bar{\alpha}(x), f, f)\Big].$$

7. **Both 1-function-quantifier forms relative to Q.** The theorems have versions relativized say to a 1-place predicate Q. For ψ the representing function of Q, we write $T_1^{Q,\alpha}(z, a, x) \equiv T_1^{1,1}(\bar{\psi}(x), \bar{\alpha}(x), z, a) \equiv T_1^{Q,1}(\bar{\alpha}(x), z, a)$. Using functions partial recursive in Q instead of partial recursive absolutely, we obtain a system of notation for ordinals constructive in Q, with notions O^Q, $<_O^Q$, $|\ |^Q$, ω_1^Q, $+_O^1$, $\sum_O^1$ in place of O, $<_O$, $|\ |$, ω_1, $+_O$, $\sum_O$ [17, 6.4; 18, §30]; the superscript for $+_O^1$, $\sum_O^1$ is "1" rather than "Q" because only the number 1 of the arguments of Q matters. Now we define $H_1^Q(a) \equiv Q(a)$; $H_{y^*}^Q(a) \equiv (Ex)T_1^{Q,H_y^Q}(a, a, x)$ if $y \in O^Q$; and $H_{3\cdot 5^y}^Q(a) \equiv H^Q_{y^Q_{(a)_1}}((a)_0)$ if $3 \cdot 5^y \in O^Q$ and $y_n^Q \simeq \Phi_1^Q(y, n)$. In the case of a statement such as "A is recursive in B" which is already relative, the direct relativization to Q is "A is recursive in Q, B". However in most such cases to be considered, Q will be recursive in the B, so "Q," need not be inserted. We have inserted the superscript "Q," on the "T_1" in the definition of $H_{y^*}^Q$ be-

cause the proofs are easier using the direct relativization, although the degrees of the predicates H_y^Q would be the same without it [17, 6.8].

XXVII. *Theorems* XIV–XXVI *hold when in place of the notions 'recursive', 'T_1^α', 'T_1^1', '$T_1^{\alpha,\beta}$', 'arithmetical', 'analytic', 'k-function-quantifier forms', 'O', '$<_O$', '$| \, |$', '$+_O$', 'H_y', 'ρ', 'σ', 'τ', 'ξ' we use 'recursive in Q', '$T_1^{Q,\alpha}$', '$T_1^{Q,1}$', '$T_1^{Q,\alpha,\beta}$', 'arithmetical in Q', 'analytic in Q', 'k-function-quantifier forms relative to Q', 'O^Q', '$<_O^Q$', '$| \, |^Q$', '$+_O^1$', 'H_y^Q', 'ρ^1', 'σ^1', 'τ^1', 'ξ^Q', respectively, except that ρ^1, σ^1, τ^1 are primitive recursive simply, in* XIV *$\rho^1(u, y)$ is a Gödel number of H_u^Q from Q, H_y^Q (and similarly σ^1 in* XV*) but also there is a function $\rho^Q(u, y)$ partial recursive uniformly in Q which gives a Gödel number of H_u^Q from H_y^Q simply, and in* XXIV *we use recursiveness simply in H_y^Q* (call them then XIV*–XXVI*).

Proof. For XIV*, cf. [17, 6.6]. For XXIV*, we relativize the proof of XXIV directly, using ρ^1, σ^1 and a primitive recursive function $v^1(a, w)$ which gives a Gödel number of $\lambda P_0^Q(a, w)$ from Q, $H^Q_{\xi^Q(a,w)}$. Thus we establish that the $P(a)$ is recursive in Q, H_y^Q for some $y \in O^Q$; but by the version of XIV* with $\rho^Q(u, y)$, Q is recursive in H_y^Q (with Gödel number $\rho^Q(1, y)$).

Let us say a predicate P is *hyperarithmetical* (*hyperarithmetical in* Q), if P is recursive in H_y for some $y \in O$ (H_y^Q for some $y \in O^Q$), or equivalently by XX, XXI and XXIV (XX*, XXI*, and XXIV*) if P is expressible in both 1-function-quantifier forms (both 1-function-quantifier forms relative to Q). We choose the name because the predicates recursive in H_y for some $y \in O$ constitute the extended arithmetical hierarchy.

The relation 'P is hyperarithmetical in Q' is obviously reflexive; and it is transitive, by use of the case of XX* for $k=0$ and with a free function variable α [17, 5.2], followed by contraction of adjacent like function quantifiers $(\alpha)(\beta)$ or $(E\alpha)(E\beta)$. Hence 'P is hyperarithmetical in Q and vice versa' is reflexive, symmetric and transitive, and gives rise to a notion of ***hyperdegree*** analogous to the notion 'degree' for 'P is recursive in Q and vice versa'. The hyperdegree of P is *less than* that of Q, if P is hyperarithmetical in Q but not vice versa. Predicates of the same degree are a fortiori of the same hyperdegree. If P is of lower degree than Q, a fortiori it is of the same or lower hyperdegree. A predicate hyperarithmetical in a hyperarithmetrical predicate is hyperarithmetical (using XX with free α and contraction).

If A is hyperarithmetical in B, and R^A is recursive in A, then $(\alpha)(Ex)R^A(a, \alpha, x)$ is expressible in the form $(\alpha)(Ex)R^B(a, \alpha, x)$ with R^B recursive in B (using XX* with free α and contraction). Hence if further $\mathfrak{C}^B$ is a complete predicate of the form $(\alpha)(Ex)R^B(a, \alpha, x)$ (cf. VII, XIX*), then $(\alpha)(Ex)R^A(a, \alpha, x)$ is recursive in $\mathfrak{C}^B$. Applying this in two directions to predicates Q_1, Q_2 of the same hyperdegree, the degree of a complete predicate $\mathfrak{C}^Q$ of the form $(\alpha)(Ex)R^Q(a, \alpha, x)$ with R^Q recursive in Q (which is the maximal degree for predicates of the form) depends only on the hyperdegree of Q. The degree and hyperdegree of $\mathfrak{C}^Q$ are greater than those of Q (using XVIII*).

Using 'hyperarithmetical' instead of 'arithmetical', not only does the analog of VIII come to hold (cf. the remarks preceding XX), but also the following analog of half of XI holds.

XXVIII. *For each* $k \geq 0$: *The predicates hyperarithmetical in predicates expressible in the k-function-quantifier forms of* (b) *are expressible in both the* $k+1$*-function-quantifier forms.*

Using XX with free α and contraction. Similarly we have a XXVIII* for the forms relative to Q.

At the present writing we do not know whether the analog of the converse part of XI holds. Consequently we do not know whether the hierarchies analogous to the two of XII are equivalent with respect to degrees (after their first two members). Thus let $\mathfrak{N}_0(a) \equiv \mathfrak{L}_0(a) \equiv a = a$; let $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$ be the predicates of the upper row of XVIII; and let $\mathfrak{L}_{k+1}(a) \equiv (\alpha)(Ex)T_1^{\mathfrak{L}_k,\alpha}(a, a, x)$.

XXIX. *For each* $k \geq 0$: $\mathfrak{N}_{k+1}$ *is expressible in a* $k+1$*-function-quantifier form of* (b), *is of maximal degree for predicates so expressible, and is of higher degree and hyperdegree than* $\mathfrak{N}_k$. $\mathfrak{L}_{k+1}$ *is expressible in a* $k+1$*-function-quantifier form of* (b), *and also in the form* $(\alpha)(Ex)R^{\mathfrak{L}_k}(a, \alpha, x)$ *with* $R^{\mathfrak{L}_k}$ *recursive in* $\mathfrak{L}_k$, *is of maximal degree for predicates expressible in the latter form, and is of higher degree and hyperdegree than* $\mathfrak{L}_k$. (*So* $\mathfrak{L}_{k+1}$ *is of degree* $\leq$ *the degree of* $\mathfrak{N}_{k+1}$.)

An XXIX* refers similarly to XVIII* ($\mathfrak{N}_0^Q(a) \equiv \mathfrak{L}_0^Q(a) \equiv Q(a)$, etc.).

XXX. *There is a primitive recursive function* $\nu^1(a)$ *such that, for any* 1*-place predicate* Q, $Q(a) \equiv \nu^1(a) \in O^Q$.

We make $\nu^1(a)$ a notation for ω exactly when $Q(a)$ [17, 6.4].

By XXV, $a \in O$ is of maximal degree for predicates $(\alpha)(Ex)R(a, \alpha, x)$ with R recursive. By XXX, Q is recursive in $a \in O^Q$; so by XXV*, $a \in O^Q$ is of maximal degree for predicates $(\alpha)(Ex)R^Q(a, \alpha, x)$ with R^Q recursive in Q. The degree of $a \in O^Q$ depends only on (and the hyperdegree of H_y^Q for any $y \in O^Q$ is) the hyperdegree of Q; for Q hyperarithmetical, the degree of $a \in O^Q$ is that of $a \in O$ (and H_y^Q for $y \in O^Q$ is hyperarithmetical, i.e. of minimal hyperdegree). Now $a \in O$, $a \in O^O$ (where $O^O = O^{\lambda a\, a \in O}$), $a \in O^{O^O}$ (where $O^{O^O} = O^{\lambda a\, a \in O^O}$), $\cdots$ are of the same respective degrees as $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3, \cdots$.

Instead of successive systems of notations for ordinals of the first and second number classes O, O^O, O^{O^O}, $\cdots$, the first constructive and the others each constructive in the predicate of being a notation of the preceding system, we can introduce higher constructive number classes in the manner indicated in Church-Kleene [5]. The relations between these two ways of extending the representation of ordinals (cf. [17, end 6.4] and Spector [28, §8]) and the correlation of predicates to ordinals are being investigated.

Hierarchies can be set up similarly to the analytic hierarchy but using function variables of higher finite types (at least). For these we have established the analogs of a number of the theorems considered above for the arithmetical and analytic hierarchies. We plan to discuss these hierarchies in a paper to be entitled "Analytic predicates and function quantifiers of higher finite types." That new number-theoretic predicates are definable by use of successively higher finite types of variables was already known from Tarski [29].

Mostowski [22] compared the arithmetical hierarchy with the hierarchy of projective sets studied by Lusin and others. Under his comparison the analogy was imperfect (cf. our [14]). There are further possibilities for the utilization of analogies in this direction, which J. W. Addison, Jr. plans to discuss.

BIBLIOGRAPHY

1. L. E. J. Brouwer, *Beweis, dass jede volle Funktion gleichmässig stetig ist*, Neder. Akad. Wetensch. vol. 27 (1924) pp. 189–193.

2. ———, *Über Definitionsbereiche von Funktionen*, Math. Ann. vol. 97 (1927) pp. 60–75.

3. A. Church, *An unsolvable problem of elementary number theory*, Amer. J. Math. vol. 58 (1936) pp. 345–363.

4. ———, *The constructive second number class*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 224–232.

5. A. Church and S. C. Kleene, *Formal definitions in the theory of ordinal numbers*, Fund. Math. vol. 28 (1936) pp. 11–21.

6. M. Davis, *On the theory of recursive unsolvability*, Thesis, Princeton, 1950.

7. ———, *Relatively recursive functions and the extended Kleene hierarchy*, Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, Providence, American Mathematical Society, 1952, vol. 1, p. 723.

8. K. Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, Monatshefte für Mathematik und Physik vol. 38 (1931) pp. 173–198.

9. ———, *On undecidable propositions of formal mathematical systems*, Notes on lectures at the Institute for Advanced Study, mimeographed, Princeton, 1934, 30 pp.

10. S. C. Kleene, *General recursive functions of natural numbers*, Math. Ann. vol. 112 (1936) pp. 727–742.

11. ———, *On notation for ordinal numbers*, J. Symbolic Logic vol. 3 (1938) pp. 150–155.

12. ———, *Recursive predicates and quantifiers*, Trans. Amer. Math. Soc. vol. 53

(1943) pp. 41–73. Section 15 is affected by the error in [13].

13. ———, *On the forms of the predicates in the theory of constructive ordinals* Amer. J. Math. vol. 66 (1944) pp. 41–58. An error in this paper is pointed out in [15, p. 685; 16, p. 527], and a corrected treatment will appear in [18].

14. ———, *A symmetric form of Gödel's theorem*, Neder. Akad. Wetensch. vol. 53 (1950) pp. 800–802 (also Indagationes Mathematicae vol. 12 (1950) pp. 244–246).

15. ———, *Recursive functions and intuitionistic mathematics*, Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, Providence, American Mathematical Society, 1952, vol. 1, pp. 679–685.

16. ———, *Introduction to metamathematics*, New York, Van Nostrand, Amsterdam, North Holland, and Groningen, Noordhoff, 1952, 10+550 pp.

17. ———, *Arithmetical predicates and function quantifiers*, to appear in Trans. Amer. Math. Soc.

18. ———, *On the forms of the predicates in the theory of constructive ordinals (second paper)*, to appear in Amer. J. Math.

19. S. C. Kleene and E. L. Post, *The upper semi-lattice of degrees of recursive unsolvability*, Ann. of Math. (2) vol. 59 (1954) pp. 379–407.

20. G. Kreisel, *On the interpretation of non-finitist proofs*, J. Symbolic Logic vol. 16 (1951) pp. 241–267 and vol. 17 (1952) pp. 43–58.

21. ———, *A variant to Hilbert's theory of the foundations of arithmetic*, Brit. J. Philos. Sci. vol. 4 (1953) pp. 107–129.

22. A. Mostowski, *On definable sets of positive integers*, Fund. Math. vol. 34 (1946) pp. 81–112.

23. ———, *A classification of logical systems*, Studia Philosophica vol. 4 (1951) pp. 237–274.

24. E. L. Post, *Finite combinatory processes—formulation I*, J. Symbolic Logic vol. 1 (1936) pp. 103–105.

25. ———, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 284–316.

26. ———, *Degrees of recursive unsolvability*, Bull. Amer. Math. Soc. Abstract 54-7-269.

27. T. Skolem, *Über die Zurückführbarkeit einiger durch Rekursionen definierten Relationen auf "arithmetische,"* Acta Sci. Math. Szeged vol. 8 (1936–37) pp. 73–88.

28. C. Spector, *Recursive well-orderings*, to appear in J. Symbolic Logic.

29. A. Tarski, *Einige Betrachtungen über die Begriffe der ω-Widerspruchsfreiheit und der ω-Vollständigkeit*, Monatshefte für Mathematik und Physik vol. 40 (1933) pp. 97–112.

30. A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, Proc. London Math. Soc. (2) vol. 42 (1936–37) pp. 230–265. *A correction*, ibid. vol. 43 (1937) pp. 544–546.

31. ———, *Systems of logic based on ordinals*, Proc. London Math. Soc. (2) vol. 45 (1939) pp. 161–228.

UNIVERSITY OF WISCONSIN

THE FEBRUARY MEETING IN NEW YORK

The five hundred eleventh meeting of the American Mathematical Society was held at Columbia University in New York City on Saturday, February 26, 1955. The meeting was attended by about 220 persons, including 191 members of the Society.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings Professor Harish-Chandra of Columbia University delivered an address entitled *Representations of semisimple Lie groups* at a general session presided over by Professor Claude Chevalley. Sessions for contributed papers were held in the morning and afternoon, presided over by Professor Everett Pitcher, Dr. R. L. Sternberg, and Professor D. J. Struik.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title. Where a paper has more than one author, the paper was presented by that author whose name is followed by "(p)". Mr. G. A. Baker, Jr., was introduced by Professor H. A. Arnold, and Mr. Shôshichi Kobayashi by Professor C. B. Allendoerfer.

ALGEBRA AND THEORY OF NUMBERS

383. S. S. Abhyankar: *Splitting of valuations in extensions of local domains*. II.

Let R be a local domain of dimension bigger than one with quotient field K and maximal ideal M. Let K^* be a finite separable extension of K. In a previous joint note (Proc. Nat. Acad. Sci. U.S.A. vol. 41 (1955)), Zariski and the author have proved that: if R is regular and has the same characteristic as its residue field, then there exist infinitely many real discrete valuations of K having center M in R which split in K^*. In the present note the following generalization is proved: If either (a) R admits a necleus (and hence in particular if R is a geometrical local ring) or if (b) R is regular and has the same characteristic as its residue field, then there exist infinitely many real discrete valuations v of K having center M in R such that v has $[K^*:K]$ distinct extensions to K^*. Case (b) is proved by applying Zariski's recent results on normal sequences of local rings (Rend. di Mat. Roma (5) vol. 13 (1954) pp. 1–38) to the quadratic sequence along a valuation of K with center M in R which splits in K^* in order to obtain a quadratic transform of R which splits in K^*, and by passing to a corresponding splitting field. Case (a) is proved by passing to completions and applying case (b). This theorem gives a new proof of a recent result of Nagata for local domains admitting a necleus. (Received January 4, 1955.)

384*t*. H. W. Becker: *Lebesgue quadrilaterals with elements in common*.

There are obvious interpretations of (A) $x^2+y^2+z^2=t^2$ as quadrilaterals, parallelopipeds, or quadrirectangular tetrahedra; hence, of such with common els. LQ with a leg in common (generalizing Hero ▲) are (B) $y=Y=2(mn-pq)(MN-PQ)$, $z=2(mp+nq)(MN-PQ)$; t, $x=(m^2\pm n^2\pm p^2+q^2)(MN-PQ)$; T, $X=(mn-pq)(M^2\pm N^2\pm P^2+Q^2)$; $Z=2(mn-pq)(MP+NQ)$. It is easy then to commonize another el., especially $z=Z$. (A) with (C) $2y^2+z^2=s^2$ has solution t, $x=5a^2\pm b^2$; s, y, $z=6ab$, $4ab$, $2ab$, or more generally, t, $x=(a^2+c^2)(5a^2-4ac+c^2)\pm b^2$; s, y, $z=2b(3a^2-2ac+c^2)$, $4ab(a-c)$, $2b(a^2+2ac-c^2)$. General solution of LQ with hypotenuse t in common is based on Euler's forms (1) Dickson's *History* II, p. 277. LQ with hyp. and one leg in common are furnished by permutation or sign change of Lebesgue's parameters, ibid. p. 265. Ibid. p. 270 erratum: "We may express 1521 as a $3\square$ in 7 ways" should be 8 ways. If (D) $X^2+Y^2+Z^2=T^2$, $T^2+U^2+V^2=W^2$ (generalizing Eels quads.) then $X, Y, Z, T, U, V, W=x_1x_2, x_1y_2, x_1z_2, x_1t_2, y_1t_2, z_1t_2, t_1t_2$, the subscripted letters satisfying (A). (D) transforms: $X, Y, Z, T, U, V, W\to TX, UX, VX, WX, WY, WZ, WT$; $T^2, TU, TV, TW, UW, VW, W^2$; $X^2, XY, XZ, XT, YT, ZT, T^2$. Solutions of $t^2\mp(y^2+z^2)=x^2, w^2$ are $t, y, z, x, w=41, 24, 12, 31, 49$; $97, 96, 12, 7, 137$; $97, 84, 48, 7, 137$; $185, 132, 120, 49, 267$; $185, 168, 60, 49, 267$ (also $49^2+168^2=\square$). Brahmagupta quads. generalize to octagons: $X, T, Z, Y=x_1x_2, t_1x_2, z_1x_2, y_1x_2$; $X', T', Z', Y'=X, x_1t_2, x_1z_2, x_1y_2$; $X'', T'', Z'', Y''=y_1y_2, t_1y_2, z_1y_2, Y'$; $X^*, T^*, Z^*, Y^*=X'', y_1t_2, y_1z_2, Y$. A closed chain of 4 LQ each pair having 2 legs in common is an O'Riordan vector, ibid. pp. 502–505. (Received January 10, 1955.)

385t. H. W. Becker: *Pythagorean 4-facets*. Preliminary report.

These are Diophantine solutions of (A) $u^2+w^2=r^2$, $x^2+y^2=u^2=u^2$, $z^2+o^2=w^2$, with (B) $y^2+z^2=v^2$. The Pyth. parameters are $\langle a, b; c, d; e, f; g, h\rangle$. By E. T. Bell, Scripta Math. vol. 12 (1946) pp. 88–89, the general solution of (A) is r, $u=k(a^2\pm b^2)(c^2+d^2)(e^2+f^2)$; w, $z=2kab(c^2+d^2)(e^2\pm f^2)$; $x=k(a^2-b^2)(c^2-d^2)(e^2+f^2)$, $y=2kcd(a^2-b^2)(e^2+f^2)$, $o=4abefk(c^2+d^2)$. Also satisfying (B), but proscribing (C) $x^2+o^2=q^2$ (a rearranged Brahmagupta quad., RBQ) there are 13 primitive numerical solutions with $r<1000$, the smallest being the mavericks $\langle 8, 5; 3, 2; 2, 1; 2, 1\rangle$, $\langle 13, 2; 3, 1; 3, 2; 10, 1\rangle$, $\langle 13, 8; 2, 1; 3, 2; 5, 2\rangle$. 3 of them generalize to the parametric solutions $\langle m^4-m^2n^2+n^4, (mn)^2; m^2, n^2; m^2-n^2\pm mn; m, n\rangle$, $\langle m^4+n^4, 2(mn)^2; m, n; m^2, n^2; m, n\rangle$, $\langle (m^2+n^2)^2, (2mn)^2; m^2+n^2, 2mn; m, n; m, n\rangle$. There are numerous transforms to RBQ, like $x, y\ z, o, r, u, v, w\to xu, yu, yw, xw, ru, u^2, yr, wu$; $ou, zu, zw, ow, rw, uw, rz, w^2$; $yu, yw, zw, zy, rv, ry, vw, rz$; $yu, zu, zw, yw, rv, uv, rz, yv$; $xy, xz, yz, y^2, uv, xv, uz, vy$; $yo, yz, z^2, zo, vw, yw, zv, zw$; $xz, xo, yo, yz, uw, xw, ou, yw$: under which $\langle a, b; c, d; w, f; g, h\rangle\to\langle a, b; c, d; c\pm d; a, b\rangle$, etc. A RBQ further satisfying (D) $x^2+y^2+z^2=t^2$ is the Pyth. hexahedron $x, y, z, o=XY, Y^2, YZ, ZX$, where (E) $V^2+X^2=U^2+Z^2=T^2$, $V^2-Z^2=U^2-X^2=Y^2$ is a Pyth. tet. If additionally to (A, B, D), (F) $y^2+z^2+o^2=s^2$ is met, the system would be a Pyth. decahedron (all 10 facets Pyth. ▲) but it is conjectured this is impossible. If instead of (B), (B′) $y^2-z^2=v'^2$, such 4-facets are rare: e.g. $x, y, z, o, r=204, 85, 36, 48, 229$; $104, 195, 48, 36, 229$. (Received January 10, 1955.)

386t. J. L. Brenner: *Relations among the minors of a matrix with dominant principal diagonal.*

If the absolute value of the principal element of each row of a square matrix exceeds the sum of the absolute values of the nonprincipal elements of that row, it is known that the matrix is invertible. This article establishes general inequalities among the minors of such a matrix. Two corollaries of the main theorem are the following. First, a lemma of Ostrowski, which he describes as a remarkable inequality [Proc. Amer. Math. Soc. vol. 3 (1952) p. 26]. Second, the absolute value of the principal minor on any set of rows exceeds the absolute value of any nonprincipal minor on the same set of rows. The main theorem is stronger and more exact. Credit is due the Office of Ordnance Research for sponsoring this work. (Received January 10, 1955.)

387*t*. Eckford Cohen: *Congruences in algebraic number fields involving sums of similar powers.*

Let F be a finite extension of the rational field, and suppose P to be a prime ideal of F of norm p^f. Denote by $Q_s(\rho)$ the number of solutions of the congruence $\rho \equiv \alpha_1 X_1^m + \cdots + \alpha_s X_s^m \pmod{P^\lambda}$, where ρ is arbitrary, $(m, p) = 1$, and $(\alpha_i, P) = 1$, $i = 1, \cdots, s$. A general formula for $Q_s(\rho)$, involving the generalized Jacobi sum, is obtained, and approximations to $Q_s(\rho)$ are deduced. These estimates are used to prove the following result: *If* $s \geqq 3$, *then* $Q_s(\rho) > 0$ *for all* P *of sufficiently large norm that satisfy the above restrictions.* The case $s = 2$ is treated in detail and explicit formulas for $Q_2(\rho)$ are determined under certain conditions. (Received January 12, 1955.)

388*t*. Eckford Cohen: *An extension of Ramanujan's sum.* II. *Additive properties.*

Let $(a, b)_k$ denote the greatest common kth power divisor of a and b. The function $c_k(n, r) = \sum \exp(2\pi inx/r^k)$, where x ranges over the integers (mod r^k) such that $(x, r^k)_k = 1$, was introduced by the author in *Part* I (Duke Math. J. vol. 16 (1949) pp. 85–90). In the present paper the following basic orthogonality property is proved: *If* $d|r$, $e|r$, *then* $\sum c_k(a, d)c_k(b, e) = r^k c_k(n, r)$ *or* 0 *according as* $d = e$ *or* $d \neq e$, *the summation being over* a, b *such that* $n \equiv a + b$ (mod r^k). This property is used to obtain the number $Q_k(r)$ of solutions x_i (mod r), y_i (mod r^k), of the congruence $n \equiv a_1 x_1^k y_1 + \cdots + a_s x_s^k y_s \pmod{r^k}$, where $(a_i, r) = 1$. The result for $Q_k(r)$ is given in terms of a singular sum involving $c_k(n, r)$. An equivalent form of the above orthogonality property is also derived, and a generating function of $Q_k(r)$ is deduced. (Received February 17, 1955.)

389*t*. Eckford Cohen: *Some totient functions.*

Using the notation of the preceding abstract, the function $\phi_k(r)$ is defined to be the number of integers b of a complete residue system (mod r^k) such that $(b, r^k)_k = 1$. The characteristic properties of $\phi_k(r)$ are deduced arithmetically, and the function is shown to be equivalent to the totient functions of Jordan (*Traité des substitutions* (1870) pp. 95–97) and of von Sterneck (Monatshefte Math. Phys. vol. 5 (1894) pp. 255–256). The number of solutions of the semilinear congruence discussed in the preceding abstract is then obtained in terms of ϕ_k. Two proofs are given, one based on the Jordan function and the other on the von Sterneck function. Finally, a divisor relation involving ϕ_k is deduced, resolving a question proposed by Métrod (L'Intermédiaire des Math. vol. 20 (1913) pp. 148–149; Dickson, *History*, vol. I, 1918, p. 155). (Received February 17, 1955.)

390*t*. Eckford Cohen: *An extension of Ramanujan's sum.* III. *Connections with generalized totients.*

This paper concludes a study (*Part* II, abstract above) of the function $c_k(n, r)$, emphasizing its relation with generalized totients. In particular, it is proved that $c_k(n, r) = \phi_k(r)\mu(r/d)\phi_k^{-1}(r/d)$ where $d^k = (n, r^k)_k$, μ denotes the Möbius function, and ϕ_k is the totient of the immediately preceding abstract. This result reduces to a theorem of O. Hölder in case $k = 1$ (Prace Mat.-Fiz. vol. 43 (1936) pp. 13–23). The orthogonality properties of *Part* II above are shown to reduce, in special cases, to certain simple arithmetical relations involving divisor sums. Some of these results are then applied in deriving a formula for the number of solutions of the linear congruence, $n \equiv a_1x_1 + \cdots + a_sx_s \pmod{r^k}$, $(a_i, r) = 1$, in $x_i \pmod{r^k}$ such that $(x_i, r^k)_k = 1$. The formula obtained generalizes a theorem of Vandiver and Nicol in the case $k = 1$, but the methods of proof are quite different (Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 825–835). Finally, the Dirichlet series $\sum_{n=1}^{\infty} c_k(n, r)n^{-s}$, $s > 1$, is summed by the methods of the present paper. (Received February 17, 1955.)

391*t*. Casper Goffman: *A definition of completeness for partly ordered groups*. Preliminary report.

Let G be an ordered group. A set $S \subset G$ is locally bounded above if for every $x > e$, the identity of G, there are $y, z \in S$ such that xy and zx are upper bounds of S. G will be called complete if every set which is locally bounded above has a least upper bound. For a class of abelian groups G, including the abelian 1-groups, associated complete groups are obtained as follows: A section in G is a pair of nonempty sets (A, B) such that $x \in A$, $y \in B$ implies $x < y$. It is maximal if $C \supset A$, $D \supset B$ and (C, D) a section implies $C = A$, $D = B$. It is dedekindean if it is maximal and if for every $y > 0$, the identity of G, there is an $x \in A$ such that $x + y \in B$. With trivial identifications, the dedekindean sections form a complete ordered group $\overline{G}$. If G is complete then $\overline{G}$ is isomorphic with G. For integrally closed groups, this completion agrees with the usual one, and for totally ordered groups it agrees with one given by Cohen and Goffman (Trans. Amer. Math. Soc. vol. 67 (1949) pp. 310–319). (Received December 3, 1954.)

392. I. N. Herstein: *Jordan homomorphisms*.

A Jordan homomorphism of one ring R into another ring R' is an additive mapping of R into R' which preserves squares. R' is said to be a prime ring if $x'R'y' = (0)$ implies $x' = 0$ or $y' = 0$. In this paper it is proved that a Jordan homomorphism of a ring R onto a prime ring R' of characteristic different from 2 and 3 is either a homomorphism or an anti-homomorphism. In particular, Jordan automorphisms of simple and primitive rings of characteristic not 2 or 3 are automorphisms or anti-automorphisms. This greatly extends results of Jacobson-Rickart, Kaplansky, Hua and Ancochea. Additive mappings preserving nth powers in case R is a ring with unit element and R' is a prime ring of characteristic larger than n are shown to be homomorphisms or anti-homomorphisms multiplied by $(n-1)$st roots of unity lying in the center of R'. (Received January 10, 1955.)

393*t*. B. W. Jones and D. B. Marsh: *A proof of a theorem of Meyer on indefinite ternary quadratic forms*.

By free use of matrices and the modern theory of quadratic forms including the Hasse symbol, the authors obtain a relatively simple proof of a theorem of Adolf Meyer which is fundamental in the theory of quadratic forms and the determination of class number of primitive indefinite ternary genera. After demonstrating that two forms in the same genus are equivalent if they represent a binary form with certain properties, the authors determine two conditions under which a ternary form represents all binary forms of given determinant; they complete the proof by establishing the satisfaction of these conditions under the hypothesis of the theorem. (Received December 15, 1954.)

394. D. H. Lehmer: *The determination of large isolated values of Ramanujan's function.*

The function $\tau(n)$ of Ramanujan, defined as the coefficient of x^{n-1} in the power series expansion of the 24th power of Euler's product $(1-x)(1-x^2)(1-x^3) \cdots$, has been the subject of many investigations. It is a multiplicative function whose values for successive powers of a fixed prime p form a second order recurring series. Unfortunately one of the coefficients in this recurrence is $\tau(p)$, so that this leaves the knowledge of $\tau(n)$ incomplete to the extent that one does not know this function for prime values of its argument. Unlike most other well known numerical functions, $\tau(p)$ is not a polynomial function of p, although it is a polynomial with respect to various moduli, some of which are fairly large. Tables of $\tau(n)$ extend only to $n \leqq 2500$. It thus becomes important to be able to obtain exact isolated values of $\tau(p)$. A formula for doing this is derived from the fact that $\tau(n)$ enters into the error term of certain divisor functions. This formula, too lengthy to be given here, has been used to compute many values of $\tau(p)$ for p as large as 13127 and 15359. One application of the formula for $p=3967$ gives $\tau(p)/p^{11/2}=1.9514$ which is the largest value for this ratio yet found. According to the Ramanujan hypothesis this ratio does not exceed 2 in absolute value. (Received February 17, 1955.)

395t. Irving Reiner: *Unimodular complements.*

The following theorem is due to Steinitz (Math. Ann. vol. 71 (1911) pp. 328–354): Let R be the ring of algebraic integers in an algebraic number field, and let $a_1, \cdots, a_n$ be relatively prime elements in R. Then there exists an $n \times n$ matrix with first row $(a_1 \cdots a_n)$ whose determinant is a unit in R. A simpler proof, valid for any Dedekind ring R, is given in this note. (Received January 13, 1955.)

396t. Kenneth Rogers: *Extremal lattices of convex bodies in complex space.*

In complex space C^n are defined star-bodies, lattices over complex quadratic number-fields, also admissible, extremal and critical lattices, all as natural analogues of the notions in R^n. Points P, Q are said to be nonassociated if the equation $P=\epsilon Q$ is false for all units ϵ of $k(\theta)$. As analogue of H. P. F. Swinnerton-Dyer's result (Proc. Cambridge Philos. Soc. vol. 49 (1953) p. 161), it is shown that for a bounded, closed convex body K in C^n, any extremal lattice in $k(\theta)$ has at least $n(n+1)$ nonassociated points on the boundary of K. As analogue of results of Korkine and Solotaryov (Math. Ann. vol. 11 (1877) pp. 242–292), it is shown that if M is the minimum of an extremal positive definite Hermitian form, then n^2 of the representations of M can be found which uniquely fix the form. This is used to find the least upper bound for the minimum of a positive definite ternary Hermitian form of determinant D, with variables in $k(i)$. The best possible result is $M \leqq (4D)^{1/3}$, and there is just one class of forms for which the equality sign is necessary. (Received January 13, 1955.)

397t. Kenneth Rogers and H. P. F. Swinnerton-Dyer: *Lattices in complex quadratic fields.*

For lattices over complex quadratic fields, a complete analogue of Mahler's theory (Proc. Roy. Soc. London vol. A 187 (1946) pp. 151–187) for real lattices is given. In particular, the existence of a critical lattice for a star-body of finite type is proved, there being essentially new difficulties only when the class-number of the field is not 1. For the body in C^2 given by $|x| \leqq 1$, $|y| \leqq 1$, a finite process for determining all critical lattices is given, thus giving a general solution of a problem Minkowski solved for the Gaussian and Eisenstein fields. In terms of forms, the result for $k(i5^{1/2})$ is: for linear forms ξ, η, of determinant 1 in variables u, v, there exist integers $(u, v) \neq (0, 0)$ of $k(i5^{1/2})$ such that $\max(|\xi|, |\eta|) \leq \{2/(-3+(15)^{1/2})\}^{1/2}$. This is best possible, there being one class of critical forms. Other applications of the general theory are given. (Received January 13, 1955.)

398t. R. L. San Soucie: *A characterization of a class of rings.*

Let F be a field of characteristic two having an automorphism α of order two and an element f_0 in F such that $f_0\alpha = f_0$ and f_0 not a square in F. Define the additive endomorphism θ of F by $f\theta = f\alpha + f_0 f$, all f in F. If R is the set of all couples (f, g) with f, g in F, if equality and addition in R are defined componentwise, and if multiplication in R is defined by $(^*)(f, g)(h, k) = (fh + g \cdot k\theta, fk + gh)$, with θ as defined above, then R. H. Bruck has shown that R is a not alternative, right alternative division ring of characteristic two. [See R. L. San Soucie, *Right alternative division rings of characteristic two*, to appear soon in the Proceedings of this society]. The present paper characterizes a class of right alternative division rings of characteristic two, including the rings mentioned above. It is proved that if R is a not alternative, right alternative division ring of characteristic two, if R satisfies the identity $(wx, y, z) = w(x, y, z) + (w, y, z)x + (w, x, (y, z))$, and if all commutators are in the left nucleus of R, then R is two-dimensional over some field F and multiplication in R is defined as in $(^*)$, where the θ is *some* additive endomorphism of F. (Received December 27, 1954.)

399. Ernst Snapper: *Integral closure of modules and complete linear systems.*

Let Σ/k be a finitely generated field extension, i.e., Σ and k are fields and $\Sigma = k(a_1, \cdots, a_n)$, where $a_1, \cdots, a_n$ are elements of Σ. A module L is a subgroup of the additive group of Σ, which is closed under multiplication by elements of k. One says that L has a finite number of generators if L consists of the linear combinations $c_1s_1 + \cdots + c_ms_m$, where $s_1, \cdots, s_m$ are fixed elements of Σ and $c_1, \cdots, c_m$ vary freely in k. The product LM of two modules L and M is the smallest module which contains all the products $\lambda\mu$, where $\lambda \in L$ and $\mu \in M$. *The integral closure* $|L|$, *of a module* L *is defined as the module which consists of those elements* b *of* Σ *for which there exists a nonzero, finitely generated module* M, *such that* $bM \subset ML$. We prove that, *if* L *is finitely generated, then* $|L|$, *is finitely generated*. It is shown that the finite generation of $|L|$, is the field-theoretic equivalent of the geometric fact that every linear system of $(r-1)$-dimensional cycles of an irreducible, r-dimensional algebraic variety is contained in a complete linear system. (Received January 13, 1955.)

400. Ruth Rebekka Struik: *On associative products of groups.*

Continuing the work of O. N. Golovin [Mat. Sbornik vol. 27 (69) (1950) pp. 427–453], a doubly-infinite number of associative products of groups can be constructed in the following manner: Let $G = A_1 * A_2$ be the free product of the groups A_1 and A_2. Let (A, B) be the subgroup generated by all $a^{-1}b^{-1}ab$, $a \in A$, $b \in B$. Let R be a subgroup of G and ${}_kR_G$ the $(k+1)$st member of the lower central series of R, that is, ${}_kR_G$ is the normal subgroup generated by R and ${}_kR_G = ({}_{k-1}R, G)$. Let $A_1 \circ A_2 = (A_1 * A_2)/({}_mA_{1_G}, {}_nA_{2_G}) \cdot ({}_nA_{1_G}, {}_mA_{2_G})$ where m, n are any two fixed non-negative integers. Then $\circ$ is associative, that is, $(A_1 \circ A_2) \circ A_3 \cong A_1 \circ (A_2 \circ A_3) \cong (A_1 * A_2 * A_3)/\prod^3_{i,j=1,i\neq j}({}_mA_i, {}_nA_j)$ under the obvious mapping, for all groups A_1, A_2,

A_3. Not all commutative products are associative. It can be shown that ∇ is not associative, where $A_1 \nabla A_2 = (A_1 * A_2)/((A_1, A_2), (A_1, A_2))$. The proof of associativity utilizes identities and inequalities of subgroups of free products, and in particular ${}_kR_F \leq {}_kR_G({}_kA_{3F} \cap {}_1G_F)$, where $F = A_1 * A_2 * A_3$. (Received January 10, 1955.)

401t. N. A. Wiegmann: *On symmetric matrices with real quaternion elements.*

It is known that a matrix A with complex elements is symmetric if and only if there exists a complex unitary matrix U such that UAU^T is a real diagonal matrix (where U^T denotes the transpose of U); here possible analogs for a matrix with real quaternion elements are considered. If U is unitary and quaternion, necessary and sufficient conditions that U^T is unitary are obtained, and certain related results are stated. It is shown that if A is quaternion and symmetric, there exists a real orthogonal matrix U such that UAU^T is quaternion and diagonal if and only if the hermitian polar matrix of A is real. Matrices of the form UDU^T are then considered where D is quaternion and diagonal and U is, in turn, complex unitary and quaternion unitary (where U^T is also unitary). (Received January 5, 1955.)

ANALYSIS

402t. R. P. Boas: *Integrability of trigonometric series.* IV.

Let $\{\lambda_n\}$ be a sequence of bounded variation (i.e., $\sum |\Delta\lambda_n| < \infty$). Form $\mu_n = \sum' \{(-1)^{n-m} - 1\}(n-m)^{-1}\lambda_m$ where the sum omits the term $m = n$. Then if $\{\mu_n\}$ is also of bounded variation, the λ_n are Fourier coefficients of a Lebesgue integrable function. A closely related theorem is that if both $f(x)$ and $f(x)$ sgn x have absolutely convergent Fourier series over $(-\pi, \pi)$, then $f(t)/t$ is Lebesgue integrable. These theorems imply various older results of a similar character. (Received January 7, 1955.)

403t. F. E. Browder: *On the positivity and semi-boundedness of the Dirichlet forms associated with an elliptic differential operator.*

Algebraic conditions in terms of the characteristic form of a Dirichlet form, associated with an elliptic differential operator K on a domain G of Euclidean n-space, are established which are necessary and sufficient for its semi-boundedness on the space $C^\infty(G)$. (Received January 13, 1955.)

404t. H. S. Collins: *Completeness, full completeness, and k-spaces.*

Let E be a completely regular T_1 topological space and $C(E)$ be the real l.t.s. of continuous real-valued functions on E, with the compact-open topology. Theorem 1: E is a k-space if and only if $C(E)$ is complete and E with its k-topology (= strongest topology on E which agrees with the given topology on each compact set) is completely regular. Theorem 2: If E is either pseudo-finite or hemicompact then E with its k-topology is completely regular (by definition, E is pseudo-finite if every compact set is finite, and is hemicompact if the compact sets have a countable base). Theorem 3: For E either pseudo-finite or hemicompact the following conditions are equivalent: (1) $C(E)$ is complete; (2) E is a k-space; and (3) $C(E)$ is fully complete (in Bull. Amer. Math. Soc. Abstract 59-1-10, full completeness is termed Property A). In addition, for E pseudo-finite, each of the following is equivalent to each of (1)–(3): (4) E is discrete; (5) E is locally compact; (6) $C(E)$ is the cartesian product of E copies of the reals, with the product topology. (Received December 22, 1954.)

405*t*. R. B. Davis: *Asymptotic solutions of a fourth order elliptic partial differential equation.* Preliminary report.

The extension of Levinson's asymptotic study of second-order elliptic partial differential equations (Ann. of Math. vol. 51 (1950) pp. 428–445) to fourth-order equations encounters this difficulty: there is no directly analogous maximum principle available. One may circumvent this obstacle in either of two ways: by using integral norms (following a suggestion of Berg and Lax), or by restricting the equation and choosing boundary values in a suitable fashion. The present paper follows the second course, and establishes an asymptotic theorem analogous to Levinson's, for the case where the operator factors ($Lu = L_1L_2u = f$), and where boundary values are given for L_2u, as well as for u. (Received January 13, 1955.)

406. Jesse Douglas: *Comparative considerations concerning the Dirichlet principle and the Douglas principle.*

The title and contents of a recently published book entitled *Dirichlet's principle*, as well as the general tenor of some recent mathematical writings, have seemed to promote among non-connoisseurs the impression that one can solve the Plateau problem with this classic principle. All such attempts, however, extending over an 80 year period in mathematical history (1850–1930), met with failure. The fact is, of course, that the Plateau problem presents an essentially new difficulty, of higher order than the fundamental problems in complex variable and conformal mapping whose solution, following the lead of Riemann, has been based on the Dirichlet principle. An entirely fresh approach was necessary. The principle introduced by the author for the solution of Plateau's problem is: $A(g) =$ minimum, where the functional $A(g)$ is defined as the mean square of the ratio of corresponding chords of the given contour Γ and the unit circumference C (times π). The argument g is an *arbitrary* one-one continuous *map* of C onto Γ, and corresponding chords are those whose end points are g-related. In the main form of Dirichlet's principle, on the other hand, the argument ϕ in the basic functional $D(\phi)$ is a *numerical*-valued function on a given plane region with a *prescribed* value at each boundary point. The contrast between the two minimum principles is further elaborated. (Received January 12, 1955.)

407. Jim Douglas, Jr. and T. M. Gallie, Jr. (p): *Variable time steps in the solution of the heat flow equation by a difference equation.*

The numerical solution of the boundary value problem $u_{xx} = u_t$ $(0 < x < 1,\ t > 0)$, $u(0, t) = u(1, t) = 0$, $u(x, 0) = g(x)$, is treated using the backwards difference equation $\Delta_t w_{i,n+1} = (w_{i,n+1} - w_{in})/\Delta t_n$, where $x_i = i\Delta x$, $t_n = \Delta t_0 + \cdots + \Delta t_{n-1}$, $f_{in} = f(x_i, t_n)$, $\Delta_x^2 f_{in} = (f_{i+1,n} - 2f_{in} + f_{i-1,n})/(\Delta x)^2$. It is well known that if the ratio of Δt to $(\Delta x)^2$ is held constant, the solution of the difference equation converges to that of the differential equation with the truncation error being $O((\Delta x)^2)$. It is shown that the conclusion holds for $\Delta t_n/(\Delta x)^2 = \alpha + \beta t_n$ or $\Delta t_n/(\Delta x)^2 = \gamma \exp (\delta t_n)$, $0 < \delta < \pi^2/2$. This results in a great reduction in total calculation necessary to complete the numerical solution. (Received January 10, 1955.)

408. Albert Edrei: *Gap properties of entire functions of finite order, bounded on a radial path.*

Let (1) $F(z) = F(0) + \sum_{n=1}^{\infty} c_n z^{\lambda_n} [c_n \neq 0]$ be an entire function of finite order ρ, bounded for positive values of the variable z. Then, defining the lower logarithmic density of the sequence $\{\lambda_n\}$ by $\delta = \liminf_{x\to\infty} (\log x)^{-1} \sum_{\lambda_n \leq x} \lambda_n^{-1}$, one has $2\rho\delta \geqq 1$. Conversely, given any sequence $\{\lambda_n\}$ of positive increasing integers and of lower logarithmic density $\delta(>0)$, there exists an entire function, of the form (1), of order $(2\delta)^{-1}$, which remains bounded on the positive axis. Slight variations in the arguments used to prove the above proposition enable the author to answer completely the following question: if a real entire function, of finite order ρ, is bounded for $z>0$, how frequent must be the changes of sign of the coefficients of its expansion $\sum a_n z^n$. (Received January 12, 1955.)

409*t*. H. G. Eggleston: *A property of bounded analytic functions.*

$f(z)$ is a bounded regular function in $|z| < 1$. The set of points θ, at which the radial limit $\lim_{r\to 1-} f(re^{i\theta})$ exists is denoted by F. The set of θ at which the cluster set of $f(z)$ reduces to a single point is denoted by G. It is shown that $F \subset G \cup H$ where H is of first category. It follows that there exist functions $f(z)$ for which F is of first category. (First category is relative to the set $0 \leqq \theta \leqq 2\pi$.) (Received January 10, 1955.)

410*t*. A. G. Fadell: *Accessibility in Euclidean n-space.*

Let there be given a point $p = (p^1, \cdots, p^n)$ in a subset S of Euclidean n-space. The accessible set $A[p, s]$ of p is defined to consist of all points $(x^1, \cdots, x^n)$ in S which satisfy the condition (vacuous for $n=1$) that the points $(x^1, \cdots, x^{k-1}, p^k, \cdots, p^n)$ be in S for $k=2, \cdots, n$. The accessible set $A[p, S]$ is open, closed, or Borel according as S is open, closed, or Borel. Let $S(p^k, \cdots, p^n)$, $2 \leqq k \leqq n$, denote the set of points $(x^1, \cdots, x^{k-1})$ in Euclidean $(k-1)$-space such that $(x^1, \cdots, x^{k-1}, p^k, \cdots, p^n)$ is in S. Then $A[p, S]$ is L_n-measurable if S is L_n-measurable and $S(p^k, \cdots, p^n)$ is L_{k-1}-measurable for $k=2, \cdots, n$. The main result of the paper is the following: If p is a point of an L_n-measurable set S such that $S(p^k, \cdots, p^n)$ is L_{k-1}-measurable for $k=2, \cdots, n$, then p is a point of density for $A[p, S]$ if and only if p is a point of density for S and $(p^1, \cdots, p^{k-1})$ is a point of density for $S(p^k, \cdots, p^n)$ for $k=2, \cdots, n$. As a corollary we have that almost every point of an L_n-measurable set is a point of density for its accessible set. The latter theorem and corollary have application in relating partial conditions to full conditions, especially in differentiability questions. (Received January 3, 1955.)

411*t*. A. G. Fadell: *Characterizations of the existence of a total and an approximate total differential.*

Let f be a real-valued function defined on a subset S of Euclidean n-space, and let $M = \limsup |f(p)-f(x)|/\|p-x\|$. Then f is termed *of bounded linear distortion* (b.l.d.) at a point p in S if $M < \infty$ as x approaches p through S. Also, f is termed of *approximately* b.l d. at p in S if p is a point of density for a subset E of S for which $M < \infty$ as x approaches p through E. Application of the writer's results concerning accessibility (see preceding abstract) yields a proof of the n-dimensional analog of the Rademacher-Stepanoff Theorem, namely, that a real-valued L_n-measurable function f on an L_n-measurable subset S of Euclidean n-space has a (Stolz) total differential a.e. in S if and only if f is a.e. of b.l.d. in S. Application of the latter theorem along with accessibility enables the following theorem: Given a real-valued L_n-measurable function f on an L_n-measurable subset S of Euclidean n-space, the following conditions are equivalent: (1) f has an approximate total differential a.e. in S; (2) f has approximate first partial derivatives a.e. in S; (3) f is a.e. of approximately b.l.d. in S; (4) f is a.e. partially of approximately b.l.d. in S; (5) f has a total differential a.e. on a sequence of L_n-measurable sets S_k with $L_n(S - \bigcup_{k=1}^{\infty} S_k) = 0$; (6) f is Lipschitzian on a sequence of L_n-measurable sets S_k with $L_n(S-\bigcup_{k=1}^{\infty} S_k)=0$. The equivalence of (1) and (2) for $n=2$ was first proved by Stepanoff. (Received January 3, 1955.)

412t. A. G. Fadell: *A characterization of the existence of a regular approximate total differential.*

A real-valued function f defined on a subset S of Euclidean n-space is said to have a regular approximate (or asymptotic) total differential at p in S if there exist finite constants A_j, $j=1, \cdots, n$, such that $\lim \left|f(p)-f(x)-\sum_{j=1}^{n} A_j(p^j-x^j)\right| / \|p-x\|$ exists and equals zero as x approaches p through a subset E of S having the following two properties: (1) E is the intersection of S and the union of oriented n-cubes having p as center and (2) p is a point of density for E. The main result of the paper is the following: A real-valued function f defined and continuous on an oriented n-cube Q, $n \geqq 2$, has a regular approximate total differential a.e. in Q if and only if f is of bounded linear distortion a.e. in Q with respect to each set of $(n-1)$ variables. Also, for an L_2-measurable function f on a planar L_2-measurable set the existence almost everywhere of a regular approximate total differential is equivalent to the existence almost everywhere of an approximate total differential. (Received January 3, 1955.)

413. Sigurdur Helgason: *A multiplier problem.*

Let $A=\{x\}$ be a semi-simple, self-adjoint Banach algebra represented by the algebra $\hat{A}=\{\hat{x}\}$ of continuous functions on the maximal ideal space $\mathfrak{M}$. Let $C^{\infty}(\mathfrak{M})$ denote the set of bounded continuous functions on $\mathfrak{M}$ and $C(\mathfrak{M})$ denote those functions in $C^{\infty}(\mathfrak{M})$ that vanish at infinity on $\mathfrak{M}$. A function $f \in C^{\infty}(\mathfrak{M})$ is called a multiplier if $\hat{x}f \in \hat{A}$ for every $\hat{x} \in \hat{A}$. Let $\tilde{A}$ denote the algebra of all multipliers. Then $C(\mathfrak{M}) \subset \tilde{A}$ implies $C^{\infty}(\mathfrak{M})=\tilde{A}$ if at least one of the two following conditions is satisfied: (i) A is regular and every closed ideal in A is contained in a regular maximal ideal. (ii) A is reflexive. This is proved by using the derived algebra A_0 (Proc. Nat. Acad. Sci. U.S.A. vol. 40, pp. 994–995) to get a decomposition of every linear functional $F \in A'$ into homomorphisms. Thereby one can to every $F \in C^{\infty}(\mathfrak{M})$ construct an endomorphism T_f of the second dual A'' leaving A invariant and such that the effect of T_f on A corresponds to the multiplication of $\hat{A}$ by f. (Received January 13, 1955.)

414t. A. J. Lohwater: *Beurling's theorem for quasi-conformal mappings.*

Let $w=f(z)$ be a topological mapping of $|z|<1$ with bounded dilatation coefficient. It is shown that $f(z)$ possesses unique boundary values "in angle" for all $e^{i\theta}$ on $|z|=1$ except for a possible set of logarithmic capacity zero, and that these boundary values cannot be the same on a set of positive capacity. Extensions are made to pseudo-analytic functions which are not necessarily univalent in $|z|<1$. These results extend certain results of Beurling [Acta Math. vol. 72 (1940) pp. 1–13] for a class of analytic functions. (Received January 6, 1955.)

415. Paul Malliavin: *Reduction of some questions of uniqueness to a Watson's problem.*

Some problems of uniqueness, gathered by S. Mandelbrojt in a same theory by the method of the adherent series, are reduced to a same problem, generalizing the Watson's problem. (Received January 11, 1955.)

416. C. B. Morrey, Jr. (p) and James Eells, Jr.: *A variational method in the theory of harmonic integrals.*

In this paper a variational method is carried through to prove the following orthogonal decomposition theorem: *Any r-form ω in L_2 on a compact orientable smooth manifold can be uniquely expressed in the form $\omega = H + \delta\alpha + d\beta$, where H is a harmonic form and α and β are forms in* $\mathfrak{p}_2$ (their components are absolutely continuous along lines and their partial derivatives are in L_2; see *Functions of several variables and absolute continuity* by Calkin and Morrey, Duke Math. J. vol. 6 (1940) pp. 170–215) *with $d\alpha = \delta\beta = 0$; if $\omega \in \mathfrak{p}_2$, then $\delta\alpha$ and $d\beta$ are also in $\mathfrak{p}_2$.* The methods and results are extended to orientable manifolds of class C'-Lip (g_{ij} Lipschitzian). Harmonic forms are found to be Holder-continuous throughout; further differentiability properties of H, α, and β are proved. (Received January 13, 1955.)

417. L. A. Rubel: *An extension of Carlson's theorem.*

For $0 < \alpha \leqq \pi$ the Carlson class $C(\alpha)$ is defined as the class of all entire functions $f(z)$ satisfying, for some $a < \infty$, $c < \alpha$, and all $\epsilon > 0$, the growth restriction: $f(z) = O(1) \exp (a|x| + c|y| + \epsilon|z|)$. A set A of positive integers is said to be total for $C(\alpha)$ if the only function $f \in C(\alpha)$ such that $f(n) = 0$ for all $n \in A$ is the null function $f(z) = 0$. The celebrated theorem of Carlson states that the set I of all positive integers is total for $C(\pi)$. The principal result of the present paper is that a set A is total for $C(\pi)$ if and only if $\overline{D}(A) = 1$ where $\overline{D}(A)$ is the upper asymptotic density of A defined by: $\overline{D}(A) = \limsup A(t)/t$ as $t \to \infty$. Here $A(t)$ denotes the number of integers $n \in A$ such that $n \leqq t$. For any $\alpha < \pi$ the condition $\overline{D}(A) \geqq \alpha/\pi$ is known to be necessary but not sufficient that A be total for $C(\alpha)$. (Received January 10, 1955.)

418*t*. J. B. Serrin: *A characterization of regular boundary points for second order linear elliptic differential equations.*

Consider the elliptic equation (*), $L(u) = \sum a_{ij} u_{ij} + \sum b_i u_i + cu = f$, where the coefficients are Hölder-continuous functions of $x = (x_1 \cdots x_n)$ in E_n, $c \leqq 0$, and $u_{ij} = \partial^2 u/\partial x_i \partial x_j$, $u_i = \partial u/\partial x_i$. Let T be a bounded domain in E_n with boundary t, and let continuous values of u be assigned on t. Then exactly as for the Laplace equation one can associate to each such set of assigned values a generalized solution of the corresponding Dirichlet problem for T (see, for example, Schauder, Math. Zeit. vol. 38 (1934)). Let the assigned values be denoted by ϕ and the associated solution by $u_\phi(x)$, $x \in T$; if a point p of t has the property that, for all ϕ, $\lim_{x \to p} u_\phi(x) = \phi(p)$, then p is said to be regular (for T) with respect to (*). We show that *a boundary point of T is regular with respect to* (*) *if and only if it is regular with respect to the Laplace equation*. It follows that regular points for the Dirichlet problem for (*) are characterized by Wiener's series criterion (special cases of this result, namely when $L(u)$ has an adjoint and when $a_{ij} = \delta_{ij}$, have already been obtained by Püschel, ibid. vol. 34, and Tautz, ibid. vol. 39). The proof given here, which is somewhat simpler than these earlier ones, makes essential use of the concept of a "barrier." (Received January 12, 1955.)

419*t*. V. L. Shapiro: *On the integral representation of continuous periodic functions.*

Let $F(x)$, where $x = (x_1, \cdots, x_n)$, be a continuous periodic function of period 2π in each variable in Euclidean n-space, $n \geqq 2$, and designate the integral mean of F on the surface of the n-dimensional sphere with radius t and center x by $L(F; x; t)$. Also let T_n designate the torus $\{x = (x_1, \cdots, x_n): -\pi < x_i \leqq \pi, i = 1, \cdots, n\}$ and let Z be a closed set of capacity zero contained in T_n. Suppose that $\limsup_{t \to 0} |L(F; x; t)|/t^2 < \infty$ in $T_n - Z$ and that $\lim_{t \to 0} 2n[L(F; x; t) - F(x)]/t^2$ is equal to $f(x)$ almost everywhere in T_n where f is an integrable on T_n. Then with the help of the Green's function introduced by Bochner for T_n, namely $G(x) \sim -\sum_{m \neq 0} e^{i(m,x)}/(m, m)$ where m is a lattice point and (m, x) is the scalar product, an integral representation for $F(x)$ in terms of $f(x)$ is obtained. In particular it is shown that for almost all x in T_n, $F(x) = (2\pi)^{-n}\int_{T_n} G(x-y)f(y)dy + (2\pi)^{-n}\int_{T_n} F(x)dx$. The proof of this result makes use of the integral representation obtained previously by Rudin for continuous functions which are not necessarily periodic in terms of their generalized Laplacians. This new integral representation of continuous periodic functions obtained here enables one to prove the following theorem in the uniqueness of multiple trigonometric series: Let the multiple trigonometric series $S = \sum_m a_m e^{i(m,x)}$ be $(C, 1)$ spherically summable to $f(x)$ almost everywhere in T_n where $f(x)$ is integrable on T_n and suppose that $\sum_{m \neq 0} a_m e^{i(m,x)}/(m, m)$ is the Fourier series of a continuous periodic function. Also suppose that $\limsup_{R \to \infty} |\sigma_R(x)| < \infty$ in $T_n - Z$ where $\sigma_R(x)$ is the $(C, 1)$ spherical mean of rank R of S. Then S is the Fourier series of $f(x)$. (Received January 6, 1955.)

420. C. J. Standish: *On the representation of a function as a Poisson transform.* I.

Necessary and sufficient conditions are obtained for a function to be represented in the form $f(x) = (1/\pi)\int_{-\infty}^{+\infty} d\beta(t)/1+(x-t)^2$ where $\beta(t)$ is (a) of bounded variation and (b) monotone and bounded, and in the form $f(x) = (1/\pi)\int_{-\infty}^{+\infty} g(t)dt/1+(x-t)^2$ where $g(t)$ is Lebesgue integrable on $(-\infty, +\infty)$. The conditions are phrased in terms of a linear differential operator of infinite order employed by H. Pollard (Trans. Amer. Math. Soc. to appear) in constructing a real inversion formula for the Poisson transform. (Received January 6, 1955.)

421. Kurt Strebel: *On the maximal dilation of quasiconformal mappings.*

Let $w(z)$ be a quasiconformal mapping of G_z onto G_w, i.e. topological with finite maximal dilation $K = \sup_Q m'/m$, where Q is any quadrilateral in G_z, m its module, and m' the module of its image in G_w. It is shown that $w(z)$ is absolutely continuous on almost every line $y = ax + b$ for arbitrarily fixed a $(z = x + iy)$. If E is any closed set in G_z and K_0 denotes the maximal dilation of $w(z)$ outside the set E, then $K_0 = K$ if E is of two-dimensional measure zero. If $w(z)$ is only known to be topological in G_z with finite maximal dilation K_0 outside E, then the maximal dilation of $w(z)$ in G_z is equal to K_0 if E is a closed set of Σ-finite linear measure or the union of such a set with a function theoretic nullset 0_{AD}. (Received January 12, 1955.)

422t. John Wermer: *Polynomial approximation on an arc in the space of three complex variables.*

We construct an arc Γ in the space of 3 complex variables z_1, z_2, z_3 with the following property: Not every continuous function on Γ is uniformly approximable on Γ by polynomials in z_1, z_2, z_3. In fact any approximable function maps Γ onto a Peano curve. It follows from this that the Banach algebra of all complex-valued continuous functions on the unit interval has a proper closed subalgebra which separates points and contains the constant 1. (Received January 13, 1955.)

APPLIED MATHEMATICS

423t. G. A. Baker, Jr.: *Equivalence theorem for quantum mechanics.*

Integral formulas are derived for the Hermitian operators of the Schrödinger formulation of quantum mechanics, thereby giving the operators precise meaning. It is then shown that the Schrödinger formulation is completely equivalent to a formulation based on a quasi-probability distribution of the canonical coordinates and momentum describing the corresponding classical system. The distribution is shown to satisfy an integro-differential equation which corresponds to Schrödinger's wave equation. The quasi-probability distribution is not a probability distribution in the usual sense of the word as it is not in general everywhere non-negative. The transformation theory of this representation of quantum mechanics is investigated. In this paper there has been no need for the customary recourse to the solution of eigenvalue equations in complete orthonormal sets or infinite matrices; however, extensive use has been made of Fourier transform methods. (Received January 6, 1955.)

424*t*. H. W. Becker: *Diophantine parallel circuits and unbalanced Wheatstone bridges.*

An efficient algorithm for producing all Diophantine circuits, s.p. or bridge, is $t=(wA+B)/(wC-D)=\{A+(BC+AD)/(wC-D)\}/C=(A+F^2/K)/C$. Where $R=t\|w$, it yields the classic results, Dickson's *History* II, pp. 688–691. All solutions are tabulated through $R=155$. Where $R=t\|r\|w$ and $C=r-R$, $t=(Rr+R^2r^2/K)/C$ and $w=(Rr+K)/C$. The procedure is to run through all $1\leqq C\leqq 2R$. There are always Diophantine solutions for $C=1, 2, 3, 5$, but none for $C=4$, $R\equiv 1 \bmod 12$. All sols. are tabulated through $R=12$. The Wheatstone bridge $R={}^{t}_{v}r^{w}_{u}=D/N$, $D=r(t+u)(v+w)+tu(v+w)+vw(t+u)$, $N=r(t+u+v+w)+(t+v)(u+w)$, Amer. Math. Monthly vol. 54 (1947) p. 599. Then $t=\{(R-u)(r+v)+(Rr+uv)^2/K\}/(S-R)$, $w=\{(R-v)\cdot(r+u)+K\}/(S-R)$, $S=u+r+v$. The procedure is to run through all compositions of $S\geqq 3$ with $2\leqq R\leqq S-1$. The condition for solutions symmetrically disposed about the median is $A\equiv D \bmod C$. The only *DUWB* whose 5 branches are consecutive integers is ${}^{2}_{5}3^{6}_{4}=4$. The *DUWB* of smallest branches and R all different is apparently ${}^{1}_{7}2^{13}_{3}=4$. R. M. Foster (communication of March 16, 1948) proposed the problem of *DUWB* whose 6 values are different primes, and supplied the example ${}^{29}_{7}71^{19}_{11}=13$. In an iterative Foster bridge, $R=$ a branch value. The smallest *IFB* (calling unity a prime) is ${}^{1}_{5}3^{7}_{2}=3$. The smallest *FB* is apparently ${}^{3}_{13}2^{19}_{5}=7$. There's nothing corresponding to *FB* in *DPC*. Let $E_n(R)$ or $e_n(R)$, $E_{\overline{W}}(R)$ or $e_{\overline{W}}(R)$ enumerate the n-branch *DPC* and *DUWB* of joint or branch resistance R. Then $E_n(R)>e_n(R)$; but $e_{\overline{W}}(R)=\infty\neq E_{\overline{W}}(R)$, e.g. $E_{\overline{W}}(2)=9$. $E_{\overline{W}}(R, S)$ is tabulated through $S=12$: 2152 *DUWB*. (Received December 3, 1954.)

425*t*. Jim Douglas, Jr.: *The solution of the diffusion equation by a high order correct difference equation.*

The numerical solution of $u_{xx}=u_t$, $0<x<1$, $t>0$, subject to $u(x, 0)=f(x)$, $u(0, t)=g(x)$, $u(1, t)=h(t)$ has been treated by means of several difference equations. Among the implicit methods used have been the backwards difference equation and the Crank-Nicolson equation. It is well known that the error in either case is $O(\Delta t)$, if $\lambda=4\Delta t/(\Delta x)^2$ is held constant as $\Delta t\rightarrow 0$. The modification of the Crank-Nicolson equation is proposed: $(1-2/3\lambda)\Delta_x^2 w_{i,n+1}+(1+2/3\lambda)\Delta_x^2 w_{in}=2(w_{i,n+1}-w_{in})/\Delta t$, where $w_{in}=w(i\Delta x,\ n\Delta t)$ and $\Delta_x^2 w_{in}=(w_{i+1,n}-2w_{in}+w_{i-1,n})/(\Delta x)^2$. The above difference equation is high order correct not only in the t-direction as is the Crank-Nicolson but also in the x-direction. It is shown that, for any $\lambda>0$, the error is $O((\Delta t)^2)$ as $\Delta t\rightarrow 0$. The method is quite practical, as it involves many fewer steps to reach a given time with a specified maximum allowable error and only twice as much work per time step as the forward difference equation. (Received January 10, 1955.)

426. Arthur Erdélyi (p), Maurice Kennedy, J. L. McGregor, and C. A. Swanson: *Asymptotic forms of Coulomb wave functions.* Preliminary report.

Asymptotic representations are obtained for the regular and irregular Coulomb wave functions, $F_L(\rho, \eta)$ and $G_L(\rho, \eta)$, and for their derivatives. It is assumed that L is fixed, $\eta \rightarrow +\infty$, and ρ/η is positive and unrestricted. With one exception (to be noted below) the asymptotic forms are expressed in terms of Airy integrals and their derivatives. The asymptotic forms are obtained by a comparison of differential equations, and the constants are determined by means of the leading terms in the known expansions of F_L and G_L for small and large ρ: the integral representations of Coulomb wave functions are not used. Asymptotic representations holding uniformly in the respectively ranges of ρ/η are obtained as follows. For G_L, there is a single asymptotic form for $\rho/\eta \geqq \epsilon > 0$, and this form remains valid if η is fixed and $\rho \rightarrow \infty$. For F_L, there is an asymptotic form for $\rho/\eta \geqq \epsilon > 0$; there is a second asymptotic form for $\rho/\eta \geqq 2$, and this second form remains valid if η is fixed and $\rho \rightarrow \infty$; and there is a third asymptotic form, in terms of modified Bessel functions of order $2L+1$, for $0 \leqq \rho/\eta \leqq 2-\epsilon < 2$, and this form remains valid if η is fixed and $\rho \rightarrow 0$. (Received January 13, 1955.)

427t. Isadore Heller: *Geometric characterisation of cyclic permutations.*

P is the set of all n by n permutation matrices $p, q, \cdots$ which are interpreted as points in n^2-space. The convex hull C of P is a polyhedron of dimension $(n-1)^2$ and has P as the set of vertices. The note proves: 1. Two permutations p and q are neighbors of order k—that is, they are on the same k-dimensional face of C and have no faces of lower dimension in common—if and only if $p^{-1}q$ is a product of k disjoint cycles. 2. (Corollary) The range of k consists of all integers in the closed interval $[0, n/2]$. 3. (Corollary) The cyclic permutations are the order-one-neighbors of the identity permutation. Cycles of length s (i.e. leaving $n-s$ elements fixed) have distance $(2s)^{1/2}$ from the identity. Geometrically the set of order-one-neighbors of any permutation is equivalent to the set of cyclic permutations. For related results see T. S. Motzkin, *The assignment problem*, Proc. 6th Appl. Math. Symp., McGraw-Hill, 1955. (Received January 10, 1955.)

428t. Jacob Korevaar: *Distributions and their Laplace transforms defined from the point of view of applied mathematics.*

Define $f^{(-1)}(t) = \int_0^t f(u)du$, $f^{(-2)}(t) = \int_0^t f^{(-1)}(u)du$, etc. A sequence of integrable functions $\{f_n(t)\}$, $n=1, 2, \cdots$, is called fundamental if for some positive integer p the sequence $\{f_n^{(-p)}(t)\}$ is uniformly convergent on every finite interval $(0, A)$. A fundamental sequence defines a distribution ϕ on the half-line $t \geqq 0$. An integrable $f(t)$ is identified with the distribution $[f]$ defined by $f_n(t) = f(t)$, $n=1, 2, \cdots$. For $\tau \geqq 0$ the sequence $f_n(t) = 0$ $(0 < t < \tau,\ t > \tau + 1/n)$, $f_n(t) = n(\tau < t < \tau + 1/n)$ defines the distribution δ_τ (δ-"function" or unit impulse "function"). Equality of distributions is defined, and it is shown that every distribution ϕ may be redefined by a fundamental sequence of continuous functions $h_n(t)$ which have a piecewise continuous derivative for $t \geqq 0$ while $h_n(0) = 0$. The derivative ϕ' of ϕ is then defined by the fundamental sequence $\{h_n'(t)\}$. Examples. $\delta_\tau = U_\tau'$, where U_τ is the unit step function. $[f]' = [f'] + f(0)\delta_0$ when f is continuous, f' piecewise continuous. A distribution ϕ is said to be of exponential type k if it may be given by a fundamental sequence $\{f_n(t)\}$ such that $f_n(t) = 0$ for $t > c_n$ while for some M, k and $q \geqq 0$, $|f_n^{(-q)}(t)| \leqq Me^{kt}$ $(n=1, 2, \cdots; 0 \leqq t < \infty)$. The

Laplace transform $L(\phi)$ of a distribution $\phi:\{f_n(t)\}$ of exponential type k is defined as $L(\phi)=\lim_{n\to\infty}L(f_n)$, $L(f_n)=\int_0^\infty f_n(t)e^{-st}dt$ (Re $s>\max(k, 0)$). Thus $L(\delta_\tau)=se^{-\tau s}$. One has $L(\phi')=sL(\phi)$. A uniqueness theorem is proved, and applications are made to differential equations. (Received January 10, 1955.)

GEOMETRY

429. C. C. Hsiung: *A theorem on surfaces with a closed boundary.*

Let S (S^*) be an orientable surface of class C'' in a three-dimensional Euclidean space with positive Gaussian curvature and a closed boundary C (C^*). Suppose that there is a one-to-one correspondence between the points of S and S^* such that the third fundamental forms and the sums of the principal radii of curvature of S and S^* at corresponding points are respectively equal. It is proved that if the two boundaries C and C^* are congruent, then S and S^* are congruent or symmetric. For closed S and S^*, this is a classical theorem of E. B. Christoffel. (Received February 21, 1955.)

430*t*. Shôshichi Kobayashi: *Holonomy groups of hypersurfaces.*

The author obtained the following theorem: the homogeneous holonomy group of an n-dinensional compact orientable Riemannian manifold M, imbedded isometrically in an $n+1$ dimensional Euclidean space E, is the entire proper orthogonal group $SO(n)$. The proof is based on the following facts: (1) the Gauss spherical mapping $\tilde{h}$ of M into an n-dimensional unit sphere S induces a bundle map h of the bundle of orthogonal frames on M into the bundle of orthogonal frames on S and the Riemannian connection on M can be induced by $\tilde{h}$ from the Riemannian connection on S; (2) if M is compact, there is at least one point x_0 of M such that h is a homeomorphism of a neighborhood U of x_0 onto $h(U)$; (3) the local homogeneous holonomy group of M at x_0 is equal to the local holonomy group of S at $h(x_0)$; (4) the local homogeneous holonomy group of S is $SO(n)$ at every point of S. The theorem is true, even if M is non-orientable: the restricted homogeneous holonomy group of M is $SO(n)$. As an immediate consequence, it can be proved that the restricted nonhomogeneous holonomy group of M is the group of all proper motions in an n-dimensional Euclidean space. Another consequence is that it is impossible to imbed isometrically an n-dimensional compact pseudo-Kaehlerian manifold into an $n+1$-dimensional Euclidean space. (Received December 13, 1954.)

431*t*. J. F. Nash: *Extension of the solution of the imbedding problem to open Riemannian manifolds.*

By means of a simple device one can show that an open Riemannian manifold can be imbedded isometrically in Euclidean space if this can be done for spheres. If an n-dimensional sphere generally requires $N(n)$ dimensions, then any n-manifold can be imbedded in $E^{(n+1)N(n)}$, and with the same differentiability as for the spheres, up to C^∞. An open n-disk can be mapped onto the complement of a single point on an n-sphere so that the mapping is C^∞ and has all derivatives zero at the periphery. Then if the sphere has a nonsingular positive metric this corresponds to a metric on the disk that goes to zero in a C^∞ manner at the periphery. One can cover a C^∞ manifold (and any differentiable manifold can be regarded as a C^∞ manifold) by a collection of C^∞ images of the open n-disk in such a way that there are $n+1$ classes of images and no two of the same class meet. Then with an appropriate procedure it is easy to assign a metric to each disk image that corresponds to a nonsingular positive metric on the sphere which, with a point deleted, corresponds to the disk. And this can be done so that the metric tensors on the disk images (and the spheres) have the same differentiability as the metric tensor on the manifold, with it being their sum. Now one imbeds each n-sphere isometrically in $E^{N(n)}$ and arranges for the point that is deleted to be the origin. Then these imbedding functions correspond to good functions on the manifold that are zero outside of the corresponding disk image. If one uses the same imbedding functions within each of the $n+1$ classes of disk images one obtains an isometric imbedding of the manifold in $E_1^{N(n)} \times E_2^{N(n)} \times \cdots \times E_{n+1}^{N(n)} = E^{(n+1)N(n)}$. For 2-manifolds it is possible one might arrange that the induced metric on the spheres have positive curvature everywhere and then use the work that has been done on the problem of Weyl to get an imbedding, perhaps in 9-space. (Received January 17, 1955.)

432. Valdemars Punga: *The concept "composite space" and its differential geometry.*

The author defines the composite space as a set of objects called elements, where each object consists of an n-dimensional point $(x^1, x^2, \cdots, x^n)$ associated with r tensors ${}^ip^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_p}$, $i=1, 2, \cdots, r$. The most investigated example of a composite space is the so-called Cartan space of line elements (x^α, p^α), where $(x^\alpha, cp^\alpha) = (x^\alpha, p^\alpha)$, $c>0$, and p^α is a contravariant vector. The Kawaguchi space is also an example of a composite space whose elements are $(x^\alpha, {}^1p^\alpha, {}^2p^\alpha, \cdots, {}^rp^\alpha)$, where ${}^ip^\alpha$ are r pseudo-vectors not completely arbitrary, but satisfying some differential equations. The König space is an example of a composite space, where the point and the associated tensors are of different dimensions. Also we can associate with an n-dimensional point any geometric object (not necessarily a tensor) and consider this combination as an element of a composite space. The author discusses the differential geometry of such spaces in general and in particular the composite space $(x^\alpha, p^\alpha, w_\alpha)$, where p^α and w_α are contra- and covariant vectors respectively. The concept of a composite space is in particular very convenient for the development of Finsler geometry and its generalizations. (Received January 12, 1955.)

STATISTICS AND PROBABILITY

433. Jerome Blackman: *On the approximation of a distribution function by an empiric distribution.*

Let $x_1, x_2, \cdots, x_n$ be independent random variables with the common distribution function $F(x)$ and the empiric distribution function $F_n(x)$. Let a_n be the value which minimizes $\int_{-\infty}^{\infty} \left| F(x-a) - F_n(x) \right|^2 dF^{(x-a)}$. The asymptotic distribution of $n^{1/2}a_n$ is obtained under the hypothesis that $F(x)$ has three continuous and bounded derivatives. (Received December 20, 1954.)

434t. K. L. Chung and C. Derman: *Non-recurrent random walks.*

A necessary and sufficient condition is proved for every infinite set of states in a (denumerable) Markov chain (with stationary transition probabilities) to be visited infinitely often with probability one. In the case of sums of independent, identically distributed, integer-valued random variables with span 1 every infinite set of positive (negative) integers is visited infinitely often with probability one if the mean is finite and positive (negative). A similar result is given when the common distribution is absolutely continuous. (Received January 17, 1955.)

435. J. M. Shapiro: *On sums of truncated random variables.*

Let (x_{nk}) $(k=1, 2, \cdots, k_n; n=1, 2, \cdots)$ be a double sequence of infinitesimal random variables such that for each n, $x_{n1}, \cdots, x_{nk_n}$ are independent. Let $S_n = x_{n1} + \cdots + x_{nk_n}$ and let $F_n(x)$ be the distribution function of S_n. Let $F(x)$ be any infinitely divisible distribution and let $G(x)$ be the bounded nondecreasing function determined by the Levy-Khintchine formula for the representation of the characteristic function of $F(x)$. For any $a>0$ ($\pm a$ a continuity point of $G(x)$) let x^a_{nk} be defined by $x^a_{nk}=x_{nk}$ if $|x_{nk}| \leq a$, $x^a_{nk}=0$ otherwise, and let $F^a_n(x)$ be the distribution function of $S^a_n = x^a_{n1} + \cdots + x^a_{nk_n}$. It is shown that if $F^a_n(x) \to F(x)$ it is necessary that (1): $G(x)=G(+\infty)$ for $x \geq a$ and (2): $G(x)=G(-\infty)$ for $x \leq -a$ (whether $F_n(x) \to F(x)$ or not); and if $F_n(x) \to F(x)$ then (1) and (2) are also sufficient for $F^a_n(x) \to F(x)$. In particular if $F(x)$ is the Poisson distribution and if $F_n(x) \to F(x)$, then for any $a>1$, $F^a_n(x) \to F(x)$. In this case it is also shown that the variances of $F^a_n(x)$ approach the variance of $F(x)$. (Received January 12, 1955.)

TOPOLOGY

436*t*. R. L. Blair: *A note on lattices of continuous functions.* Preliminary report.

If X is a topological space and K is a chain endowed with its order topology, then $C(X, K)$ denotes the set of all continous functions from X to K. Kaplansky (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 617–623) has shown that if X is compact Hausdorff then, as a lattice, $C(X, R)$ characterizes X (here R denotes the real numbers). On the other hand (cf. Gelfand and Kolmogoroff, C. R. (Doklady) Acad. Sci. URSS. vol. 22 (1939) pp. 11–15) if X is completely regular then, as a ring, $C(X, R)$ characterizes the Stone-Čech compactification $\beta(X)$ of X. These results are here subsumed in the following theorem. If X is completely regular then, as a lattice alone, $C(X, R)$ characterizes $\beta(X)$. A key step in the proof is a sharpened version of Kaplansky's result (loc. cit., Theorem 1) which may be formulated as follows: If $\alpha, \beta \in K$, write $\alpha \ll \beta$ in case $\alpha \leq \beta$ and in fact $\alpha < \beta$ unless α is maximal or β is minimal in K. A sublattice B of $C(X, K)$ *separates* X in case for any $f, g \in B$ and any disjoint closed sets F, G of X there is an $h \in B$ with $h(x) \ll f(x)$, $g(y) \ll h(y)$ for all $x \in F$, $y \in G$. Then if X_i is compact Hausdorff and if B_i is a sublattice of $C(X_i, K)$ which separates X_i $(i=1, 2)$, any lattice isomorphism between B_1 and B_2 induces a homeomorphism between X_1 and X_2. (Received January 14, 1955.)

437*t*. L. M. Blumenthal and V. L. Klee: *On metric independence and linear independence.*

Suppose M is a metric space with distance function ρ, and for each $x \in M$ let f_x be the function $\rho(x, y) \mid y \in M$. A subset A of M is said to be "metrically dependent in M" provided the set of functions $\{f_a : a \in A\}$ is linearly dependent over M; otherwise, A is "metrically independent in M." Results of I. J. Schoenberg, A. H. Stone, and C. H. Dowker are used to show that every metric space has a bounded homeomorph in which every subset A is metrically independent in A. From this follows: (I) If M is a metric space, then M is homeomorphic with a linearly independent subset of CM, the space of all bounded real continuous functions on M. (II) Every separable metric space is homeomorphic with a linearly independent subset of $C\,[0, 1]$. (With the words "linearly independent" omitted, (I) and (II) are well known.) It is proved also that a metric quadruple Q is metrically dependent in Q if and only if Q is pseudo-linear. (Received December 13, 1954.)

438. J. F. Nash: *A path space and the Stiefel-Whitney classes.*

A very simple proof of the topological invariance of the Stiefel-Whitney classes of a differentiable manifold may be obtained as a direct corollary of a general theorem on a bundle of paths. Consider the class of all parametrized paths in a manifold which have the property of never returning to their initial points. Using the initial point to define the projection mapping to the manifold, this becomes a fibre space of the manifold. By a direct, and quite elementary, construction of the two appropriate mappings this can be rather easily seen to be homotopically equivalent as a fibre space to the fibre space of tangents. Since the Stiefel-Whitney classes depend only on the fibre homotopy type of the tangent bundle, and since the path fibre space is obviously topologically invariant, their topological invariance follows. (Received January 13, 1955.)

439*t*. L. E. Ward, Jr.: *Fixed point theorems for special types of continua.*

In a previous paper (Bull. Amer. Math. Soc. Abstract 60-5-632) the author has defined a *tree* to be a continuum (=compact connected Hausdorff space) in which any two points are separated by some third point. R. H. Bing has defined a *tree-like continuum* (Duke Math. J. vol. 18, p. 653). The tree-like continua include the dendrites (metric trees) and certain indecomposable continua. A space has f.p.p. if it has the fixed point property for continuous functions. The following theorems are proved. I. A tree has f.p.p. II. A tree-like continuum has f.p.p. III. If X is a continuum which admits a continuous linear order, if I denotes the closed unit interval, and if T is a separable space such that $I\times T$ has f.p.p., then $X\times T$ has f.p.p. II settles a question raised by Bing. I and II are separate generalizations of the fixed point theorem for dendrites and are proved by order-theoretic methods. (Received December 1, 1954.)

440*t*. Chien Wenjen: *Quasi-equicontinuous sets of functions.*

A sequence of functions $\{f_n(x)\}$ from a topological space X to a metric space Y is said to be ϵ-related if for any $\epsilon>0$ there exists a neighborhood $U(x)$ of x such that for any point x' in $U(x)$ there is a number $N_\epsilon(x, x')$ satisfying: $\rho[f_n(x), f_n(x')]<\epsilon$ if $n>N_\epsilon(x, x')$. A family of continuous functions F from the space X to the space Y is said to be quasi-equicontinuous if in every infinite subset Q of F and at any point $x\in X$ there is a sequence contained in Q which is ϵ-related for any $\epsilon>0$. Theorem: If X is a locally separable space and Y is metric, a set of functions $F\subset Y^X$, where Y^X denotes the set of all continuous functions from X to Y, is compact under p-topology if and only if (1) F is closed in Y^X, (2) $F(x)=\bigcup_{f\in F} f(x)$ is compact for any $x\in X$, (3) F is quasi-equicontinuous. An immediate consequence of the theorem is: Let F be a family of continuous functions from a separable space X to a metric space Y. The necessary and sufficient conditions that it be possible to select a subsequence converging pointwise to a continuous function from any sequence of functions of F are: (1) $F(x)$ is countably compact for any $x\in X$, (2) F is quasi-equicontinuous. (Received December 30, 1954.)

R. D. SCHAFER,
Associate Secretary

BOOK REVIEWS

Eine neue Methode in der Analysis und deren Anwendungen. By Paul Turán. Budapest, Akadémiai Kiadó, 1953. 195+1 pp. and errata. 80.00 Ft.

This is a connected account, with extensions, of a method developed by the author in a series of papers beginning in 1941. The general theme is the application of Diophantine approximation to analysis and analytical theory of numbers. In the extensive literature that has grown up round this subject during the present century, arithmetic and analysis have become linked in a mutual relationship. Arithmetical theorems on the simultaneous approximation to real numbers by integers have been applied to the proof of inequalities for trigonometrical sums

$$f(t) = \sum_{j=1}^{h} a_j e(\lambda_j t) \qquad [\lambda_j \text{ real}; e(x) = e^{2\pi i x}];$$

and, conversely, arithmetical theorems have been derived from inequalities for suitably chosen sums of this type. Thus, the classical theorems of Dirichlet and Kronecker stand in this mutual relationship to theorems about the solubility in t of the inequality $|f(t)| > \theta f^*(0)$ for a given $\theta < 1$, where $f^*(t)$ is $f(t)$ with a_j replaced by $|a_j|$. For Dirichlet's theorem the appropriate $f(t)$ have a_j real and positive, but to compensate for this restriction the solutions t can be "localized"; thus, if $\theta = \cos(2\pi/\omega)$, where $\omega > 4$, there is a solution in any given interval $\tau \leqq t \leqq \tau\omega^h$ $(\tau > 0)$. For Kronecker's theorem no restriction is placed on the a_j, but the λ_j must be supposed linearly independent (over the field of rational numbers); and there is no localization, except in special cases where the degree of linear independence can be estimated quantitatively (as, for example, when the λ_j are logarithms of primes).

By way of introducing his own point of view the author makes two comments on this situation; firstly, that the analytical theorems have equal right with the arithmetical to be considered fundamental; and, secondly, that for many purposes close localization is more useful than a strong inequality. He is thus led to formulate, as basic tools for direct application, a series of inequalities for a generalized $f(t)$ with complex λ_j. The enunciations involve only integral values of t, and for such values we may write unambiguously, after rearrangement of the terms,

$$f(t) = f_k(t) = \sum_{j=1}^{k} b_j z_j^t; \qquad |z_1| \geqq |z_2| \geqq \cdots \geqq |z_k| > 0.$$

The main results of Part I may now be stated as follows: For any $m \geqq 0$ the closed interval $[m, m+k]$ contains an integer μ such that

$$(1) \qquad |f_k(\mu)| \geqq \left(\frac{Ak}{m+k}\right)^k |z_k|^\mu |f_k(0)|,$$

and an integer ν such that

$$(2) \qquad |f_k(\nu)| \geqq \left(\frac{Bk}{m+2k}\right)^k |z_1|^\nu \min_{j=1,\cdots,k} |f_j(0)|,$$

where A, B are positive absolute constants for which possible values are $A = 1/(2e)$, $B = 1/(24e^2)$. The proof of (1) is essentially algebraical, but (2) involves some function theory as well. An example is quoted to show that these inequalities, which are developments of Littlewood's "inequality for a sum of cosines," are essentially best possible, apart from the values of A and B.

In his preface the author says that the title of his book is very pretentious, and appeals for justification to the number and variety of the applications in Part II. These applications cover such topics as: almost periodic (trigonometric) polynomials, Dirichlet series with gaps, quasi-analytic functions, boundary values of analytic functions, integral functions, differential equations, approximate solution of algebraic equations, and a variety of questions in the theory of the Riemann zeta function and the distribution of primes, with extensions to other functions defined by Dirichlet series. On the face of it this list should satisfy the most exacting critic, and it certainly leaves no doubt about the *interest* of the method. But a ready assessment of its *power* is more difficult, because its impact on existing theory is apt to take the form of new problems or modifications of old ones rather than direct improvements of known results. It is not easy to locate a conspicuous example of success where serious attempts by traditional methods have notoriously failed, though this may well be due to the very failure of such attempts. Perhaps the best way of trying to make a comparison will be to examine one particular application where it is easy to isolate the characteristic difficulty and to describe the author's method of overcoming it.

In the study of the error term in the prime number theorem, statements of the form

$$(3) \qquad \zeta(s) \neq 0 \quad \text{for} \quad \sigma > 1 - \eta(t), \quad t > t_0\ (>0),$$

where $\zeta(s)=\zeta(\sigma+it)$ is the Riemann zeta function, are linked with statements of the form

$$\Delta(x) = O(xe^{-H(x)}) \qquad (x \geqq 1), \tag{4}$$

where $\Delta(x)$ is defined by

$$x + \Delta(x) = \psi(x) = \sum_{p^m \leqq x} \log p = \sum_{n \leqq x} \Delta(n)$$

(p running through primes, m and n through positive integers). Here $\eta(t)$ is a positive decreasing function, $H(x)$ a positive increasing function, of sufficiently smooth behaviour. A known $\eta(t)$ leads to a possible $H(x)$ by standard routine. The general pattern of the subject suggests that the correspondence thus set up is essentially "right," and that a converse inference from $H(x)$ to $\eta(t)$ should be possible. The author shows that this is so in the special case

$$\eta(t) = c(\log t)^{-\gamma}, \qquad H(x) = C(\log x)^{1/(1+\gamma)} \qquad (0 < \gamma \leqq 1)$$

(the most interesting in our present state of knowledge), if we disregard the particular values of the positive constants c and C. The procedure is as follows. By familiar methods for "explicit formulae" it is proved that, for $\sigma > 1$, $t \geqq 0$, $\xi \geqq 1$, $h > 0$ (h integral),

$$\begin{aligned} &\frac{1}{h!}\sum_{n \geqq \xi} \frac{\Delta(n)}{n^s} \log^h \frac{n}{\xi} - \frac{\xi^{1-s}}{(s-1)^{h+1}} \\ &\qquad\qquad = -\sum_{\rho} \frac{\xi^{\rho-s}}{(s-\rho)^{h+1}} + O\left(\frac{\log(t+2)}{\xi}\right), \end{aligned} \tag{5}$$

where ρ runs through the complex zeros of $\zeta(s)$, and the symbol O implies an absolute constant and an inequality valid for the full range of the relevant variables (s, ξ, h here). (This seems to be the correct form for the denominators, rather than that of Appendix V.) On the assumption that (4) is satisfied and that there is a zero $\rho^*=\sigma^*+it^*$ with t^* large and $1-\sigma^*$ small, estimates are obtained for the left-hand side of (5), for the contribution to the sum $\sum$ on the right-hand side of terms ρ outside a small neighborhood of ρ^*, and for the number of terms in the remaining finite sum $\sum'$, when ξ and h are taken fairly large and $s=\sigma+it^*$ fairly near ρ^*. The resulting estimate of $\sum'$ would yield the desired inequality $1-\sigma^* \geqq \eta(t^*)$ by proper choice of the parameters, if $\sum'$ consisted of the single term $\rho=\rho^*$. But such complete isolation is not to be expected, and there is a danger of interference from other terms. For a *given* h we have no safeguard; but by suitable use of (2) (with $b_j = 1$) it is proved that for *some h* of the appropriate order of magnitude $\sum'$ is not of significantly lower order (for the present purpose) than the single term $\rho=\rho^*$. By more traditional methods the problem may be reduced to the study of a finite sum $\sum\Re(s-\rho)^{-1}$ of positive terms presenting no interference difficulty; but without some modification this does not seem to give the precise result desired.

A similar chain of argument is used in connection with the "quasi-Riemannian hypothesis." This is the assertion that $\Theta<1$, where Θ is the upper bound of the real parts of the zeros of $\zeta(s)$. But the author prefers the equivalent statement that there is a $\vartheta<1$ such that $\zeta(s)$ has at most a finite number of zeros in $\sigma>\vartheta$, because he is concerned with necessary and sufficient conditions in terms of estimates of sums

$$\sum_{N_1\leqq\nu\leqq N_2} e^{-itf(\nu)},$$

and the parameters in these estimates are linked more naturally with ϑ than with Θ. The basic case is $f(x)=\log x$, but equivalence relations are found for other $f(x)$ also. These investigations were inspired by the hope that an independent estimate of these sums by Vinogradov's elementary method might shed new light on the existence of ϑ; but the author reports that the hope has so far proved illusory in that this method fails in the basic case $f(x)=\log x$ and gives insufficiently precise results in other cases.

A particularly interesting instance of the oblique impact of the author's method on existing theory is in the estimation of $N(\sigma, T)$, the number of zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ with $\beta\geqq\sigma$, $0<\gamma\leqq T$. There are various known estimates of the form

$$N(\sigma, T) = O(T^{\lambda(\sigma)(1-\sigma)}\log^B T) \qquad (1/2 \leqq \sigma \leqq 1) \tag{6}$$

(uniformly in σ as $T\to\infty$), where $\lambda(\sigma)$ is bounded in $1/2\leqq\sigma\leqq1$ and B is a positive constant. The Russian school has given the name "density method" to the use of such estimates in number-theory, and "density hypothesis" to the assertion that (6) is true with $\lambda(\sigma)=2$ (and some B). In the absence of a proof, and in view of possible applications, special interest attaches to near approaches to the density hypothesis. On the Lindelöf hypothesis

$$\zeta(\alpha+it) = O(t^{\epsilon}) \qquad (\alpha = 1/2;\ t\to\infty), \tag{7}$$

it is known that (6) is true with $\lambda(\sigma)=2+\epsilon$ (where ϵ denotes generally an arbitrarily small positive constant). Without hypothesis (and with special reference to the neighborhood of $\sigma=1$) the author proves (6) with

$$\lambda(\sigma) = 2 + A(1 - \sigma)^a \qquad (1 - b \leqq \sigma \leqq 1),$$

where A, a, b are positive constants (e.g., $A = 600$, $a = 1/100$). He also states that, if (7) is replaced by a similar hypothesis with a fixed α in $1/2 < \alpha < 1$, then (6) holds with

$$\lambda(\sigma) = 2 + 3\epsilon^{1/3} \qquad (\alpha + \epsilon^{1/3} \leqq \sigma \leqq 1).$$

But the proof is unfortunately omitted. (To reconcile the contradictory announcements in lines 1 and 22 on p. 162, read XXXVIII for XXXIX in line 22.)

In work of this kind it is perhaps inevitable that the central idea should sometimes be submerged by details. To reduce this danger the author has relegated some of the incidentals to six Appendices. Something more might be done in the same cause by a systematic use of integrals instead of sums. Thus, it would be possible to reduce to a few lines the elaborate argument used to estimate the difference on the left-hand side of (5), by writing this difference as

$$-\frac{1}{k!}\int_{\xi}^{\infty} \Delta(x) d_s\left(\frac{1}{x^s} \log^k \frac{x}{\xi}\right),$$

and using the estimate $\Delta(x) = O(xe^{-H(x)})$ $(x \geqq \xi)$.

In commending this stimulating book to the notice of analysts, and particularly to those interested in the analytical theory of numbers, the reviewer may perhaps be allowed two final comments on small points of wording. On pp. 131–132 it is stated that a proof of the "quasi-Riemannian hypothesis" $(\Theta < 1)$ would entail essentially all the consequences of a proof of the Riemann hypothesis $(\Theta = 1/2)$. This applies, of course, only to a restricted class of "consequences"; thus, the order of the error in the prime number theorem would be affected by one pair of zeros $\beta \pm i\gamma$ with $\beta > 1/2$. On p. 141 reference is made to the method created by Vinogradov for the "solution of Goldbach's problem." This is a momentary lapse from more accurate descriptions elsewhere, but, since this form of words has become current in the literature, it seems opportune to point out that Goldbach's problem has not been solved. It is no disparagement of Vinogradov's outstanding achievement with the sum of three primes to state that the basic problem of two primes remains a major challenge.

A. E. INGHAM

The foundation of statistics. By L. J. Savage. New York, Wiley, 1954. 16+2+294 pp. $6.00.

This book is an exposition of some of the leading ideas and techniques in modern statistics. They are supposed to hang together under the awning of *personal probability*,[1] which theory is expounded in the first four, and in a ramified way in the first eight chapters. This *personalistic* view of probability, as distinguished from the *objectivistic* (repeated events) and the *necessary* (logical implication) view, is an old one and holds that probability measures the degree of confidence a particular person has in the truth of a particular proposition. The present account, couched in *behavioristic* rather than *verbalistic* terms, is derived from the work of Bruno de Finetti. Its main outline may conveniently be gleaned from the end papers, at either end of the book. Briefly stated it is a theory of rational *decision* in a *grand world* (or a *small world*) based on a personal *preference* among *acts* in view of their *consequences* resulting from the *true state* of the world. Mathematically, an act is a function from state to consequence and it is simply ordered by the relation preference. This ordering of acts carries with it a natural ordering of sets of states or *events*. In the beginning there are inequalities; but if the ordering is *fine* and *tight* then these inequalities squeeze so hard on each other that there emerges a finitely additive *probability measure* on the events which *agrees* with the *qualitative probability*, thus making it *quantitative*, and we have something recognizable as an ordinary probability. (For the mathematician a rough picture would be a sort of Archimedean arithmetic without all the real numbers.) It is said that some subjectivists do not care for this strict quantification, as it tends to make life too easy and takes the show away from non-numerical probability. Such persons are in for another disillusionment, for just as probability measure arithmetizes the comparison of events, *utility*, a concept due to von Neumann and Morgenstern and discussed with some detail in Chapter 5, arithmetizes the comparison of acts. Now the acts are simply ordered according to the mathematical expectations of their utilities, and why not indeed, as they say it in French, confound the act with its utility? It thus turns out that an act is a random variable, and the recommended decision is to supremize its mathematical expectation. The rest is easier sailing, at least on the verbal level. Chapter 6 discusses how an *observation* affects the decision by conditioning the personal (I almost said *a priori*) probabilities. There is a good illustrative example there. Chapter 7, entitled "partition problems," treats the classical case where the decision is solely determined by an unobservable random variable or rather its distribution, which in the finitely-valued case reduces to a *partition*. This chapter contains a discussion of sufficient statistic, likelihood ratios, and sequential analysis. Chapters 9 to 14 are all concerned with Wald's decision theory, here called the *minimax theory*. It is first explained without reference to the personalistic theory, and then reinterpreted with it. The parallelism between the minimax theory and the theory of zero-sum two-person games is noted; von Neumann's fundamental theorem is stated without proof, and some recent work is mentioned. Various objections to the minimax rule are discussed in Chapter 13. Chapter 15 treats point estimation. As many as nine criteria for an estimate are discussed. A common defense for these criteria which seems to have escaped attention is that they lead to some sort of mathematics. It might be somewhat frivolous to say with Landau that they have also led to dissertations,[3] but surely it is not too much to suppose that even a hard-headed statistician may play around a little with a formula or a theorem without regard for the grand world? Chapter 16 is on testing and Chapter 17 on interval estimation for which the author seems to have little use. The curtain falls quickly on the mention of fiducial probability. There are two appendixes on expected value and convex functions. A third appendix is on bibliographic material and lists about 170 items with some comments.

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[1] Italicized terms refer to those used in the book.

Following the author's good example I shall allow myself a few personal observations. A book like this is necessarily part philosophy, and one who is not philosophically bent, as Mr. Savage clearly is, is often hard put to tell between what is critical thinking and what is quibbling about words. To such a person a good part of the discussion of the foundations of probability is typified by the following two examples. 1. Re probability: when a coin is tossed there is besides head and tail the possibility of the coin's standing on its edge or disappearing into a crevice. (For a variation on this theme see p. 15 on whether a rotten egg spoils an omelet.) 2. Re utility: some people gamble for a monetary loss in order to kill time or to cultivate good relations. (For a variation see p. 101 on the show-off flier.) I do not know how to draw a line between such bull-session stunts and more serious argumentation, but I should add that on the whole the author shows restraint in exercising his verbal felicity. As to clever discourse in general on the foundations of probability, I am in agreement with someone (whom I forget) that Poincaré has said most of the brilliant things, and what is left has been said by Borel. Indeed, it is a little surprising that Borel's book *Valeur pratique et philosophie des probabilités* (the last volume of his well-known collection), which also expounds a theory of subjective probability based on betting behavior (cf. p. 60 of the book under review), is not referred to at all.

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[3] This view was expressed to me by P. L. Hsu nine years ago.

The author writes extremely well and obviously enjoys writing. To the very conservative his florid personal style may be at times disconcerting, but seeing that many mathematicians are such careless writers I think his literariness is to be commended. The adult reader may ignore his advice on doing all the exercises etc., he may even gloss over the nuances of personalisticism, but if he can overcome the initial barrage of a new terminology[3] he will enjoy most of what the author has to say and the way he says it.

K. L. CHUNG

Tables of integral transforms. Vol. II. Prepared under the direction of A. Erdélyi, New York, McGraw-Hill, 1954. 16+451 pp. $8.00.

This volume is divided into two parts of somewhat different character. The first, Chapters VIII through XV, follows the same organization plan as Volume I. [For a review thereof see this Bulletin vol. 60 (1954) pp. 491–493.] That is, the integrals are classified as transforms under the following types:

Hankel	$\int_0^\infty f(x)J_\nu(xy)(xy)^{1/2}dx$
Y-transform	$\int_0^\infty f(x)Y_\nu(xy)(xy)^{1/2}dx$
K-transform	$\int_0^\infty f(x)K_\nu(xy)(xy)^{1/2}dx$
H-transform	$\int_0^\infty f(x)H_\nu(xy)(xy)^{1/2}dx$
Kontorovich-Lebedev	$\int_0^\infty f(x)K_{ix}(y)dx$
Fractional integrals	$\int_0^y f(x)(y-x)^{r-1}dx$
Stieltjes	$\int_0^\infty \frac{f(x)}{(x+y)^r}dx$
Hilbert	$\int_{-\infty}^\infty \frac{f(x)}{x-y}dx.$

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[3] It is hoped that the brief recapitulation given in the first paragraph of this review may be of some help there.

Here J_ν and Y_ν are Bessel functions of first and second kinds, respectively; $K_\nu(x)$ is the modified Bessel function of the third kind, and $H_\nu(x)$ is Struve's function. The second part of the volume contains integrals of higher transcendental functions, most of which could not properly have come into the foregoing tables of transforms. They are classified under the following titles: orthogonal polynomials, gamma and related functions, Legendre functions, Bessel functions, and hypergeometric functions. As in Volume I there is an appendix for notations and definitions and an index thereto.

It is the fate of all tables to be incomplete, and in spite of the ambitious scope of the present set most users will probably spot omissions. For example, the reviewer would have welcomed a chapter on the Weierstrass (or Gauss) transform. The omission is not serious since this transform can easily be related to the Mellin or Fourier transform. The appearance of this second volume confirms the reviewer's earlier opinion that these tables will ultimately be among the indispensable tools of many analysts and applied mathematicians.

D. V. WIDDER

Geometrie der Zahlen. By O. H. Keller. Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. I_2, No. 11, Part III. 2d ed. Leipzig, Teubner, 1954. 84 pp. 8.80 DM.

The title *Geometrie der Zahlen* was introduced by Minkowski over half a century ago. Its subject matter has naturally expanded greatly, particularly in the last twenty years, and the author commendably presents a more modern account of the subject. The aim of the *Enzyklopädie* is "to find a middle road between the . . . historical presentation and the . . . systematic presentation." The reviewer's impression is that the work tends to be more systematic than historical, but the few isolated remarks that follow can in no way detract from the debt owed to the author for his pioneering compilation.

One major pre-Minkowskian development is the study of the minima of quadratic forms in n variables, elegantly interpreted as the densest lattice packing of equal spheres in n dimensions. The discovery, essentially, that the symmetric tetrahedral packing (with $n+1$ mutually tangent spheres) is no longer densest when $n=4$ is one of the earliest indications that n-dimensional "Euclidean geometry" would depend on n with "number-theoretic" irregularity. Strangely enough, the footnote reference to this result (footnote 184a) does not refer to its discovery (by Korkine and Zolatareff in 1872) but rather to a series of papers starting in 1948. The footnotes catch up in the long run, if the reader uses his powers of deduction. But more difficult is the deduction that the aforementioned result led to the development of extremal methods and polyhedron packings.

The Minkowskian synthesis (i.e., the packing of space with general convex bodies), certainly receives due attention, although a sense of continuity with earlier work, particularly with that of Hermite, is not made apparent. Likewise the contact between the geometry of numbers and algebraic number theory seems lost in the mass of details again. The reviewer refers to the contact occurring when the algebraist had to work with inequalities not only to find bounds on norms but even to prove the discriminant theorem, which implies the following: "Every irreducible monic polynomial of degree at least two and with integral coefficients contains a repeated (polynomial) factor modulo some prime." Conversely in the last twenty years (owing largely to the influence of Mordell, Davenport, and Mahler), the intense desire for unimprovable bounds in algebraic number theory caused the replacement of the convex body by the more general star body. One contrast in presentation that the reader will immediately note is that while the older methods (Minkowskian and earlier) are often presented with enough detail (diagrams, tables, etc.) to excite the non-specialist, the methods developed in the last twenty years are given in little detail, with emphasis instead on lists of definitions, theorems, and conjectures, and with a valuable bibliography drawn up for 1951. Incidentally, a more modern bibliographical format (alphabetically arranged, and containing the titles of the papers) probably would be more useful.

The chapter titles are as follows: A. Convex bodies. B. Star bodies. C. Linear forms. D. Minima of homogeneous forms. E. Inhomogeneous forms. F. Quadratic definite forms. G. Continued fractions. H. Algebraic numbers. I. Partitions and lattice point figures. The reviewer feels that the inclusion of the last chapter is hardly justified by the historical continuity presented, but that otherwise the titles suggest that the geometry of numbers still would not be beyond Minkowski's recognition.

HARVEY COHN

Economic activity analysis. Ed. by O. Morgenstern. New York, Wiley; London, Chapman and Hall, 1954. 18+554 pp. $6.75.

This is a collection of essays by members of the Princeton University Economics Research Project who were studying the mathematical structure of American type economies with support from the

Office of Naval Research. It is in three parts, prefaced by a valuable summary by the editor. I: Economic properties of the input-output system; II: Mathematical properties of linear economic systems; and III: Metaeconomics.

I. Among the models popular at present is the (static) Leontief system and a dynamic generalization of it that has been studied by the USAF. The basic data for such a system is an input-output matrix (a_{ij}) where the indices correspond to particular industries and a_{ij} represents the effect of an industry i on an industry j. More precisely a_{ij} is the fraction of the output of industry i to industry j. Suppose that the total output of industry i is x_i and suppose that the outside demand for industry i (e.g., for foreign trade) is y_i. Then for equilibrium we have

$$x_i - \sum_j x_j a_{ij} = y_i, \qquad i = 1, 2, \cdots, n,$$

or

$$(I - A)x = y.$$

Thus, given y and knowing $A = (a_{ij})$, we can find $x = (I - A)^{-1}y$.

The coefficients a_{ij} can be obtained from official statistics. The detail in which an economy can be studied depends on the computational power available; early models had $n = 10$, recent models had n about 200, and data are available for n about 500. One of the first problems to be considered is the effect of aggregation, i.e. which industries can be lumped together without serious loss of significance. Effects of various aggregations in reduction of a case of $n = 18$ to a case $n = 8$ are studied in computational detail. Another problem is the effect of uncertainties in the a_{ij}.

II. The a_{ij} just introduced clearly satisfy $a_{ij} \geqq 0$, $\sum_j a_{ij} \leqq 1$. Matrices (a_{ij}) with this property have been called stochastic matrices. They are here called Minkowski-Leontief matrices. The matrix $I - (a_{ij})$ is then called a matrix of Leontief type. It is easily seen that in such a matrix the diagonal element in each row is never smaller than the negative sum of the other elements. Thus matrices of Leontief type are a special case of so-called matrices with dominant main diagonal. These matrices are of great interest in many connections. The Minkowski-Leontief matrices belong to the class of positive or non-negative matrices on which a number of important classical results exist. These theories are here reported and extended in various ways by M. A. Woodbury and Y. K. Wong.

III. The first contribution to this section is a translation of two papers by K. Menger[1] concerned with the validity of the law of diminishing returns. This is: *Let the product of applying* y(e.g., *dollars*) *to* x(e.g., *acres*) *be* $E(x, y)$. *Then for* $h>0$

$$E(x, y_2 + h) - E(x, y_2) < E(x, y_1 + h) - E(x, y_1)$$

provided $y_2>y_1>y=y(x)$. Various properties of production functions E such as monotony, super-additivity, super-homogeneity, dependence are introduced. Examples are presented to show how these are related among themselves and with the above law and similar propositions. These relations are summarized graphically. This essay can be recommended by mathematicians to their friends in social science as an example of how one discipline can be used in another.

The concluding essay by Morgenstern is concerned with the role of experiment and computing in economics. Apart from discussions of a more philosophical nature there are indications of some computational experiments of immediate interest to economists.

OLGA TAUSSKY

Mathematics and plausible reasoning. By G. Pólya. Volume I, *Induction and analogy in mathematics*, 16+280 pp., $5.50; volume II, *Patterns of plausible inference*, 10+190 pp., $4.50. Princeton University Press, 1954. 2 vols., $9.00.

There are many so-called "popular" books on mathematics. Some of them turn out to be of interest to professional mathematicians only (or, perhaps, to professional mathematicians *in ovo* as well). Others are so non-technical as to be well within the reach of any educated layman, and, consequently, their subject hardly deserves to be called mathematics. Most of the time Pólya manages to steer an admirable course between these two extremes. The two volumes under review are, however, not uniform in this respect; the first is more the mathematician's volume and the second the philosopher's. Since this review is addressed to mathematicians, it will discuss the first volume in more detail, and, it may well be charged, with more sympathy, than the second.

The book as a whole is organized around the central thesis that a good guess is quite as important as a good proof. As in his little book *How to solve it*, Pólya advocates that the mathematician should think and talk (at his desk and in the class room) about the theory of guesses as well as the theory of proofs. "Certainly, let us learn proving," he says, "but also *let us learn guessing.*" The function of the first volume in the theory of plausible reasoning is to provide some concrete mathematical raw material; the second volume is more interested in an abstract philosophical discussion of the patterns that the first volume indicates.

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[1] *Bemerkungen zu den Ertragsgesetzen* and *Weitere Bemerkungen zu den Ertragsgesetzen* in Zeitschrift f. Nationalökonomie vol. 7 (1936) pp. 25–26 and 388–397.

More than half of the first volume is occupied by problems (at the end of each chapter) and their solutions (at the back of the book). The organization of these problems is similar to that of the well known problem collection by Pólya and Szegö. Sometimes the very statement and sometimes the solution of one problem suggests the next problem; by working through a graded series of problems of this sort the reader gets a very good insight into their subject. The problems are almost always non-trivial. The first problem in the first chapter is to guess the rule according to which the successive terms of the following sequence are chosen: 11, 31, 41, 61, 71, 101, 131, $\cdots$. (For the convenience of the reader of this review the solution is given below.) The last problem in the last chapter concerns some computations involving the torsional rigidity of a beam with square cross-section. Many of the problems invite the reader to guess an answer, to examine a situation, or to conjecture a theorem.

The problems, by the way, are not called problems; they are called examples and comments. Roughly speaking, the examples are mathematical exercises and the comments are philosophical discourse about them. The quality of the comments is quite variable. Some are hardly more than weak jokes (there is one about the differences among logicians, mathematicians, physicists, and engineers), while others are penetrating remarks on mathematical methodology (e.g., a discussion of the effect of strengthening the hypothesis of a theorem to be proved by induction).

The greater portion of the body of volume I (i.e., of the material distinct from the problems and their solutions) is mathematical exposition of unusually high caliber; only about 40 pages (sprinkled through the volume) are devoted to outright philosophizing. An idea of the contents can be gained from the following list of the main mathematical ideas that occur in the various chapters. (The titles of the chapters would be of little use here; Chapter VI, for instance, is entitled *A more general statement.*) I. Goldbach's conjecture. II. The Pythagorean theorem; Euler's method of evaluating $\sum_{n=1}^{\infty} n^{-2}$. III. The Euler formula, $V-E+F=2$. IV. Sums of squares; in particular, the four-square theorem. V. If $a_n > 0$ ($n=1, 2, 3, \cdots$), then

$$\limsup_n ((a_1 + a_{n+1})/a_n)^n \geq e.$$

VI. Euler's recursion formula for the sum of the divisors of n. VII. Mathematical induction. Some non-routine applications, such as the following one: "If the polygon P is convex and contained in the polygon Q, the perimeter of P is shorter than the perimeter of Q." VIII. Maxima and minima, with, again, several ingenious, non-routine applications. IX. Minimum principles from optics and mechanics. X. The isoperimetric problem. XI. Miscellaneous problems. Sample: if the intersection of a solid sphere with a solid cylinder whose axis passes through the center is removed from the sphere, find the volume of the remainder in terms of the radius of the sphere and the height of the cylindrical hole.

The second volume is quite different from the first. The problems occupy less space (less than a third of the volume) and play a less important role. Their character is also different. Sample: "Check Heron's formula [for the area of a triangle in terms of the sides] as many ways as you can." Chapter XII discusses the "fundamental inductive pattern" and some of its variations. (If A implies B, then the discovery that B is true makes A more credible.) There is an interesting discussion of the author's teaching methods, centered around the formula for the area of the lateral surface of the frustum of a right circular cone. Chapter XIII continues in the same vein. By way of an example there is a long discussion of judicial proof with many details of a murder case. The best part of the volume for the mathematician is about a dozen pages (in the examples and comments on Chapter XIII) that discuss some of Pólya's own work with historical and psychological side lights. Chapters XIV and XV concern probability, mostly from a non-mathematical point of view. The main emphasis of the last chapter, Chapter XVI, is on the applications of the preceding discussion to pedagogic methods.

Neither volume has an index; there is, instead, a very detailed analytical table of contents. The physical appearance of the volumes is excellent. The style throughout is informal and charming.

PAUL R. HALMOS

P.S. The sequence 11, 31, 41, 61, 71, 101, 131, $\cdots$ consists of the primes ending in 1.

Elements of algebra. By Howard Levi. New York, Chelsea, 1954. 160 pp. $3.25.

This book is addressed to beginning students of mathematics; it is, roughly speaking, a text on freshman algebra. It is a text on freshman algebra in the sense that it talks about the removal of parentheses and the solution of quadratic equations (among other things). The level of the book, however, is so unusually high, mathematically as well as pedagogically, that it merits the attention of professional mathematicians (as well as of professional pedagogues) interested in the wider dissemination of their subject among cultured people. The book has its faults. The terminology, the exposition, the statements of the theorems, and the choice of subjects included and excluded are not always perfect. Despite its faults, however, the book is a closer approximation to the right way to teach mathematics to beginners than anything else now in existence.

The quickest way to summarize the contents of *Elements of algebra* is to say that it is *Grundlagen der Analysis* for freshmen. (The reference is, of course, to Landau's well-known derivation of the real number system from the Peano axioms.) Levi (unlike Landau) starts by describing some of the terminology of elementary set theory; he goes on to introduce the symbols $\cup$ and $\cap$ and to give a precise definition of the concept of function. The second chapter introduces what are unfortunately called *cardinal numbers*. (Perhaps *natural numbers* would have been more in accord with custom and less likely to confuse the student who subsequently finds out that infinite cardinals exist also.) The definition is along the lines of von Neumann's theory of ordinals. The number 0 is the empty set, the number 1 is the singleton $\{0\}$, the number 2 is the pair $\{0, 1\}$, etc. Based on this definition the elementary arithmetic operations are defined and their properties are derived.

Chapters III, IV, and V are a digression from the Landau program. In Chapter III *expressions* are defined (they are "well-formed formulas" involving cardinal numbers, variables, addition, multiplication, and parentheses), and a concept of *equality* is introduced for expressions (two expressions are equal if they represent the same function of numbers). Chapter IV defines a *monomial* as an expression not involving addition and a *polynomial* as a sum of monomials. The theorem that every expression is equal to some polynomial is offered as the meaningful substitute for the mysterious process of "simplifying" expressions by "removing parentheses." Chapter V concerns the unusual concept of a *number system*; this is defined to be a set with two commutative and associative binary operations satisfying one distributive law.

In Chapters VI, VII, and VIII the integers and the rational numbers are constructed by the usual method of ordered pairs. Chapter IX discusses equations; it includes the factor theorem and the solution of quadratic equations. Chapters X and XI construct the real number system (by means of infinite decimals) and prove that it has the upper-bound-property. An appendix makes brief mention of a few related matters (such as infinite cardinals and the definition of a group).

The book contains many striking phrases designed to eliminate some common sources of confusion. Here are two examples. (1) "It should be stated emphatically that there is nothing undesirable about parentheses. Our efforts at removing them are directed toward devising a test to determine if two given expressions are equal." (2) "The reader is urged to distinguish between his everyday use of the words [*greater* and *less*] and the mathematical one; otherwise he will not be impressed by their similarities nor take seriously their differences."

There are also examples of bad didactic technique. They are all of the same type: some difficult concepts are not sufficiently motivated. Thus the empty set is introduced without so much as a by-your-leave and the possibility of a set being a member of itself is casually (and needlessly) referred to. The construction of the integers via ordered pairs is introduced by the following sentence: "The reader who finds our definitions arbitrary and somewhat bizarre should be informed that they are the product of a long evolution of which our exposition gives only the final stage." (The patronizing tone of this sentence recurs frequently.) The fact that the author is thinking of (7, 11) as $7-11$ is kept a secret from the reader.

Mathematically, the author's treatment of statements involving variables ("if x is even, then $x+1$ is odd") is probably unobjectionable but certainly peculiar. The assertion that "we shall only accept those statements that are definitions and those statements that can be proved to be logical consequences of the definitions" is somewhat startling, to say the least.

Enough has been said to communicate the flavor of the work. The book can be useful to a beginner as an outline of territory whose detail maps are not available to him. As such, the reviewer recommends it, but he recommends also that an experienced guide be taken along on the tour.

PAUL R. HALMOS

Theory of games and statistical decisions. By D. Blackwell and M. A. Girshick. New York, Wiley, 1954. 12+355 pp. $7.50.

Statistical decision theory originated in Wald's 1939 paper (Ann. Math. Statist. vol. 10, pp. 299–326), whose interest is now almost purely historical. It was designed to embrace all problems of statistical inference which are the raison d'être of statistics; with inessential modifications it still does. In its simpler form the statistician has to construct a (decision) function $\delta(x)$ from Euclidean n-space $X=\{x\}$ to a decision space $D=\{d\}$. Each point x represents a possible value of a chance variable (a sample) whose distribution function F is unknown to the statistician but is assumed to be one of a class Ω (which contains at least two and often infinitely many members). The "goodness" of δ for F is measured by a functional $r(F, \delta)$ of δ and F; in practice $r(F, \delta)$ is usually $\int W(F, \delta(x))dF(x)$, where W is a functional of F and $d=\delta(x)\in D$. A partial ordering among the δ is defined by: δ_1 "as good as" δ_2 if $r(F, \delta_1)\leqq r(F, \delta_2)$ for all $F\in\Omega$, and δ_1 "better than" δ_2 if also the inequality holds for at least one F in Ω. A class of decision functions is called complete if, for any decision function outside the class, there exists a better one in the class. A class of decision functions is called essentially complete if, for any decision function outside the class, there exists one in the class which is as good. In actual practice it is usually necessary to choose an "optimal" element or elements from an essentially complete class; for example, according to the "minimax principle" δ^* is optimal if

$$\sup_F r(F, \delta^*) = \inf_\delta \sup_F r(F, \delta)$$

and there exists no δ better than δ^*. In application to physical phenomena the physical scientist will want to use an optimal decision function, say δ^*, and will make decision $\delta^*(x)$ when x is the complex of his observations (his sample). In the general Wald theory the mathematical problems are enormously complicated by allowing the observations to be obtained sequentially, by having the theory indicate the choice of chance variables to be observed (design of experiments), and by permitting randomized decision functions.

The value of Wald's ideas for statistical theory can hardly be overestimated. They introduced a unifying clarity into the entire mathematical theory. A clear framework for the formulation of old and new problems was provided and the stimulus was given for a whole host of new ones. Decision theory, if understood by the practical statistician, would serve him in good stead in obtaining working solutions even for problems for which the optimal theory has not yet been developed.

We shall discuss below the theorems of decision theory on the relation between complete classes and Bayes solutions, and on the behavior of Bayes solutions; δ_ξ is a Bayes solution for the (a priori) distribution ξ on the space of F's if $\int r(F, \delta_\xi)d\xi=\min_\delta \int r(F, \delta)d\xi$. The other principal results, which should also be of interest to mathematicians in general, are the theorems on the elimination of randomization (Ann. Math. Statist. (1951) pp. 1–21 and 112–114) which gave rise to, and arose from, generalizations of Liapounoff's theorem in measure theory (Pacific Journal of Mathematics (1951) pp. 59–74, and Proc. Amer. Math. Soc. (1951) pp. 390–395), and the solution of an outstanding problem in sequential analysis, the proof of the optimum character of the Wald sequential probability ratio test (A: Ann. Math. Statist. (1948) pp. 326–339).

Wald's 1950 book (*Statistical decision functions*) contains five chapters. Chapter I gives the general formulation of the problems of statistics in terms of decision theory. Chapter II is on the theory of games. Chapter III is the heart of the book and contains theorems on complete classes, Bayes solutions, and minimax solutions. Most of the difficulties, many of them of measure-theoretic character, are caused by the general formulation of the problem. The proofs testify to Wald's great mathematical powers and ingenuity, but they are not of a polished and final sort. In fact (see discussion below) it is almost certain that in final form the approach will be different. Chapter IV is essentially the paper in Proc. Nat. Acad. Sci. U.S.A. (1949) pp. 99–102, hereafter cited as B, and the paper in Ann. Math. Statist. (1950) pp. 82–99. The final Chapter V contains complete classes, and more rarely, also minimax solutions, for problems designed to illustrate the theory. While most of these problems are interesting and ingenious, relatively few are important or vital. Thus a serious question could be raised whether the state of the theory was such as to justify the publication of Wald's book, not only because of the state of Chapter III, but also because of the paucity of vital practical applications of the theory.

The present book by Blackwell and Girshick eliminates the heart of the difficulties of Wald's Chapter III by a drastic reduction in the mathematical generality, in part by a restriction to discrete distributions. It contains elaborate discussions of the formulation of the decision problem, and several chapters on the theory of games. Chapter IV of Wald's book is essentially reproduced, and there are ad hoc examples of the type of Wald's Chapter V. In addition there is some standard statistical material on sequential analysis, sufficiency, and point estimation, which is not usually classified under decision theory. The book concludes with results of one of the authors on comparison of experiments in the case of discrete distributions and finitely many alternatives.

Whatever doubts may be raised about the justification for Wald's book apply a fortiori to the present book, published four years after Wald's. Very few essential results not included in Wald's book are to be found in the present book. The practical statistician will not find in either book answers to the problems which confront him. He would be introduced to an approach of immense value to him, but it is obvious that Wald's book is mathematically inaccessible to most practical statisticians, and this is so, in the opinion of the reviewer, with the present book. It might be expected then that the present book would appeal to mathematicians, but the reviewer has misgivings on this score. The book is intended primarily as a textbook for first year graduate students in statistics. It is very formal in presentation; there are numerous definitions, very formally stated. The reviewer counted twenty-four rolling definitions in Chapter 3 and only eight theorems, many of the latter of very modest content. A boring effect may be created among mathematicians by the practice of elevating minor and obvious arguments to the level of theorems and lemmas, with all that this implies in notation and in obscuring the main results; for example, Lemma 8.4.1 is a typical example of a most common mathematical argument which would ordinarily rate a casual sentence. Thus, the presentation is too imposingly formal for practical statisticians, and may be too boringly formal and the mathematical content may be too dilute to be interesting to mathematicians.

One of the achievements of the Wald theory is the proof of the optimum character of the Wald sequential probability ratio test. In the original proof given in A, the first two lemmas yield a convexity result which is more expeditiously given in B. The proof in the present book follows the lines of that of A with the improvements from B just noted. In their proof (page 292) the present authors appear to effect an economy in the argument by omitting the arguments of §§5 and 6 of A. The purpose of these sections was to enable one to conclude that, for any given a priori distribution, there exist non-negative weights w_{12} and w_{21} (Blackwell and Girshick's notation) which will make any given sequential probability ratio test a Bayes solution. The present authors argue that this conclusion follows from the fact that equations (10.3.7) and (10.3.8) are linear in w_{12} and w_{21} and so can be solved for w_{12} and w_{21}. However, the crucial question is whether such w_{12} and w_{21} exist. Equations (10.3.7) and (10.3.8) are not homogeneous in w_{12} and w_{21}, and the authors have not proved that these equations have a solution (are always consistent). Not only this, but the authors have failed to prove that there is a non-negative solution, which is essential for the argument. Thus the authors have achieved an "economy" of argument at the expense of an essential gap in the proof; this gap is present also in their paper in Econometrica (1949) pp. 213–244. (A minor detail not related to the above:

The proof is also not valid unless one postulates, as is done in A, that $E(n|S; H_1)$ and $E(n|S; H_2)$ are both finite.)

The authors' view is that decision theory is the study of what they call statistical games, and a branch of the general theory of games. The essence of a zero-sum game (which is obviously the kind of game meant) is the conflict caused by the opposing interests of the players. What makes (von Neumann's) main theorem for the zero-sum two-person game so pretty is its ingenious resolution of this conflict. What is a reasonable resolution of the conflict in the case of the n-person game is still under debate. It is misleading to think of statistical theory as a zero-sum two-person game. The conflict which is the essence of the game is not present (i.e., there is no basis or need for regarding Nature as a malevolent opponent of the statistician), and it is not especially useful to introduce the terminology of game theory in obtaining results in decision theory. Mathematically speaking, this means that decision theory is not a part of game theory, and that a book on decision theory should not make decision theory a branch of game theory. But, it will be remonstrated, is not Wald's book based on this game-theoretic approach? The answer is that, at the time Wald wrote his book, it was not known that Bayes solutions correspond simply to supporting planes for the set of risk points in the space of the F's. Not knowing this Wald used the equation

$$(*) \qquad \inf_{\delta} \sup_{\xi} r(\xi, \delta) = \sup_{\xi} \inf_{\delta} r(\xi, \delta)$$

as his principal tool for proving complete class theorems, and consequently found it indispensable to set up a structure in which he could prove this equation. In one of the very few statistical papers written after the completion of his book and before his untimely death in 1950 (Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 149–157) the true situation is understood, the formulation and the arguments involve no game theory whatever, and the complete class theorem is derived by a supporting plane argument. It is desirable, in an exposition of decision theory, to show the connection with game theory, namely that, when (*) holds, it is sometimes easier or suggestive to find a minimax δ by first finding a ξ least favorable to the statistician. It may also be considered desirable to enlarge the student's education by inclusion of a section on game theory in a textbook on decision theory. But, in the light of our present knowledge, it is misleading to consider decision theory a special branch of game theory.

The book contains a very lengthy bibliography in which the relatively few essential papers on decision theory are submerged. The theorems are numbered serially without any indication whatever of their authorship. When one considers that the entire subject is only fifteen years old, that the majority of present results are due to Wald, and that the number of other significant contributors to the theory is very small, it would seem that a due regard for history would be neither difficult nor amiss. The authors cite in the bibliography their own unpublished work on invariant procedures, but have neglected to cite earlier unpublished work by M. P. Peisakoff (*Transformation of parameters*, Princeton Thesis, 1951).

The brief chapter on utility and principles of choice is a desirable part of the book. The subject certainly deserves discussion, and more than the casual remarks accorded to it in Wald's book. The authors' discussion summarizes most of the meager results on the subject.

There are many exercises, some of which should be instructive to the student. In this respect the present book is much superior to Wald's, which contains no exercises.

The number of definitions and the amount of notation are so great that an indexed glossary would be of great help in reading the book.

We close with a word about further problems. Some work is now being done which aims at an elegant reworking of essentially Wald's Chapter III. The natural method is to try to choose a proper topologic space in which support planes will represent Bayes solutions. While there is still a certain amount of this left to do and this work has a value, it is not likely to constitute a deep advance, and there are definite limits to its mathematical interest. It seems to the reviewer that essential advances remain to be made in the direction of more detailed behavior of general Bayes solutions and of detailed description of Bayes solutions for important special problems. Methods are needed for obtaining optimal (e.g., minimax) solutions. The problem of choice and, in particular, consequences of proposed solutions to the problem of choice, require much investigation. The theory of decision functions can undoubtedly be employed to illuminate and deepen results in other branches of statistics; recent application to the maximum likelihood method is a case in point. In any case there remain many problems which are interesting and difficult from the mathematical point of view and which show that the subject has great mathematical vitality. Of its practical statistical importance there can be no doubt.

Added in proof. Dr. M. A. Girshick died in March, 1955. He was Professor of Statistics and leader of the statistics group at Stanford

University. His untimely death removes from the scene one of the early American workers in statistics.

J. WOLFOWITZ

Die Grundlagen der Quantenmechanik. By G. Ludwig. Berlin, Springer, 1954. 12+460 pp., 52 figures. Paperbound, 49 DM; clothbound, 52.60 DM.

This book gives a clear and interesting account of the quantum mechanics of a non-relativistic system with a finite number of degrees of freedom. This is a subject whose main lines, particularly consistency and uniqueness, are relatively well established. Roughly the first half of the book is devoted to a careful laying of the foundations of the subject from both the mathematical and physical sides. The second half develops the crucial applications and techniques in a thoroughgoing and readable way. The material of the book is well integrated and evenly presented, there being a good balance between the demands of intelligibility and those of brevity, with most doubtful situations being resolved in favor of the former. Though much of the mathematical and some of the physical material is essentially contained in von Neumann's well known book, the present work by its larger size and more limited scope provides a more spacious and elementary exposition. While its emphasis is on the physical side, its attitude is primarily logical, so that the mathematics is treated as a basic part of the scheme, rather than as a necessary evil. On the whole the book is solid and spirited, eclectic rather than pure, and should be unusually useful for some time to come.

The book includes somewhat condensed but readable treatments of Hilbert space and of group representation theory, in so far as they are used by it. Elsewhere in the book the mathematics makes use of assumptions that are generally clearly formulated so that the whole work is essentially quite rigorous. Important developments involving a relatively high level of mathematical sophistication are however omitted, among them von Neumann's theorem on the uniqueness of the Schrödinger operators and Kato's recent work in the Transactions on n-particle systems.

It is a significant anomaly that there is no material in this treatment of foundations on either quantum fields or relativistic invariance. The foundations that are presented are roughly as valid, from a physico-mathematical viewpoint, as the usual foundations of the Newtonian dynamics of a finite set of particles. Either theory is logically consistent as well as mathematically categorical and self-contained. But neither is ultimately physically self-contained, and this is particularly the case for quantum mechanics. Such a basic phenomenon as the dual particle-wave aspects of light and matter, treated in the first chapter, cannot be well understood without the introduction of systems with an infinite number of degrees of freedom. And it is generally believed that the nature of the interaction between light and matter can be comprehended successfully only in a relativistic theory, although as yet there is no wholly satisfactory mathematical treatment of the matter. In other terms, quantum mechanics is logically more indivisible than classical mechanics—which manifests itself in the circumstance that more of the physics is in the mathematics. But a book of reasonable size will probably never be able to go into all such matters with anything like the exceptional thoroughness and clarity with which this book illuminates the fundamentals of "classical" quantum mechanics, i.e. the part of quantum mechanics thought by most informed persons to be in fairly definitive form.

I. E. SEGAL

Methods of algebraic geometry. Vol. III. *Birational geometry.* By W. V. D. Hodge and D. Pedoe. Cambridge University Press, 1954. 10+336 pp. $7.50.

The book we are reviewing is the third volume of a series devoted to the methods of algebraic geometry. Since a common spirit animates these books, the present review would be incomplete without a glance at the three volumes.

The first part of Volume I (reviewed in this Bulletin vol. 55 (1949) pp. 315–316) contains various algebraic preliminaries, ranging from linear algebra, matrices and determinants to field-theory and the study of polynomials along classical lines. Its second part presents an account of the "linear geometry" in projective spaces, with a full analytic treatment of Grassmann coordinates, collineations and correlations, and with a long side-trip into synthetic projective geometry.

With the second volume (reviewed in this Bulletin vol. 58 (1952) pp. 678–679) begins algebraic geometry proper. Irreducibility, generic points, dimension and associated forms are discussed. A chapter on algebraic correspondences introduces a theory of intersection multiplicities which, without being as generally applicable as more sophisticated sister-theories, is however sufficient for most purposes. As an illustration we find two remarkably exhaustive chapters on quadrics and Grassmann varieties.

In the third volume the emphasis shifts from projective to affine space, from a study in the large to a local theory. After an introductory chapter on ideal theory, we find the "arithmetic" theory of varieties, i.e. the correspondence between ideals and varieties, the study of simple points, and that of normal varieties. More powerful algebraic tools, i.e. valuations, are then brought into play, and the book reaches its climax with a proof of the local uniformization theorem and of the reduction of singularities for surfaces.

At a time when abstract algebraic geometry is becoming more and more popular among mathematicians, many of us are confronted with the problem of how to learn (or teach!) it. Some good books are now available, and, for the time being, no two of them have the same purpose. Walker's *Algebraic curves* presents a carefully expounded special and significant theory. A sketchy and suggestive treatment of many important topics may be found in Lefschetz's *Algebraic geometry*. In A. Weil's *Foundations* we find a systematic and logical book, centered on intersection theory, and which some people prefer to read backwards. A book assuming from the start an exhaustive knowledge of commutative algebra is still missing, but perhaps not for long.

With Hodge and Pedoe's *Methods* we now see a quite different type of book. In its three volumes we find a leisurely description of an impressively extensive part of algebraic geometry, and of the algebraic methods which are used nowadays. Motivations are given. Examples of significant and useful varieties are numerous. All the algebra which is needed is given, and, what is more, these books tell how to translate geometry into algebra, and conversely. The powerful and useful method of associated forms (called "Cayley forms" by the authors) is given a leading part in Volume II. The authors do not hesitate to repeat the same thing twice in different words, thus easing the reading for a beginner: e.g., prime ideals and factor rings are introduced in Chapter X in a special case, and in full generality in Chapter XV. None of these features impairs the value of these books for the working geometer, and all of them enhance their value for the student.

The only things which may be regretted by the working geometer (and, mostly, for the sake of its own references!) are the restriction to characteristic 0, and the use of an intersection theory which is not the most general one. The authors give, in the introductions, their reasons for having chosen such a course: broadly speaking the inclusion of characteristic $p\neq 0$ would have been inconsistent with the spirit of these books, which only intend to put classical algebraic geometry on a sound basis.

We now come to a more detailed account of Volume III, the subtitle of which is *Birational geometry*. It is written for readers who have mastered the classical methods of algebraic geometry (e.g. who have studied the first two volumes), and who wish to get acquainted with the methods and the results of the modern algebraic school (read Oscar Zariski) on local problems.

The introductory chapter (Chap. XV) on ideal theory has been written "with the needs of geometry in mind," and the reviewer is convinced that the authors have successfully reached their goal. Besides classical topics (like factor rings, noetherian rings and modules, primary representation, quasi-equality, integral dependence), we find many others which algebraic geometry has recently introduced into algebra: extension and contraction of ideals, quotient rings, detailed study of integral closures.

Chapter XVI is devoted to the arithmetic theory of varieties. Since the preceding volumes dealt with projective varieties, we first encounter the passage from projective to affine space, a feature rarely found in books, and thus most welcome. Then comes the correspondence between ideals and varieties, based upon Hilbert's zero theorem (proved in Volume I by using elimination theory), and upon the primary representation of ideals. Quotient rings of simple points are studied, and uniformizing parameters introduced. A careful study of the behavior of quotient rings by ground-field extension leads to Zariski's characterization of simple subvarieties. Affine normal varieties (i.e. affine varieties whose coordinate ring is integrally closed) are introduced, and the correspondence between an affine variety and a normal model is studied. Projective varieties having an integrally closed homogeneous coordinate ring are called "projectively normal," a name which seems excellent to the reviewer; useful features of this section are the detailed study of the integral closure of a graded integral domain $K[x_0, \cdots, x_n]$, and the characterization of projectively normal varieties by the completeness of their systems of plane sections. At many points in this chapter the connection with associated forms is made.

Chapter XVII presents for the first time in book form the kind of valuation theory that is needed by algebraic geometers. Besides classical topics, like linearly ordered group and the extension of a valuation to an algebraic extension, it contains the study of the rank, and that of the compounded valuations. In the case of a function-field, the dimension of a valuation is introduced, and its relations with rank and rational rank are carefully studied. Some helpful words are written about the connection between valuations and branches. Then the authors define the center of a valuation on a projective model, and give many properties relating valuations and their centers. The behavior of valuations under ground-field extension is studied. The reviewer was, however, slightly disappointed not to find any clear-cut theorem giving the existence of valuations with given center (a very comprehensive result of Zariski's is quoted, but without proof); the difficulty lies perhaps in the fact that, instead of the powerful theorem about the extension of a specialization to a place, the authors proved only that any integrally closed domain is an intersection of valuation rings.

In the last chapter (Chapter XVIII) valuation theory is applied to geometric problems. Corresponding subvarieties in a birational correspondence are defined in a geometric way, and characterized by means of valuation theory. Transforms and total transforms of subvarieties are discussed, and a weak form of Zariski's "main theorem" is proved. The authors derive from a more general result the fact that, if P is a normal point and if the total transform of P under a rational mapping F consists of a finite number of points, then F is regular at P. After an introduction devoted to monoïdal transformations, comes the climax of this book, i.e. the proof of the local uniformization theorem. This proof follows, broadly speaking, Zariski's method, but a valiant effort has been made toward greater intelligibility. The existence of finite resolving systems for a function-field F is then proved without topologizing the Riemann surface of F. And, as in Zariski's "simplified proof," the method of replacing, in a resolving system, two varieties by a single one, leads to the reduction of singularities for surfaces.

A one-page bibliographical note tells more about the history of the subject than could many pages in the usual historical style, and the authors should not apologize about that in their introduction. Here, as in the remainder of their books, the authors should be commended for having given the facts, many useful facts, in a straightforward way. They should also be commended for having successfully steered a course which is equally remote from bashfulness about using algebra and from oversophistication in its use. In writing their books they have rendered an invaluable service to the mathematical community.

P. SAMUEL

The printing of mathematics. Aids for authors and editors and rules for compositors and readers at the University Press, Oxford. By T. W. Chaundy, P. R. Barrett, and Charles Batey. Oxford University Press, 1954. 10+105 pp. $2.40.

Mathematics in type. Richmond, Va., The William Byrd Press. 12+58 pp. $3.00; $1.50 to staff members of educational institutions.

Printing is a necessary evil: there is substantial agreement among mathematicians that an alleged piece of mathematics has no standing until it has appeared in print for all interested people to read. There is also a general impression that editors make arbitrary and unreasonable rules about the form of manuscripts; and that printers impose absurd restrictions on the symbolism which may be used. These two books are intended to dispel these impressions and give helpful advice to authors. Both of them give detailed explanations of how mathematics actually gets into type; understanding this, the mathematical author can understand why one of two equivalent notations is more economical than another, and generally what he should do (or not do) in order to help the printer. One book or the other should be required reading for anyone who is going to write a mathematical paper. The Oxford one is considerably more detailed about the mechanics of monotype composition and proof correction; in some other respects it may be misleading to an American reader, since of course it represents Oxford practice, which is not entirely typical even of British practice. The Byrd guide is somewhat safer for an American reader to follow, but part of its discussion is based on methods used at the Byrd Press, and not in general use, which make it possible to set on a machine many common combinations which ordinarily require hand work. A minor point, frequently overlooked, is clarified by understanding the mechanics of printing: manuscripts should not be marked in the same way as proofs, since they are handled by the printer in a different way.

Some of the differences between British and American printing customs are interesting. The Americans insist on typewritten copy; the British are quite happy with legible handwriting, and even prefer it under some circumstances. In Oxford practice displayed formulas are numbered on the right; "A very few distinguished mathematicians have numbered their equations on the left: this is exceptional"—but of course is the American standard. Here the British practice is more economical of space if one is willing to agree that a numbered formula need not be displayed, or conversely that ordinary words are admissible in a display. Oxford prefers $\binom{n}{r}$ to ${}_nC_r$ for reasons of style, but hopefully suggests $(n|r)$ as a distinctive replacement for either. The British prefer $\frac{1}{2}$ for looks; the Americans prefer 1/2 for legibility. The Oxford book suggests several other notational innovations, for instance $\surd a+b\backslash$ for $\surd(a+b)$ (making the radical sign serve as its own bracket, as $|$ does in an absolute value); $\exp_a x$ for a^x in the case of a complicated x; $y_\sharp$ and $y_\flat$ for max y and min y (here there would be trouble if the y's had subscripts: perhaps $\sharp y$ and $\flat b$ would be preferable, but in any case $\flat$ is hard to *write* distinctively. One may well object to any notation that is not convenient both to write and to print clearly). I add one suggestion of my own: write $\lfloor x+y!$ for $\Gamma(x+y+1)$. Anyone who is repelled by such innovations may recall that the solidus (/) was an innovation less than a century ago.

Both books pay a deserved amount of attention to matters of style, although here again the Oxford book is fuller. There are two aspects of mathematical style, only one of which has to do with the mechanics of printing. This is the fact that some symbols don't combine happily. For instance, consider a_B, where the subscript is almost as big as the main letter; or j_k, which, especially when the whole thing is a subscript, is much harder for the eye to take in than is k_j. Appearance and ease of reading are improved if some care is taken to adjust the symbolism to the demands of printing.

The other aspect of style is essentially literary, and applies even if the paper is not to be printed from type. Both books stress such frequently overlooked points as that a formula is a phrase or sentence of the same language as the rest of the paper, and should be arranged and punctuated as such. They explain the rules governing spacing of symbols, breaking formulas, etc. The Oxford book discusses in detail the preferred usage of punctuation marks, of "I" versus "we," of "assume," "arbitrary," "only," and so on, and compares the relative merits of alternative ways of saying the same thing. These rules are not arbitrary rules, but a summary of the current usage of writers who write clearly and considerately. One can usually recognize good writing, even if one is not aware of the characteristics which make it so; the authors have isolated some of these characteristics. Some of the spirit of this discussion can be felt from the following quotations.

"A good mathematical presentation is one in which the essential information admits of being 'immediately apprehended'; it should not be sufficient merely to say that it is 'all there' for anyone who has the patience and skill to disengage it."

"Some mathematicians (including the writers) will maintain that symbolism can be overdone; that a remorselessly symbolic mathematics need not be the more intelligible. The passage from mind to mind must be made through the reader's eye, and a microscopic notation, all 'jots and tittles,' indices and subscripts, may be as illegible as a macroscopic exposition relying largely on words and phrases. The ideal lies between these, in which an occasional word punctuates the symbolism and a formula or a little knot of symbols breaks the flow of words."

"Mathematicians who are writing in English are asked not to forget the dignity and traditions of the language. What they write purports to be English prose, even though symbols have replaced many of its words; it should be both readable and speakable as well as printable. Thus symbols such as '∴,' '∵,' or end-tags like 'q.e.d.,' 'q.e.f.' are best left behind in the schoolroom. What they say can be as well said in plain English. When 'with respect to' grows tedious by repetition, it need not be cut to 'w.r.t.,' which is not current English. The preposition 'in' will generally serve."

I recommend this part of the Oxford book especially to all of us who feel that doing research is so much more fun (and claim that it is so much more important) than writing it down carefully for others to comprehend.

Both books contain long lists of available characters; there is a wealth of choice available for anyone who is imaginative enough to do something besides varying a few letters by covering them with hats of various shapes. The Oxford University Press is, however, not an entirely safe guide for authors, since, for example, it is willing to allow, and indeed has allowed, a Chinese character as a mathematical symbol. Other, less well-equipped, presses would disagree; and in general it seems that notations should, if possible, be chosen so that they can be reproduced by all reasonable printers.

R. P. BOAS, JR.

BRIEF MENTION

Higher algebra. By H. Hasse. Vols. 1 and 2 in one volume. Trans. from the 3d rev. German edition by T. J. Benac. New York, Ungar, 1954.

Exercises to higher algebra. By H. Hasse and W. Klobe. Trans. from the 2d rev. and enlarged German edition by T. J. Benac. New York, Ungar, 1954. 212 pp. $4.00.

For reviews of earlier editions (in German) see this Bulletin vol. 33, p. 251; vol. 34, p. 672; vol. 40, p. 370; vol. 41, p. 476; vol. 44, p. 178.

Contributions to the theory of partial differential equations. Ed. by L. Bers, S. Bochner, and F. John. (Annals of Mathematics Studies, no. 33.) Princeton University Press, 1954. 8+257 pp. $4.00.

This volume contains 15 papers read at a conference held in October, 1952.

Opere. By U. Dini. Vol. 2. Ed. by the Unione Matematica Italiana with the assistance of the Consiglio Nazionale delle Ricerche. Rome, Cremonese, 1954. 6+509 pp. 4500 lire.

For vol. 1 cf. this Bulletin vol. 60, p. 288. Vol. 2 contains papers on functions of a real variable and expansions in series, the Dini-Neumann problem, and analytic functions.

Table of the gamma function for complex arguments. (National Bureau of Standards Applied Mathematics Series, no. 34.) Washington, Government Printing Office, 1954. 16+105 pp. $2.00.

The natural logarithm of $\Gamma(x+iy)$ is tabulated for $x=0(.1)10$, $y=0(.1)10$, to 12 decimals.

Table of sine and cosine integrals for arguments from 10 *to* 100. (National Bureau of Standards Applied Mathematics Series, no. 32.) Washington, Government Printing Office, 1954. 16+187 pp. $2.25.

This is a reissue of Table 13 of the Mathematical Tables Project (this Bulletin vol. 49, p. 32).

Lineare Operatoren im Hilbertschen Raum. By W. Schmeidler. Stuttgart, Teubner, 1954. 6+89 pp. 7.80 DM.

This little book is intended as an introduction to the theory of Hilbert space and linear operators on Hilbert space. It is essentially self-contained and, although the text contains only pure mathematics, is clearly motivated by applications. The approach is classical. Fundamental theorems for Hilbert space are proved first for the space l_2 of sequences and then extended by the representation theorem to abstract Hilbert space, which by definition is separable and infinite dimensional. The book is divided into three parts: I, The Hilbert space $\mathfrak{H}$. II, Linear operators in $\mathfrak{H}$. III, Spectral theory. Part II emphasizes the completely continuous operators and contains, among other things, the Schmidt normal form and the Fredholm theory for such operators. Part III is mainly concerned with the spectral theorem. All linear operators are bounded until the end of Part III where the spectral theorem is extended to unbounded Hermitian operators. Each of the parts concludes with a section on exercises, examples, and applications which serve to broaden considerably the scope of the book.

C. E. Rickart

Erratum

In the review of *Tables of binomial coefficients*, published by the Cambridge University Press [this Bulletin vol. 61 (1955) p. 91], the price was incorrectly given as $5.50. The price is $6.50.

NOTES

The following individuals were among those who received doctorates during 1954 in the mathematical sciences and related subjects from universities in the United States and Canada. The university, month in which the degree was conferred, minor subjects (other than mathematics), and title of the dissertation are given in each case if available.

C. E. Abraham, Texas, May, *Partial integro-differential equations and the H-R transform.*

J. E. Adney, Ohio State, June, *On the power of a prime dividing the order of the group of automorphisms.*

S. G. Allen, Stanford, *A class of minimax tests for one-sided composite hypotheses.*

S. L. Andersen, North Carolina State, June, *Robust tests for variances and effect of non normality and variance heterogeneity on standard tests.*

F. W. Anderson, Iowa, August, *A lattice characterization of completely C-spaces.*

Louis Auslander, Chicago, June, *Contributions to the curvature theory of Finsler spaces.*

Maurice Auslander, Columbia, May, *Relative cohomology theory of groups and continuations of homomorphisms.*

R. W. Bagley, Florida, June, minor in philosophy, *The topolattice and permutation group of an infinite set.*

A. V. Balakrishnan, Southern California, May, *On powers of the infinitesimal generators of groups and semigroups of linear bounded transformations.*

W. E. Barnes, British Columbia, May, minor in philosophy, *Primal ideals and isolated components in non-commutative rings.*

Lida K. Barrett, Pennsylvania, June, *Regular curves and regular points of finite order.*

G. E. Baxter, Minnesota, March, minor in physics, *An application of stochastic processes to certain problems in the Brownian motion of continuous media.*

Evelyn M. Bender, Massachusetts Institute of Technology, June, minor in modern languages, *Some generalisations of the Krull ramification theory for rings.*

Dean C. Benson, Iowa State, December, minor in physics, *Regularization of certain systems of differential equations.*

Donald C. Benson, Stanford, June, *Extensions of a theorem of Loewner on integral operators.*

W. J. Berger, Carnegie, June, *Special calculational methods in matrix theory.*

D. R. Bey, Illinois, June, minor in education, *Modified discriminant analysis of school organizations.*

Allan Birnbaum, Columbia, April, *Characterizations of complete classes of tests of some multiparametric hypotheses, with applications to likelihood ratio tests.*

R. C. Blanchfield, Princeton, October, *Intersection theory of manifolds with operators with applications to knot theory.*

Isadore Blumen, North Carolina, *The estimation of the means of the multivariate normal distribution: minimax solutions.*

R. N. Bradt, Stanford, *On the design and comparison of certain dichotomous experiments.*

Barron Brainerd, Michigan, June, *An algebraic theory of probability with applications to analysis and mathematical logic.*

G. U. Brauer, Michigan, June, *Some abelian semigroups of linear transformations of Hausdorff type.*

Leo Breiman, California, Berkeley, June, *Homogenous processes.*

E. H. Brown, Jr., Massachusetts Institute of Technology, September, minor in humanities, *Finite computability of the homotopy groups of finite groups.*

W. P. Brown, Michigan, February, *An algebra related to the orthogonal group.*

C. F. Brumfiel, Purdue, August, minor in physics, *Noetherian minimal bases and equations with prescribed groups.*

B. F. Bryant, Vanderbilt, December, minor in physics, *Unstable self-homeomorphisms of a compact space.*

D. A. Buchsbaum, Columbia, May, *Exact categories and duality.*

C. C. Buck, Michigan, February, *The algebraic aspect of integration in space.*

M. D. Burrow, McGill, May, *A generalisation of the method of Young operators and its use in constructing primitive idempotents for the representation of* $GL(2, q)$.

J. B. Butler, Jr., California, Berkeley, June, *On the behavior of an analytic operator in Banach space in the neighborhood of a singular point.*

S. G. Campbell, Syracuse, June, *On a class of non linear partial differential equations.*

K. H. Carlson, Wisconsin, June, *The application of some inequalities in the theory of surfaces.*

J. H. Case, Tulane, August, *Some relations between induced homomorphisms on homology and cohomology groups.*

Anna Chandapillai, Wisconsin, June, *The distribution of characteristic values of a linear boundary problem of the second order based on a differential equation with a turning point.*

C. L. Chiang, California, Berkeley, January, *On regular best asymptotically normal estimates with an application to a stochastic process.*

N. A. Childress, Florida, June, minor in education, *Surfaces obtained from involutions generated by homographies of periods three, five and thirteen.*

R. R. Christian, Yale, June, *On integration with respect to a finitely additive measure whose values lie in a Dedekind complete partially ordered vector space.*

W. J. Coles, Duke, June, minor in philosophy, *Linear and Riccati systems.*

A. H. Copeland, Jr., Massachusetts Institute of Technology, September, minor in physics, *On the composability of mapping classes.*

W. R. Cowell, Wisconsin, August, *Quasinormal kernels of loops.*

Arno Cronheim, Illinois, June, minors in logic and philosophy, *Motion groups of planes.*

R. B. Crouch, Kansas, October, minor in education, *Monomial groups.*

T. H. Crowley, Ohio State, June, *A problem in the theory of distributions.*

P. P. Crump, North Carolina State, June, *Optimal designs to estimate the parameters of a variance component model.*

H. J. Curtis, Illinois Institute of Technology, June, *A metrization problem concerning lattices.*

M. B. Danford, North Carolina State, June, *Factor analysis and related statistical techniques.*

H. F. Davis, II, Massachusetts Institute of Technology, June, minor in physics, *Decomposition theorem for the Haar integral.*

Cyrus Derman, Columbia, March, *Some contributions to the theory of Markov chains.*

J. W. Dettman, Carnegie, June, *A theory of interacting particles with variable rest masses.*

L. I. Deverall, Utah, August, minor in mechanical engineering, 1. *Solution of some problems in bending of thin plates by the finite Fourier sine transform.* 2. *Some relations involving the special functions of mathematical physics.*

D. J. Dickinson, Michigan, February, *On Bessel and Lommel polynomials.*

R. J. Dickson, California Institute of Technology, June, minor in aeronautics, *Bounds for solutions of some non-linear parabolic problems.*

J. J. Dunne, Notre Dame, June, minor in chemistry, *Theory of games in the extensive form.*

B. M. Dwork, Columbia, May, *On root number in the functional equation of the Weil L-series.*

A. R. Eckler, Princeton, June, *Rotation sampling.*

D. E. Edmondson, California Institute of Technology, June, minor in aeronautics, *Homomorphisms of a modular lattice.*

James Eells, Jr., Harvard, June, *Geometric aspects of integration theory.*

Jacqueline P. Evans, Harvard, March, *An approximation and interpolation by functions analytic in a given region and an application to orthonormal systems.*

A. G. Fadell, Ohio State, December, *Accessability in Euclidean N-space with application to differentiability theorems.*

W. M. Faucett, Tulane, August, *Topological semigroups and continua.*

W. J. Feeney, Catholic, June, minors in physics and philosophy, *Certain unsolvable problems in the theory of cancellation semi-groups.*

Jacob Feldman, Chicago, June, *Isomorphisms of rings of operators.*

E. H. Feller, Wisconsin, June, *The lattice of submodules of a module over a noncommutative ring.*

W. J. Firey, Stanford, June, *On ballistically closed regions.*

J. E. Fischer, St. Louis, June, *Quaternary-binary representations in quadratic forms.*

Harvey Fletcher, Jr., Utah, March, minor in physics, *Trigonometric series applied to bending of thin rectangular plates.*

H. A. Forrester, Princeton, June, *The theory of semi-cubical complexes.*

G. E. N. Fox, McGill, May, *Topology in the field of p-adic numbers.*

Phyllis A. Fox, Massachusetts Institute of Technology, February, minor in electrical engineering, *On the use of coordinate perturbations in the solution of physical problems.*

Elvy L. Fredrickson, Oregon State, June, minor in the college teaching program, *Application of the Schmidt theory to nonlinear integral equations.*

R. A. Fuchs, California, Berkeley, September, *Asymptotic developments of water waves obliquely incident to a dock.*

T. M. Gallie, Jr., Rice, June, *Applications of differential operators in the theory of Dirichlet series.*

R. A. Gambill, Purdue, May, minor in physics, *On systems of linear and non-linear differential equations with periodic coefficients.*

A. O. Garder, Jr., Washington University, January, *Topics in the theory of convolution transforms.*

C. B. Gass, Nebraska, June, minor in physics, *Eigenfunction expansions associated with certain irreducible partial differential equations.*

R. K. Getoor, Michigan, June, *Some connections between operators in Hilbert space and random functions of the second order.*

Richard Goldberg, New York, June, *The slow flow of a rarefied gas past a spherical obstacle.*

R. E. Gomory, Princeton, June, *Critical points at infinity and forced oscillation.*

L. E. Grosh, Purdue, May, minor in general engineering, *Uniform distribution and round off error.*

Seymour Haber, Massachusetts Institute of Technology, September, minor in chemical engineering, *Boundary value problem for a singularly perturbed differential equation.*

J. E. Hafstrom, Minnesota, December, *Non-linear transformations in Wiener space of the form* $y(t)=x(t)+q[t, x(t_1), \cdots, x(t_n)]$.

J. D. Hankins, Missouri, August, *Metric characterizations of elliptic n-space.*

H. A. Hauptman, Maryland, June, minor in physics, *An n-dimensional Euclidean algorithm.*

Sigurdur Helgason, Princeton, June, *Banach algebras and almost periodic functions.*

Morton Hellman, New York, June, *Matrix commutator relationships arising from systems of linear ordinary differential equations.*

P. S. Herwitz, North Carolina, June, *On the approximate solutions of differential equations by difference equations.*

A. L. Hess, Wisconsin, August, minor in education, *Geometric constructions: An historical and interpretative study with implication for teachers of secondary school mathematics and for teacher training programs.*

J. W. Hollingsworth, Wisconsin, August, *Parametric methods in the numerical solution of differential equations.*

T. R. Horton, Florida, August, minor in education, *On the equivalence of quadratic forms.*

J. L. Howell, Yale, June, *A class of fourth order differential operators.*

W. M. Huebsch, Notre Dame, June, *Covering homotopy.*

C. W. Huff, Georgia, August, *A study of the exponential equation in non-commuting matrices.*

S. P. Hughart, California Institute of Technology, June, minor in physics, *Representations for dicategories.*

J. S. Hunter, North Carolina State, June, *Multi-factor experimental designs.*

D. V. Huntsberger, Iowa State, June, minor in physics, *An extension of preliminary tests of significance permitting control of disturbances in statistical inferences.*

J. R. Isbell, Princeton, June, *Absolute games.*

P. K. Ito, St. Louis, June, *On the simultaneous minimax point estimation.*

M. J. Jacobson, Carnegie, June, *Reynolds theory and journal bearing lubrication.*

T. R. Jenkins, University of Washington, August, *On fields of quotients of certain convolution algebras.*

A. K. Jennings, Kansas, June, minor in physics, *Operators in reproducing kernel spaces.*

E. R. Johnston, Minnesota, March, minor in mechanics, *A study in polynomial approximation in the complex domain.*

Sister Mary Andrea Johnston, Catholic, June, minors in physics and education, *General expressions for certain coefficients in the cyclotomic polynomial* $\psi_d(x)$.

N. D. Kazarinoff, Wisconsin, January, minor in physics, *Asymptotic forms for the Whittaker functions of large complex order.*

Elsa E. Keitzer, Carnegie, June, *Some problems of network analysis and synthesis.*

Herbert Keller, New York, June, *On systems of linear ordinary differential equations with applications to ionospheric propagation.*

Maurice Kennedy, California Institute of Technology, June, minor in physics, *Ergodic theorems for a certain class of Markoff processes.*

Hewitt Kenyon, California, Berkeley, September, *Webbing derivatives and their integrals.*

H. P. Kerfoot, Southern California, May, *Almost periodic semigroups.*

R. W. Klopfenstein, Iowa State, December, minor in electrical engineering, *Radiation patterns for slotted cylinders of arbitrary cross sections.*

H. S. Konijn, California, Berkeley, September, *On the power of some tests for independence.*

P. J. Koosis, California, Berkeley, September, *On Fourier coefficients of absolutely continuous measures.*

Bertram Kostant, Chicago, August, *The representation of a Lie algebra and its enveloping algebra which arises from a unitary representation of a Lie group.*

G. L. Krabbe, California, Berkeley, June, *Certain groups of operators in Banach space.*

C. H. Kraft, California, Berkeley, June, *On the problem of consistent and uniformly consistent statistical procedures.*

H. P. Kramer, California, Berkeley, June, *Perturbation of differential operators.*

J. B. Kruskal, Jr., Princeton, June, *The theory of well-partially-ordered sets.*

W. T. Kyner, Jr., California, Berkeley, September, *Fixed point theorems in Banach space.*

H. T. LaBorde, North Carolina, August, *A method for the numerical computation of the characteristic roots of a matrix and extensions of some theorems of P. Stein.*

R. K. Lashof, Columbia, May, *Topological group extensions and Lie algebras of locally compact groups.*

C. Y. Lee, University of Washington, December, *Domain functionals for pseudoanalytic functions.*

Karel deLeeuw, Princeton, June, *The relative cohomology theory of finite groups and algebraic number theory.*

A. B. Lehman, Florida, January, minor in physics, *On metric lattices.*

R. S. Lehman, Stanford, June, *Developments in the neighborhood of the beach of surface waves over an inclined bottom.*

C. F. A. Leth, Brown, June, *The effect of shear stresses on the carrying capacity of I-beams.*

H. A. Lindstone, Southern California, May, minor in physics, *Singular perturbations of linear differential equations in the complex plane.*

C. W. McArthur, Tulane, August, *On unconditional summability of sequences in semigroups with a topology.*

L. F. McAuley, North Carolina, June, *On the aposyndetic decomposition of continua.*

Garner McCrossen, Colorado, June, minor in philosophy, *A generalized Laplace-Stieltjes transformation.*

J. L. McGregor, California Institute of Technology, June, minor in aeronautics, *Generalized translation operators.*

R. W. McKelvey, Wisconsin, January, minor in physics, *The solutions of second order linear ordinary differential equations about a turning point of the second order.*

E. B. McLeod, Jr., Stanford, January, *An application of the Schiffer variation to the free boundary problems of hydrodynamics.*

Josiah Macy, Jr., Massachusetts Institute of Technology, September, minor in biology, *Probability model for cortical responses to successive auditory clicks.*

E. A. Maier, Oregon, June, *Matrices in a Boolean algebra.*

M. D. Marcus, California, Berkeley, June, *The application of fixed-point theorems to the perturbation of ordinary differential equations.*

D. C. B. Marsh, Colorado, June, minors in philosophy and chemistry, *An investigation of the number of classes in the genus of certain indefinite ternary quadratic forms.*

A. P. Mattuck, Princeton, June, *Abelian varieties over p-adic ground fields.*

C. H. Meng, Southern California, January, *On approximately normal operators.*

J.-P. G. Meyer, Cornell, February, *Classification of mappings of a 3-dimensional complex into a 2-dimensional projective space.*

P. L. Meyer, Stanford, *An application of the invariance principle to the Student hypothesis.*

W. A. Michael, Jr., Illinois, June, minor in philosophy, *Singular integral equations with normal kernels.*

J. W. Milnor, Princeton, June, *Isotopy of links.*

M. L. Minsky, Princeton, June, *Neural-analog networks and the brain model problem.*

Sister Irene Morvan, Catholic, June, minors in physics and chemistry, *A study of loci associated with systems of isopolar lines of lines with respect to a quadrangle.*

Mervin Muller, California, Los Angeles, September, *Some Monte Carlo methods for the Dirichlet problem.*

B. H. Murdoch, Princeton, October, *Preharmonic functions.*

H. A. Myers, Michigan State, December, *The unsymmetrically fed prolate spheroidal antenna.*

Rangaswamy Narasimhan, Indiana, February, minor in physics, Part I. *On the asymptotic stability of solutions of nonlinear partial differential equations of parabolic type.* Part II. *On bounded solutions of linear elliptic differential equations.*

T. V. Narayana, North Carolina, *Sequential procedures in probit analysis.*

C. J. Neugebauer, Ohio State, June, *Cyclic additivity.*

D. V. Newton, University of Washington, June, *A generalized form of the problem of Bolza in the calculus of variations.*

C. A. Nicol, Texas, May, *Generating functions for restricted partitions and the von Sterneck number.*

R. H. Niemann, Purdue, August, *A study of Stieltjes integral transforms of a certain class of functions.*

Walter Noll, Indiana, September, minor in physics, *On the continuity of the solid and the fluid states.*

G. C. Nooney, Stanford, January, *On the vibrations of triangular membranes.*

R. Z. Norman, Michigan, June, *On the number of linear graphs with given blocks.*

A. B. Novikoff, Stanford, January, *On a special system of orthogonal polynomials.*

R. H. Oehmke, Chicago, August, *A class of non-commutative power associative algebras.*

C. S. Ogilvy, Syracuse, January, *An investigation of some properties of asymptotic lines on surfaces of negative Gaussian curvature.*

F. R. Olson, Duke, June, minor in physics, *Arithmetic properties of Bernoulli numbers of higher order.*

R. C. Osborn, Texas, May, minor in philosophy, *Integration formulae for the hyperbolic partial differential equation with four independent variables and regions interior to the cone.*

E. C. Paige, Chicago, June, *Jordan algebras of characteristic* 2.

P. B. Patterson, Florida, June, minor in education, *Almost regular forms.*

A. H. Payne, North Carolina, June, *Some polynomials associated with Tschebycheff polynomials.*

Jacqueline L. Penez, Minnesota, June, minor in mechanics, *Approximation by boundary values of analytic functions.*

R. H. Pennington, Stanford, June, *Surface instabilities on pulsating gas bubbles.*

L. L. Philipson, California, Los Angeles, June, *The asymptotic character of the solutions of a class of ordinary linear differential equations depending on a parameter.*

K. C. S. Pillai, North Carolina, *On some distribution problems in multivariate analysis.*

R. G. Pohrer, Duke, June, minor in philosophy, *On the solution of equations in a finite field.*

E. E. Posey, Tennessee, June, minor in physics, *Almost polyhedral cells in Euclidean 3-space.*

J. H. Powell, Michigan State, December, *A mathematical model for single function group organization theory with applications to sociometric investigations.*

G. C. Preston, Minnesota, June, *On locally compact totally disconnected Abelian groups and their character groups.*

R. E. Priest, Illinois, June, minor in physics, *Limits of homotopy and cohomotopy groups.*

Valdemars Punga, George Washington, February, *Affine connection in space of line elements.*

R. A. Raimi, Michigan, June, *Equicontinuity of linear transformations.*

R. W. Randall, Jr., Rice, June, *Some aspects of Dirichlet series.*

Bayard Rankin, California, Berkeley, *The concept of sets enchained by a stochastic process and its use in cascade shower theory.*

B. A. Rattray, Princeton, January, *An antipodal-point orthogonal-point theorem.*

A. H. Read, Harvard, June, *External problems on Riemann surfaces.*

Edgar Reich, California, Los Angeles, March, *Some distortion theorems for functions analytic in the unit circle.*

R. B. Reisel, Northwestern, August, *A generalization of the Wedderburn-Malcev theorem to infinite dimensional algebras.*

W. F. Reynolds, Harvard, June, *On finite groups related to permutation groups of prime degree.*

T. D. Riney, Purdue, August, minor in engineering mechanics, *On the coefficients occurring in the asymptotic expansion of the generalized hypergeometric function.*

Rose M. Ring, Brown, June, *The complex eigenvalue problem for radio waves in a curvilinearly tapered atmosphere.*

B. V. Ritchie, Purdue, August, minor in physics, *On Galois theory for certain classes of nilpotent algebras.*

R. A. Roberts, Michigan, February, *Duality theorems for generalized manifolds.*

W. G. Rosen, Illinois, October, minor in physics, *On invariant means over topological semigroups.*

E. W. Ross, Jr., Brown, June, I. *On the stability of gas flow in a conical pipe.* II. *On the load-carrying of plastic plates.*

W. C. Ross, Jr., Iowa, June, *Certain functions of order statistics.*

J. P. Roth, Michigan, February, *An investigation in algebraic topology and Morse theory.*

R. W. Royston, Michigan, February, *A frequency function which can be transformed into a gamma type function by a quadratic transformation of the variable.*

L. A. Rubel, Wisconsin, June, *Entire functions and Ostrowski sequences.*

J. U. Russell, Illinois, June, minor in education, *Decompositions of analytic functions with respect to the singular set.*

L. W. Rutland, Colorado, June, minors in physics and education, *An optimum solution of N equations in M unknowns with N greater* *than M and an application to the adjustment of electrical network singularities.*

Jacques Saint Pierre, North Carolina, *Distribution of linear contrasts of order statistics.*

J. L. Sanders, Jr., Brown, June, *Plastic stress-strain relations based on infinitely many plane loading surfaces.*

Herbert Scarf, Princeton, June, *Differential operators on manifolds, and applications to stochastic processes.*

Binyamin Schwarz, Washington University, June, *Complex nonoscillation theorems and criteria of univalence.*

K. C. Seal, North Carolina, *On a class of decision procedures for ranking means.*

B. M. Seelbinder, North Carolina, June, *Some new results on positive solutions of linear diophantine equations.*

G. B. Seligman, Yale, June, *Lie algebras of prime characteristic.*

J. M. Shaheen, Cincinnati, September, *On the theory of Nörlund means and their application to power series and Fourier series.*

Daniel Shanks, Maryland, June, minor in physics, *Non-linear transformations of divergent and slowly convergent sequences.*

J. M. Shapiro, Minnesota, June, *An error estimate for the convergence of distributions of sums of independent random variables to infinitely divisible distributions.*

H. S. C. Sharp, Massachusetts Institute of Technology, June, minor in modern languages, *Some series solutions for finite strain in axisymmetric membranes.*

Abe Shenitzer, New York, June, *Decomposition of a group with a single defining relation into a free product.*

R. L. Shively, Michigan, February, *On pseudo Laguerre polynomials.*

A. R. Sims, Southern California, May, *Linear differential operators of the second order.*

C. J. Sinke, Purdue, May, *New methods for obtaining the coefficients in certain asymptotic expansions.*

Morris Skibinsky, North Carolina, *Some properties of a Bayes two-stage test for the mean.*

H. T. Slaby, Wisconsin, January, *Central nilpotency of commutative Moufang loops.*

M. B. Sledd, Massachusetts Institute of Technology, September, minor in electrical engineering, *On circular cylindrical shells of variable wall thickness.*

Harry Smith, Jr., North Carolina State, June, *Weighting coefficients for age-adjusted death rates.*

N. B. Smith, Iowa State, June, minor in physics, *Types of functions.*

Charles Standish, Cornell, June, *Notes on a class of ergodic transformations.*

J. M. Stark, Massachusetts Institute of Technology, June, minor in physics, *On distortion in pseudo-conformal mapping.*

Marvin Stern, New York, June, *The rolling up of a vortex sheet.*

David Stoller, California, Los Angeles, April, *Asymptotically optimum discriminant function estimates.*

H. L. Stubbs, Boston, June, *Non-normal models for the classification of speech sounds.*

G. H. Swift, Jr., University of Washington, August, *On irregular Borel measures.*

W. B. Temple, Texas, May, minor in physics, *Stieltjes integral representation of convex functions.*

W. A. Thompson, North Carolina, *On the ratio of variances in the mixed incomplete block model.*

J. H. Turnock, Jr., Illinois, October, minor in astronomy, *Geodesic correspondences between Riemannian spaces.*

R. S. Varga, Harvard, June, *Properties of a special set of entire functions and their respective partial sums.*

R. L. Vaught, California, Berkeley, September, *Topics in the theory of arithmetical classes and Boolean algebras.*

George Veronis, Brown, June, *A study of time-dependent and wind-driven ocean circulation.*

F. J. Wagner, Notre Dame, June, *Completion and compactification of topological spaces.*

D. W. Wall, Michigan, February, *Some results in the theory of algebras with radical.*

J. H. Walter, Michigan, June, *Automorphisms of the projective unitary groups.*

Daniel Waterman, Chicago, June, I. *Integrals associated with functions of class* L_p. II. *A convergence theorem.* III. *On some high indices theorems.*

E. C. Watters, Jr., Maryland, January, minors in physics and electrical engineering, *An example of subsonic flow.*

L. H. Wegner, Jr., Oregon, June, *Contributions to the several sample problem.*

Morris Weisfeld, Yale, June, *Galois theory of derivation in division rings.*

J. E. Whitesitt, Illinois, June, *Construction of the lattice of complemental ideals within the unit group.*

Joyce W. Williams, Illinois, June, minor in physics, *Singular integral equations with symmetric complex-valued kernels of class* I.

R. F. Williams, Virginia, June, *Reductions of open mappings.*

J. C. Wilson, Louisiana State, June, *Period relations for Picard integrals defined on a special class of Kahler manifolds.*

Jonathan Winson, Columbia, March, *Error analysis of electronic analogue computation; linear differential equations with constant coefficients.*

P. S. Wolfe, California, Berkeley, September, I. *Games of infinite length.* II. *A non-degenerate formulation and "Simplex" solution of linear programming problems.*

E. S. Wolk, Brown, June, *Representation of topological algebras.*

A. W. Wortham, Oklahoma A. and M., May, *On estimates of variance components.*

Z. S. Wurtele, Columbia, March, *Some properties of Bayes procedures which improve lot quality.*

Ti Yen, Illinois, June, minor in mechanical engineering, *Trace on finite A* W^**-algebras.*

R. A. Zemlin, Ohio State, June, *On a conjecture arising from a theorem of Frobenius.*

The following doctorates were conferred in 1953, but were not included in the list in the preceding volume of this Bulletin (vol. 60, pp. 292–302):

F. J. Arena, George Peabody, August, minor in history and philosophy of education, *A short history of ordinary differential equations.*

D. B. Gillies, Princeton, October, *Some theorems on N-person games.*

Jack Laderman, Columbia, April, *On statistical decision functions for selecting one of k populations.*

Jack Moshman, Tennessee, August, *A two-sample procedure for linear discrimination in normal samples.*

D. L. Wallace, Princeton, October, *Confidence regions for the location of the vertex in quadratic regression.*

RESEARCH PROBLEMS

16. Richard Bellman: *Number theory.*

Let $f(t)$ be the number of solutions of $y^3+2=tx^2$ in integers x and y. It is known that $f(t)$ is finite for each t. What bound can one obtain for $f(t)$, and what asymptotic relation, if any, holds for

$$\sum_{t=1}^{T} f(t) \quad \text{as} \quad T \to \infty ?$$

(Received January 10, 1955.)

17. Richard Bellman: *Analysis.*

Let

$$x = \sum_{n=1}^{\infty} t^n/n^s, \qquad y = \sum_{n=1}^{\infty} t^n/n^{s+1}, \qquad |t| < 1.$$

Eliminating t we have $y = \sum_{n=1}^{\infty} c_n x^n$. What is the radius of convergence of this series as a function of s for $-\infty < s < \infty$? (Received January 10, 1955.)

18. Ernest Michael: *Continuity of multiplicative linear functionals.*

Let A be a closed subalgebra of the cartesian product (product topology) of countably many commutative, complex Banach algebras. Is every complex-valued multiplicative linear functional on A continuous? (Reference: E. Michael, Memoirs of the American Mathematical Society, no. 11, 1952, section 12.) (Received January 18, 1955.)

19. E. T. Parker: *A tensor equation.*

Let $[g_{ab}]$ be a symmetric $n \times n$ matrix of real analytic functions of n real variables, with nonzero determinant over a neighborhood. Let g_{ab} be the covariant metric tensor of a Riemannian space. Find a necessary and sufficient condition on g_{ab} in order that there exist a scalar whose second covariant derivative is g_{ab} over a neighborhood. One might also find conditions for the existence of such a scalar over the space. It may be possible to obtain a result under a hypothesis weaker than that the functions of g_{ab} be analytic. The question might be of interest in non-Riemannian spaces. (Received February 2, 1955.)

20. Casper Goffman and G. M. Petersen: *Consistent matrix summability methods.*

Regular matric methods A and B are called consistent if every bounded sequence summed by both methods is summed to the same value. Show, for every set S of regular methods which are mutually consistent, that there is a set $T \subset S$ of regular methods which are mutually consistent and such that every bounded sequence is summed by a method in T. In particular, consider the case where S has two elements. (Received February 25, 1955.)

Problems Discussed by the XIth General Assembly of the International Radio Scientific Union, and Called to the Attention of Mathematicians

(Communicated to the International Mathematical Union by Professor B. van der Pol, and submitted to the Bulletin by Professor E. Bompiani, Secretary of the International Mathematical Union. Received by the Bulletin, February 7, 1955.)

1. The propagation of radio waves through a medium (i.e. troposphere, ionosphere), in which turbulence occurs causing a statistical spatial distribution of the dielectric constant, which in turn produces a scattering (irregular reflection and diffraction) of the radio waves. (Booker, Carrol, Megaw and others.)

2. The relation—if it exists—between the "frequency bandwidth" of a linear electrical circuit and its "time constants" was investigated and discussed. This problem is analogous to the mathematical problem of the geometrical relation between a given function of a real variable and the form of its Fourier cosine and sine transforms.

3. Mean spectrum of a series of impulses which in principle are identical and periodic, but are actually spaced and deformed at random. This problem has been treated by R. Fortet, L'Onde Electrique vol. 34 (1954) p. 683.

If further generalized it can be stated in the following form:

4. *Spectrum of a "random" signal compared with that of a simple signal emitted by the same system.* It is convenient, for determining the spectrum of a given emitter, to apply either a single elementary signal $E(t)$ or a periodic sequence of signals $E(t)$. But the spectrum emitted under actual operating conditions, corresponding to a sum

$$\sum_{i=1}^{n} E(t - t_i),$$

of elementary signals shifted in time, where t_i are random, does not necessarily bear a simple relation to the spectrum of a single signal or that of a periodical signal.

In all cases of amplitude modulation or pulse modulation a simple solution to the problem of comparing the two types of spectra has been effected easily. In fact, in those cases there corresponds to the sum

$$\sum_{i=1}^{n} E(t - t_i)$$

applied to the input of the system a sum

$$\sum_{i=1}^{n} S(t - t_i)$$

representing the output signal since the system effects only a linear transformation on the signal.

But the problem has yet to be resolved for the case when the transformation is not linear. In particular, in frequency modulation, to a signal of the form

$$\sum_{i=1}^{n} E(t - t_i)$$

there corresponds an output signal which can be put in complex terms in the form of a product

$$\prod_{i=1}^{n} S(t - t_i)$$

and the relationship between the various spectra does not seem to be very simple for those cases when the index of modulation is not very small.

THE APRIL MEETING IN BROOKLYN

The five hundred twelfth meeting of the American Mathematical Society was held on Thursday, Friday, and Saturday, April 14–16, 1955, at the Polytechnic Institute of Brooklyn, which was celebrating its Centennial Year. A Symposium on Mathematical Probability and its Applications (sponsored by the Society and the Office of Ordnance Research) was held on Thursday and Friday. About 500 persons registered, including 392 members of the Society.

The Symposium was divided into three sessions which met at 10:00 A.M. and 2:00 P.M. on Thursday, and at 10:00 A.M. on Friday. The first session, devoted to Diffusion Theory, was presided over by Professor Shizuo Kakutani. The following papers were presented: *Brownian motion depending on five parameters*, by Professor Paul Lévy of l'École Polytechnique, Paris; *A new look at the Dirichlet problem*, by Professor J. L. Doob of the University of Illinois; *Probability and differential equations*, by Professor William Feller of Princeton University.

At the second session of the Symposium, devoted to the Theory of Turbulence, the following papers were presented: *Functional analysis in the theory of turbulence*, by Professor Eberhard Hopf of Indiana University; *Stochastic processes of interest in statistical astronomy*, by Professor Guido Münch of the California Institute of Technology; *The singularity in the spectrum of homogeneous turbulence*, by Dr. G. K. Batchelor of Trinity College, Cambridge University. The chairman for this session was Professor Salomon Bochner.

Probability in Classical and Modern Physics was the subject of the third session at which Professor Herbert Robbins was the chairman. The papers presented were *Probability in classical physics*, by Professor Mark Kac of Cornell University; *Infinite models in physics*, by Dr. S. M. Ulam of Los Alamos, New Mexico; *Quantum theory and the foundations of probability*, by Professor B. O. Koopman of Columbia University.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor R. D. Schafer of the University of Connecticut addressed the Society at 2:00 P.M. on Friday on *Structure and representation of nonassociative algebras*, and Professor Åke Pleijel of the University of Lund and the University of Maryland addressed the Society at 2:00 P.M. on Saturday on *Eigenfunction distribution for polar differential equations*. These sessions were presided over by Professors A. A. Albert and C. B. Morrey, Jr., respectively.

Sessions for contributed papers were held at 3:15 P.M. on Friday, and at 10:00 A.M. and 3:15 P.M. on Saturday. Chairmen at these sessions were Professors J. B. Diaz, C. H. W. Sedgewick, Seymour Sherman, Wallace Givens, Drs. J. H. Curtiss, R. G. Bartle.

The ladies of the Department of Mathematics entertained at tea on Friday afternoon.

The Council of the Society met at 5:15 P.M. on Friday, reconvening after dinner.

The Secretary announced the election of the following thirty-nine persons to ordinary membership in the Society:

Mr. Eugene Albert, Brooklyn College;
Professor Norman Edward Albrecht, Hamline University;
Mr. Herbert Reeder Bailey, U. S. Naval Ordnance Plant, Indianapolis, Indiana;
Professor Herbert Barkan, Newark College of Engineering;
Mr. George Blustein, A. B. Dumont Laboratories, Passaic, New Jersey;
Mr. Jeffrey Jay Bowe, Air Force Cambridge Research Center, Cambridge, Massachusetts;
Mr. William Edward Boyce, Carnegie Institute of Technology;
Sister Theodora Breighner, O. S. U., Ursuline College, Louisville, Kentucky;
Mr. Fred C. Calabrese, Goodyear Aircraft Corporation, Akron, Ohio;
Mr. Joel Carroll, Argonne National Laboratory, Lemont, Illinois;
Miss Ellen Correl, Purdue University;
Mr. Cordell Evans, University of Pittsburgh;
Miss Catherine Mae Fitzpatrick, Rock Island Arsenal, Rock Island, Illinois;
Professor Masanori Fukaymiya, Tôhoku University, Sendai, Japan;
Mr. Herbert Zucker Hale, Astra Bent Wood Furniture Company, New York;
Miss Irene Harwick, Carnegie Institute of Technology;
Mr. Herbert Aaron Hauptman, Naval Research Laboratory, Washington, D. C.;
Mr. Michael Held, Institute of Mathematical Sciences, New York University;
Mr. Arnold Alfred Johnson, Goodyear Aircraft Corporation, Akron, Ohio;
Dr. Irving Kanter, Radio Corporation of America;
Mr. Raymond Kassler, Boland and Boyce, Inc., Belleville, New Jersey;
Mr. John David Leadley, University of Washington;
Mr. Robert Pierce Lee, Flight Determination Laboratory, White Sands Proving Ground;
Mr. Jacob Manheimer, Israeli Ministry of Defence, Tel-Aviv, Israel;
Mrs. Mary Kuskoff Neff, University of Florida;
Mr. Ernst Irvin Prasse, U. S. National Advisory Committee for Aeronautics, Cleveland, Ohio;
Mr. Leon Alexander Robbins, Park College, Parkville, Missouri;
Mr. William Henry Roudebush, U. S. National Advisory Committee for Aeronautics, Cleveland, Ohio;
Lt. Kenneth James Schlagheck, United States Air Force;
Professor Norman James Schoonmaker, University of Massachusetts;
Dr. John Robert Stock, National Carbon Research Laboratories, Cleveland, Ohio;
Miss Ellen Esther Swanson, American Mathematical Society, Providence, Rhode Island;
Mr. William Clement Swift, University of Kentucky;

Mr. Robert Tates, Syracuse University;
Mr. Paul Haynes Thrower, University of Texas;
Sergeant Bruce Ramon Vogeli, Army Chemical Center, Maryland;
Mr. Alfred Weissberg, Battelle Memorial Institute;
Mr. Kenneth Franklin Wilson, University of Florida;
Mr. C. Rex Yeaman, Ricks College, Rexburg, Idaho.

It was reported that the following twenty-eight persons had been elected to membership on nomination of institutional members as indicated:

University of Chicago: Mr. Sterling Khazag Berberian, Dr. Wilfred Halliday Cockcroft, Mr. William L. Hoyt, Mr. Harold Irving Levine, Mr. Elon Lages Lima, Dr. Teruhisa Matsusaka, Dr. Jacob Pieter Murre, Mr. David Shale, Mr. Henry P. F. Swinnerton-Dyer, and Dr. Erik Christopher Zeeman.

Haverford College: Mr. W. Wistar Comfort.

Indiana University: Mr. Robert Edwin Heaton, Dr. Dietrich Morgenstern, Mr. Patrick Shanahan, Mr. Robert Clinton Wrede, and Mr. Arthur Jay Ziffer.

Institute for Advanced Study: Dr. Edoardo Storchi.

Michigan State College: Mr. William Leonard Harkness, Mr. Don Raymond Lick, and Mr. Frank Conant Sherburne, Jr.

New York University: Miss Betty Jane Gassner.

University of Tennessee: Mr. Ceslovas Masaitis.

University of Texas: Mr. Lewis Edwin Batson, Mr. Edwin Hale Connell, Mr. Victor A. Erma, Mr. Robert Heller, Jr., Mr. Richard Paul Kelisky, and Mr. William S. Mahavier.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematiker Vereinigung: Dr. Wolfgang Gaschutz, Universitate Kiel, Kiel, Germany, Professor Hermann Karl Schmidt, Universitat Wurzburg, Klinikstr., Germany, and Professor Egon Ullrich, Justus-Liebig University, Giessen, Germany; French Mathematical Society: Professor Maurice Audin, 22 rue de Nimes, Alger, Algerie, Mr. Pierre Brousse, 20 rue Boncenne, Pointers, Vienne, France, and Professor Sylvain Monavon, Lycée Parc Imperial, Nice, France; Indian Mathematical Society: Professor Balramji Rao Bhonsle, Government Engineering College, Jalapur, India; London Mathematical Society: Dr. Hanna Neumann, The University, Hull, England, and Mr. Jacob Jack Grannek, Queen's University, Belfast, Ireland; Swedish Mathematical Society: Professor Carl-Gustav Esseen, The Royal Institute of Technology, Stockholm, Sweden; Swiss Mathematical Society: Dr. Werner Goutschi, University of California, Berkeley; Wiskundig Genootschap Amsterdam: Mr. Maarten Jan Bossen, Technological University, Delft, Netherlands.

The following appointments by the President were reported: As a committee to nominate a representative of the Society on the Policy Committee for Mathematics: Samuel Eilenberg, Chairman, Marston Morse, and Antoni Zygmund; as a committee to nominate officers and members of the Council for 1956: H. F. Bohnenblust, Chairman, H. S. M. Coxeter, E. E. Moise, Deane Montgomery, and W. M. Whyburn; as a committee on arrangements for the 1955 Annual Meeting to be held at The Rice Institute: F. E. Ulrich, Chairman, H. E. Bray, A. H. Brown, L. K. Durst, H. M. Gehman, Guy Johnson, G. R. MacLane, S. Mandelbrojt, and J. W. T. Youngs; as a Committee to Select Gibbs Lecturers for 1956 and 1957: Marston Morse, Chairman, Garrett Birkhoff, and K. O. Friedrichs; as a committee on arrangements for the meeting to be held at the University of Tennessee, November 18–19, 1955: O. G. Harrold, Chairman, J. H. Roberts, W. S. Snyder, and R. L. Wilson; as representatives of the Society on the Organizing Committee for the second part of the Symposium on Mathematical Statistics and Probability to be held at the University of California at Berkeley during the summer of 1955: J. L. Doob and George Pólya; as a joint committee of the Society and Association to investigate the possibility of holding the annual meeting at a time other than the week between Christmas and New Year's: R. M. Thrall, Chairman, H. M. Gehman, W. T. Martin, and J. M. Thomas; as a replacement for Professor T. R. Hollcroft on the Committee on the Employment Register: W. M. Hirsch.

The Secretary reported that the following persons have accepted invitations to deliver hour addresses during 1955: A. V. C. Pleijel, Brooklyn, New York, April 15–16; R. D. Schafer, Brooklyn, New York, April 15–16; A. H. Taub, Chicago, Illinois, April 22–23; Irving Kaplansky, Stanford, California, April 30; I. Barsotti, Vancouver, British Columbia, Canada, June 18; Raoul Bott, Ann Arbor, Michigan, Summer Meeting 1955; Edwin Hewitt, Ann Arbor, Michigan, Summer Meeting 1955; and Sze-Tsen Hu, Knoxville, Tennessee, November 18–19.

The following items were reported for the information of the Council: selection of W. T. Martin as Managing Editor of the Bulletin Editorial Committee for 1955; A. C. Schaeffer as Managing Editor of the Proceedings Editorial Committee for 1955; L. V. Ahlfors as Managing Editor of the Transactions and Memoirs Editorial Committee for 1955; Einar Hille as Chairman of the Mathematical Reviews Editorial Committee for 1955; Leo Zippin as Chairman of the Mathematical Surveys Editorial Committee for 1955; Salomon Bochner as Chairman of the Colloquium Editorial Committee for 1955; C. J. Rees as Chairman of the Committee on

Printing and Publishing for 1955; and Dr. L. A. MacColl as Editor of the Proceedings of the Applied Mathematics Symposium held at Brooklyn Polytechnic Institute, April 14–15, 1955.

The following appointments to represent the Society were announced: Professor J. F. Randolph at the inauguration of Clifford Cook Furnas as Chancellor of the University of Buffalo on January 7, 1955; Professor Wilfred Kaplan at the ceremonies commemorating the One Hundredth Anniversary of the founding of Michigan State College on February 12, 1955; Dr. Rufus Oldenburger at the inauguration of Leland Henry Carlson as President of Rockford College on February 27, 1955; Professor Paul A. White at the inauguration of Clark George Kuebler as Provost of Santa Barbara College on March 28, 1955.

The following actions taken by mail vote of the Council were reported: election of Professors Deane Montgomery and D. V. Widder to membership on the Executive Committee of the Council; election of the University of Alberta, Edmonton, Alberta, Canada, to institutional membership; approval of a recommendation of the Committee on Translations that the Translations Project be continued at the present level.

The Council voted to set meetings at Stanford University on April 30, 1955, and at the U. S. Naval Postgraduate School, Monterey, California, April 28, 1956, and voted to change the date of the October Meeting at the University of Maryland from October 29 to October 22, 1955.

The Council voted to approve the inclusion of the Society for Industrial and Applied Mathematics in the Combined List of Members.

The Council voted to discharge the Editorial Committee for Applied Mathematics Symposia Proceedings and to transfer the duties of this committee to the Applied Mathematics Committee.

The Council voted that the representatives of the Society on the Policy Committee for Mathematics shall be ex officio: the President, the President Elect or Ex-President as the case may be, the Secretary, and one of the Vice Presidents to be named each year by the President. The present representatives of the Society shall serve out their terms and the ex officio members shall be seated in the order named.

The Secretary reported the resignation of Professor J. W. Green as Associate Secretary for the Far West effective April 30, 1955. The Council elected Professor V. L. Klee as a replacement for Professor Green.

The Secretary reported that the Society for Industrial and Applied

Mathematics had contributed $100 toward the operation of the Employment Register. The Council voted to increase the membership of the Employment Register Committee from five to seven, two of the members to represent the Society for Industrial and Applied Mathematics.

The Council voted to approve an amendment to the by-laws increasing the membership of the Bulletin Editorial Committee from two to three.

The Council voted that it was the consensus of the Council that members of the Society should receive the Bulletin and Notices as a privilege of membership and that they should support the publication program of the Society to at least the extent they now do by subscribing for the Proceedings. The President was requested to appoint a committee to present a plan to implement this motion.

The Council approved a report from the Committee on Subsidies to Journals. Among the recommendations of this committee were the following: requests for new subsidies or for increases in present subsidies shall be addressed to the Secretary of the Society who will refer the request to the Committee on Printing and Publishing. This committee, after reviewing the request, will present its recommendation to the Council and Trustees for action; that subsidies to journals shall be computed on the basis of all papers published in the journal by individual members of the Society; that institutional dues are not to be considered as re-imbursement to the Society for publication costs but are to be based as closely as possible on the mathematical research activity on the campus of the institution.

The Council voted to write to the Senate Judiciary Committee in support of a personal bill which would exempt Professor Hermann Weyl from a provision of the McCarran-Walter Act which might cause the loss of his American citizenship.

Abstracts of the contributed papers follow. Those with "*t*" after the abstract number were presented by title. In the case of joint papers, the name of the author who read the paper is followed by (p). Dr. Stoll was introduced by Professor W. H. Gottschalk, Dr. Reiner by Professor Irving Reiner, and Dr. Kreyszig by Professor Stefan Bergman.

ALGEBRA AND THEORY OF NUMBERS

441*t*. Armand Borel and G. D. Mostow: *On semi-simple automorphisms of Lie algebras.*

This paper investigates the existence of fixed points and invariant Cartan subalgebras under automorphisms of (finite-dimensional) Lie algebras over an arbitrary field of characteristic 0. It is shown that a semi-simple (i.e. with simple elementary divisors) automorphism of a nonsolvable Lie algebra L leaves fixed an element x with non-nilpotent ad x, and which is regular if L is semi-simple. As to the second question, the main result is: Let G be a group of semi-simple automorphisms of a Lie algebra L. Then G leaves a Cartan subalgebra invariant when G and L are solvable or when L is arbitrary and G has type (MP)*. Here, a linear group G is said to be of type (MP)* if it contains an increasing sequence of *invariant* subgroups (G_i) $((0 \leq i \leq s), G_0 = \{e\}, G_s = G)$, such that G_{i+1} is generated by G_i and by an abelian subgroup H_i which either is cyclic or has a connected algebraic hull $(0 \leq i < s)$. This result generalizes a theorem of J-P. Serre and one of the authors (Comment Math. Helv. vol. 27 (1953) pp. 128–139). It is also shown that an automorphism of a semi-simple Lie algebra leaving a Cartan subalgebra invariant must be semi-simple. (Received February 7, 1955.)

442. Abraham Brind: *The summation which forms the basis for proving both Fermat theorems.*

The author proves by mathematical induction: If y and n are two arbitrary positive integers, then $y^n = \pm y + \sum_{r=1}^{y-1} \sum_{s=1}^{n-1} (-1)^{s-1} C_{n,s} r^{n-s}$, where the plus sign must be taken when n is odd and the minus sign when n is even. He then derives two important particular cases, one for $n = p$, any odd prime, and the other for $n = 2$, the only even prime. In the latter case he establishes another summation distinct from the first both in the matter of the limits of summation and in the sign of y. The application of the above to the so-called Fermat little theorem is shown to be immediate. But the author can only allude to the somewhat lengthy proof of Fermat's last theorem which he hopes to have ready for publication in the near future. The infinite descent method is not used at all. Nor is unique decomposition used in the proof of the odd (prime) case. A more basic property of integers is used to distinguish them from irrationals (the necessity of using complex numbers and fractional rational numbers is easily eliminated): the inductive property of positive integers. The proof is effected by comparing a summation derived from the Fermat equation with the above. Unique decomposition is used in the complete solution of the Pythagorean case (in which, by the way, a very significant analytical meaning of some expressions is pointed out) and in the proof of the impossibility of the equation when $n = 4$. (Received March 1, 1955.)

443t. Leonard Carlitz: *Note on the class number of real quadratic fields.*

Let d denote the discriminant and $h(d)$ the class number of the real quadratic field $R(d^{1/2})$; also let $\epsilon = (t + ud^{1/2})/2$ denote the fundamental unit of the field, $\epsilon > 1$. For $d = p$, p prime > 2, Ankeny, Artin, and Chowla (Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951) pp. 524–527) have stated the formula $2uh(p)/t \equiv (A + B)/p \pmod{p}$, where A is the product of the quadratic residues of p and B is the product of the nonresidues in the interval $1, p-1$. In the present paper the following more general result is proved. Let $d = pm > 0$, $p > m > 1$ and put $A = \prod_1^d a^{(m_0/a)}$, $B = \prod_1^d b^{(m_0/b)}$, where in the first product $(a/p) = +1$, while in the second $(b/p) = -1$, and $m_0 = (-1)^{(p-1)/2} m$. Then $2muh(d)/t \equiv (B - A)/p \pmod{p}$. (Received February 25, 1955.)

444t. Leonard Carlitz: *On the representation of an integer as the sum of twenty-four squares.*

Making use of the methods developed in earlier papers, van der Pol (Proc. Neder. Akad. Wetensch. Ser. A57 = Indagationes Math. vol. 16 (1954) pp. 349–361) has given an elegant proof of Ramanujan's formula for the number of representations of an integer as the sum of twenty-four squares. The present paper contains a short proof of the formula which assumes only certain standard results from the theory of elliptic functions. (Received February 25, 1955.)

445*t*. Leonard Carlitz and J. H. Hodges: *Distribution of matrices in a finite field with respect to characteristic polynomial and invariant factors.*

In this paper the distribution of square matrices, with elements in $GF(q)$, having certain properties is studied by examining the conditions imposed by these properties on the characteristic polynomials of the matrices. The number of nonderogatory matrices of given order is found in terms of the number of nonsingular matrices of that order and the Euler function for $GF[q, x]$, the polynomial ring in x over $GF(q)$. Matrices A and B are said to be in the same class if and only if A is similar to B. The characteristic polynomial of a matrix A of order m is $F(\lambda) = H_1(\lambda)H_2(\lambda) \cdots H_m(\lambda)$, where $H_{i+1} | H_i$ and the H_i are the invariant factors of $\lambda I - A$. The number of classes of order m having a given $F(\lambda)$ as characteristic polynomial, the total number of classes of order m, and the number of classes of order m having $H_1(\lambda)$ of degree r are all found in terms of certain types of partitions of positive integers. (Received February 25, 1955.)

446*t*. Eckford Cohen: *A class of arithmetical functions.*

Let r be an integer >1 and suppose $f(n)$ to be an arithmetical function of period r. Further place $(n, r) = k$; then $f(n)$ is defined to be an *even* function (mod r) provided $f(n) = f(k)$ for all n. The following theorem is proved: *An arithmetical function $f(n)$ of period r is even if and only if it is expressible in the form $\sum_{d|r} \alpha(d)c(n, d)$ where $c(n, d)$ is Ramanujan's sum.* The proof of this result is based on an orthogonality property of $c(n, d)$ due to the author (Duke Math. J. vol. 19 (1952) p. 120, (3.10)). It is also shown that the even functions (mod r) include the class of functions introduced by Anderson and Apostol (Duke Math. J. vol. 20 (1953) pp. 211–216). Using the above theorem, inversion formulas for even functions are obtained. In special cases these formulas reduce to results of Nicol and Vandiver (Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 825–835, Theorems V and VI). A new proof of Theorem VII of Nicol and Vandiver on restricted partitions (mod r) is also given. (Received March 31, 1955.)

447*t*. Karl Goldberg: *Bounds and asymptotic behavior of coefficients in the series* log $e^x e^y$.

In the formal power series for log $e^x e^y$ when x and y do not commute let $c_m(s_1, \cdots, s_m)$ denote the coefficient of the term $x^{s_1}y^{s_2} \cdots (x, y)^{s_m}$ with $s_i \geqq 1$ for all i. The author proves $|c_m(s_1, \cdots, s_m)| \leqq 1/2^d M$ with equality when d vanishes where $d = \sum_{i=1}^{m} (s_i - 1)$ and $M = m!/[m/2]![(m-1)/2]!$ with $[a]$ denoting the largest integer in a. It is conjectured that the constant 2 in the inequality can be replaced by the constant π. Furthermore let $C(m, d) = \max |c_m(s_1, \cdots, s_m)|$ where the s_i range over all values such that $\sum_{i=1}^{m} (s_i - 1) = d$. The author proves that for a fixed d: $\lim_{m\to\infty} MC(m, d) = 4(2^{d+2} - 1)|B_{d+2}|/(d+2)!$ if d is even and $\lim_{m\to\infty} mMC(m, d) = 8(d+2)(2^{d+3} - 1)$

$\cdot |B_{d+3}|/(d+3)|$ if d is odd where B_i is the ith Bernoulli number in the usual notation $(B_1 = -1/2)$. These theorems also hold for the coefficients $c_y(s_1, \cdots, s_m)$ of terms starting with a positive power of y since $|c_w(s_1, \cdots, s_m)| = |c_y(s_1, \cdots, s_m)|$. (Received February 17, 1955.)

448. Emilie V. Haynsworth: *Characteristic roots of certain matrices.* Preliminary report.

As one result of a recent paper by the author (*Quasi-stochastic matrices*, to be published), it was shown that any $n \times n$ matrix $A = (a_{ij})$, with $\sum_{j=1}^{k} a_{ij} = r_{11}$, $\sum_{j=k+1}^{n} a_{ij} = r_{12}$ $(i = 1, \cdots, k)$, $\sum_{j=1}^{k} a_{ij} = r_{21}$, and $\sum_{j=k+1}^{n} a_{ij} = r_{22}$ $(i = k+1, \cdots, n)$, has two roots which are roots of the matrix $R = (r_{ij})$ $(i, j = 1, 2)$. In this paper the result is generalized to matrices which can be partitioned into m^2 submatrices, $m < n$, with row-sums, r_{ij} $(i, j = 1, \cdots, m)$. Such matrices have as m of their roots the roots of the matrix $R = (r_{ij})$. It is also shown that the other $n - m$ roots of A are roots of an $(n-m)^2$ matrix whose elements can be found by inspection from the elements of A. These results are proved by using theorems from a paper by L. S. Goddard (*An extension of a matrix theorem by A. Brauer*, not yet published). (Received January 14, 1955.)

449. K. A. Hirsch: *On locally nilpotent groups.*

On another occasion (International Congress of Mathematicians, Cambridge, 1950; Mathematische Nachrichten vol. 3 (1951)) the author has proved: *Theorem* 1: A group with maximal condition for subgroups in which every proper subgroup differs from its normaliser is nilpotent. It is plausible to ask whether the maximal condition can be weakened or omitted altogether, provided the conclusion is weakened correspondingly. In this direction B. I. Plotkin (Doklady Akad. Nauk USSR vol. 76 (1951); see also Kuroš, *Teoriya Grupp*, 2d ed., Moscow, 1953, §63) has proved: *Theorem* 2: A group in which every proper subgroup differs from its normaliser is locally nilpotent. It will now be shown that a stronger form of the arguments that have been used in the proof of Theorem 1 is sufficient to give a simple proof of Theorem 2. For this purpose a theorem that can claim interest on its own account is required. *Theorem* 3: Every group G possesses a unique maximal locally nilpotent normal subgroup that contains all locally nilpotent normal subgroups of G. Theorem 3 is a consequence of: *Theorem* 4: The union of two locally nilpotent normal subgroups of an arbitrary group G is itself a locally nilpotent normal subgroup of G. A full account will appear elsewhere (Schur memorial volume of Math. Zeit.). (Received February 25, 1955.)

450t. J. H. Hodges: *Explicit values and applications of certain exponential sums in a finite field.*

In *Representations by quadratic forms in a finite field* (Duke Math. J. vol. 21 (1954) pp. 123–138) and *Representations by skew forms in a finite field* (Archiv der Mathematik vol. 5 (1954) pp. 19–31), L. Carlitz has defined certain exponential sums for symmetric and skew matrices, respectively, over a finite field. The present paper is concerned with finding the explicit values of these sums and with applications of the sums to the solution of a number of matric problems. For example, the number of ways to partition a given symmetric matrix A of order t into the sum of n $(n > 1)$ symmetric matrices of order t having assigned ranks and invariants is found. (Received February 28, 1955.)

451*t*. M. D. Marcus: *An eigenvalue inequality for the product of normal matrices.*

If A and B are n-square normal matrices with eigenvalues $\lambda_i(A)$ and $\lambda_i(B)$, $i=1, \cdots, n$, then it is proved that $\min_p \operatorname{Re} \sum_{i=1}^{n} \lambda_i(A)\lambda_{p(i)}(B) \leq \operatorname{Re} \sum_{i=1}^{n} \lambda_i(AB) \leq \max_p \operatorname{Re} \sum_{i=1}^{n} \lambda_i(A)\lambda_{p(i)}(B)$ where p runs over all permutations of the integers $1, 2, \cdots, n$. (Received February 21, 1955.)

452*t*. M. D. Marcus: *A note on matrix inequalities.*

Let $\lambda(A)$, $\lambda_m(A)$, and $\lambda_M(A)$ denote an arbitrary eigenvalue of A, the minimum, and the maximum eigenvalues of A respectively where A is an arbitrary n-square matrix. Let $\|x\|^2 = \sum_{i=1}^{n} |x_i|^2$. Results: (i) If C and D are arbitrary, $|\lambda(CD)|^2 \leq \lambda_M(CC^*)\lambda_M(DD^*)$. If there exist x_0 and x_1, unit vectors minimizing $\|C^*CDD^*x\|$ and $\|DD^*C^*Cx\|$ respectively, such that $\|DD^*x_0\| \neq 0$ or $\|C^*Cx_1\| \neq 0$ then $|\lambda(CD)|^2 \geq \lambda_m(CC^*)\lambda_m(DD^*)$. (Compare S. N. Roy, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 635–638.) (ii) Let $C = A + iB$ be Hermitian and $i_1 \leq i_2 \leq \cdots \leq i_k$, $k \leq n$ be integers. Then $k\lambda_m(C) \leq \sum_{\alpha,\beta=1}^{k} A_{i_\alpha i_\beta} \leq k\lambda_M(C)$. (iii) If C is arbitrary then it is proved that $k\lambda_m(CC^*) \leq \sum_{\alpha,\beta=1}^{k} \sum_{j=1}^{n} C_{i_\alpha j}\bar{C}_{i_\beta j} \leq k\lambda_M(CC^*)$, extending a result of W. V. Parker (Duke Math. J. vol. 15 (1948) pp. 711–715). (Received February 21, 1955.)

453*t*. G. D. Mostow: *Fully reducible subgroups of algebraic groups.*

Let G be an algebraic matric group over an arbitrary field of characteristic zero. Let $\mathcal{G}$ denote its Lie algebra and $\mathcal{N}$ the maximum ideal of nilpotent matrices. Denote by N the corresponding connected algebraic subgroup of G. *Theorem.* If M is any maximal fully reducible subgroup of G, then $G = M \cdot N$ semi-directly. The maximal fully reducible subgroups of G are conjugate under an inner automorphism. *Theorem.* Let E be the associative enveloping algebra of G. There is a Wedderburn decomposition $S+T$ for E with (a) $S \cap G$ maximal fully reducible in G; (b) $(I+T) \cap G = N$, I = identity; and (c) $T \cap \mathcal{G} = \mathcal{N}$. *Theorem.* A fully reducible group of automorphisms of a Lie (resp. associative or Jordan) algebra $\mathcal{G}$ keeps invariant a maximal semi-simple subalgebra; it keeps invariant a Cartan subalgebra if G is solvable. An intermediate result is: *Theorem.* A rational representation carries fully reducible groups into fully reducible groups. A corollary of this is: The first main theorem of the theory of invariants holds for fully reducible groups. (Received March 1, 1955.)

454*t*. G. D. Mostow: *Self-adjoint groups.*

Theorem. Let $G_1, G_2, \cdots, G_n$ be a nested sequence of fully reducible algebraic groups of automorphisms of a real linear space V. Then a positive definite inner product can be introduced on V with respect to which $G_1, G_2, \cdots, G_n$ are self-adjoint. The proof utilizes the notion of "invariant compact form" which is defined for groups which are not necessarily connected. The basic steps in the proof are: (1) A fully reducible real algebraic group has an invariant compact form; (2) If $F \subset G$ and each has an invariant compact form, then any invariant compact form for F can be extended to an invariant compact form for G. Among the intermediate results: *Theorem.* A Lie group with a finite number of connected components is topologically the direct product of a compact subgroup K and a euclidean subspace E with $kEk^{-1} = E$ for k in K. All maximal compact subgroups are conjugate under inner automorphisms. A corollary of the main theorem for the case $n=1$ is: If H is a closed connected subgroup of a Lie group G with $(\mathrm{Ad}\ H)_{G/R}$ fully reducible, then G/H admits a covariant fibering (R denoting the radical). (Received March 1, 1955.)

455t. Morris Newman and Olga Taussky: *Unimodular circulants of rational integers.*

It is shown that there are nontrivial unimodular $n\times n$ circulants with rational integers as elements for all n except 2, 3, 4, 6 (a circulant is called trivial if it contains only one nonzero element in each row). If n is a power of a prime and such a circulant is of the form AA' (A' the transpose of A) where A is an $n\times n$ matrix with rational integers as elements, then A is itself the product of a circulant, a permutation matrix, and a diagonal matrix consisting of ± 1's. These results are applied in a study of the normal basis of algebraic number fields. (Received February 10, 1955.)

456t. Irma M. Reiner: *On the two-adic density of representations by quadratic forms.*

The problem of determining $A_q(S, T)$, the number of solutions of $X'SX=T$ (mod q), where S and T are symmetric integral matrices, has been considered by C. L. Siegel (Ann. of Math. vol. 36 (1935) pp. 527–606). He obtained explicit formulas for $A_q(S, T)$ when $q=p^a$, where p is a prime not dividing $2|S||T|$. He further showed that in the case where $|S||T|$ is odd, the determination of $A_r(S, T)$ for $r=2^b$, $b\geqq 3$, reduces immediately to that of finding $A_8(S, T)$. In this paper formulas are found for $A_2(S, T)$ and $A_8(S, T)$, assuming that $|S||T|$ is odd. A combinatorial argument is used when $q=2$, and exponential sums are employed when $q=8$. The proof uses 2-adic canonical forms of B. W. Jones and Gordon Pall (Duke Math. J. vol. 11 (1944) pp. 715–727). (Received February 21, 1955.)

457t. E. V. Schenkman: *Groups with Abelian derived groups.*

The author constructs a universal semigroup S and gives its multiplication table explicitly so that if G is a group generated by elements of finite order and if G has Abelian commutator subgroup, then G is a homomorphic image of S. In particular he gives a presentation and multiplication table for the free product of finite cyclic groups modulo the second derived group of the free product. (Received February 24, 1955.)

458t. Ernst Snapper: *Higher-dimensional field theory.* I. *The integral closure of a module.*

Let E/F be a finitely generated field extension, i.e., E and F are commutative fields and $E=F(a_1, \cdots, a_n)$, where $a_1, \cdots, a_n$ belong to E. The present paper is the first of a series of three articles which have as purpose to develop an intrinsic theory for E/F, by giving a purely field-theoretic treatment of the theory of linear systems of algebraic varieties. Let L denote a finitely generated module of E/F. See the author's paper *Integral closure of modules and complete linear systems*, Bull. Amer. Math. Soc. Abstract 61-3-399, for the notion of the integral closure $|L|_i$ of L and the fact that $|L|_i$ is finitely generated. In the present paper, the theory of $|L|_i$ is further developed. In particular, it is proved that there exists an integer k_0 such that, if $k\geqq k_0$, $|L^{k+s}|_i=|L^k|_iL^s$ for all $s\geqq 0$. This fact constitutes the main theorem concerning integral closures of modules and will enable us in the subsequent articles to give a purely field-theoretic treatment of Zariski's theory of normalization. (Received March 4, 1955.)

ANALYSIS

459. Joseph Andrushkiw: *On the minimum maximorum and maximum minimorum of a polynomial whose all zeros are real.*

Let $s_1, s_2, \cdots, s_n$ denote the zeros of the polynomial $f(s)=1+a_1s+\cdots+a_ns^n$, $s_1^*, s_2^*, \cdots, s_n^*$ the zeros of the polynomial $f(s; x)=f(s)-x$, $s_{-k}=\sum_{j=1}^{n} s_j^{-k}$, $s_{-k}^*=\sum_{j=1}^{n} s_j^{*-k}$, D_k and $D_k(x)$ the determinants $|s_{-(i+j)}|$, $|s^*_{-(i+j)}|$, $i, j=1, 2, \cdots, k$; $k=1, 2, \cdots, n$, respectively. $D_n(x)$ expressed in terms of coefficients of $f(s; x)$ is a polynomial of $n-1$st degree in x and equals the discriminant of $f(s; x)$ multiplied by a factor (author's paper, Bull. Amer. Math. Soc. Abstract 58-3-289). It follows that the zeros of $D_n(x)$ are $f(\alpha_k)$ where α_k are the zeros of the derivative $f'(s)$. If all zeros of $f(s)$ are real and distinct, the greatest negative zero N of $D_n(x)$ represents maximum minimorum of $f(s)$ and the smallest positive P minimum maximorum. Since the zeros of a polynomial are continuous functions of its coefficients, it follows that $x_1<N<P<x_2$ where x_1 and x_2 are zeros of the quadratic equation $D_2(x)=0$. Cauchy's (M. Marden: *The geometry of zeros of a polynomial*, p. 98, ex. 2) and Hadamard's theorems together with the above result yield the inequality: $-(c+(c^2+4ab)^{1/2})/2b<N<-D_n\cdot[D_n+C(n, p)(n^2+n)^{n/2}M^{2n-1}]^{-1}<0<D_n[D_n+C(n, p)(n^2+n)^{n/2}M^{2n-1}]^{-1}<P<(-c+(c^2+4ab)^{1/2})/2b$, where $a>0$, $b>0$, and c are assumed to be polynomials in a_1, a_2, a_3, a_4, $p=[(n+1)/2]$ and $M=\max(1, ka_k)$, $k=1, 2, \cdots, n$. (Received February 7, 1955.)

460t. Stefan Bergman: *Bounds for the stream function of a compressible fluid.*

The author considers the stream function $\psi(\mathrm{H}, \theta)$ of compressible fluids in the H, θ-plane. (Here H is a conveniently chosen function of the Mach number and θ the angle which the velocity vector forms with the positive x-direction. (See Amer. J. Math. vol. 70, p. 859 ff., (2.1), (2.2).) ψ satisfies the equation $l(\mathrm{H})\psi_{\theta\theta}+\psi_{\mathrm{HH}}=0$, where $l(\mathrm{H})>0$, $\mathrm{H}<0$ in the subsonic region, $l(\mathrm{H})<0$, $\mathrm{H}>0$ in the supersonic region. The author assumes that $a_1^2\mathrm{H}^P<|l(\mathrm{H})|<a_2^2\mathrm{H}^P$, where $0<a_1<a_2<\infty$, P is an integer, and, for $\mathrm{H}<0$, P is odd. If $\psi(0, \theta)=\sum_{m=0}^{P} C_m\theta^m$, $\psi_{\mathrm{H}}(\mathrm{H}, \theta)_{\mathrm{H}=0}=0$, $\psi_{\mathrm{H}}=\partial\psi/\partial\mathrm{H}$, then $\psi(\mathrm{H}, \theta)\leqq\sum_{m=0}^{P}\sum_{\mu=1}^{2}(-1)^\mu((-1)^\mu C_m\theta^m)^+F[-m/2, (-m+1)/2, (P+1)/(P+2); 4a_\mu^2|\mathrm{H}|^{P+2}\cdot(P+2)^{-2}\theta^{-2}]$ where F is the hypergeometric function; $a^+=a$ if $a\geqq 0$, $a^+=0$ if $a<0$. Replacing $(-1)^\mu$ by $(-1)^{\mu+1}$ yields a lower bound. If $\psi(0, \theta)=0$, $\psi_{\mathrm{H}}(\mathrm{H}, \theta)_{\mathrm{H}=0}=\sum_{m=1}^{P} D_m\theta^m$, then it is proved under analogous assumptions for $\mathrm{H}>0$ that $\psi_\theta(\mathrm{H}, \theta)\leqq\sum_{m=1}^{P}\sum_{\mu=1}^{2}(-1)^{\mu+1}((-1)^{\mu+1}mD_m\theta^{m-1}\mathrm{H})^+F[(-m+1)/2, -m/2+1, (P+3)/(P+2); 4a_{3-\mu}^2\mathrm{H}^{P+2}(P+2)^{-2}\theta^{-2}]$. A similar formula holds for a lower bound, as well as for $\mathrm{H}<0$. (Received April 11, 1955.)

461t. H. S. Collins: *An extension theorem for completely regular spaces.*

Let E be a completely regular T_1 topological space, A and M closed subsets, and denote by $C(E)$ [by $C_E(M)$] the set of real continuous functions on E [the set of real continuous functions on M, each of which has a continuous extension over E]. In what follows, statements involving a uniformity refer to the weakest uniform structure on E for which all $f\in C(E)$ are uniformly continuous. Theorem. If A is totally bounded and f real continuous on M is such that $|f(t)|<1$, all $t\in A\cap M$, the following two conditions together are sufficient that there be $g\in C(E)$, an extension of f, such that $|g(t)|<1$, all $t\in A$: (1) $f\in C_E(M)$, and (2) either A or M is complete. If $\mathfrak{A}$ is a non-void family of subsets of E, endow $C(E)$ [$C_E(M)$] with the topology of uniform convergence on the members of $\mathfrak{A}$ [of $\{A \cap M : A \in \mathfrak{A}\}$]. Define T, for $f \in C(E)$, by $T(f) = f$ restricted to M. Corollary. T is open onto $C_{\mathfrak{B}}(M)$ if either (1) $\mathfrak{A}$ = all compact sets, or (2) $\mathfrak{A}$ = all totally bounded sets and M is complete or each closed $A \in \mathfrak{A}$ is complete. (Received February 18, 1955.)

462*t*. Albert Edrei: *On the zeros of successive derivatives.*

Consider an analytic function $f(z)$, regular at z_0, and assume (i) that if $f(z)$ is entire, it is not of exponential type; (ii) that z_0 is not a limit point of the zeros of the successive derivatives of $f(z)$. Then, either the sequence (1) $\{|f^{(n+1)}(z_0)/f^{(n)}(z_0)|\}$ tends to infinity or else, for all sufficiently large values of the positive number P, every interval $[P, PQ]$ contains a limit point of (1). The positive quantity Q is an absolute constant. It is possible to deduce, from the above result, that if $f(z)$ is a real entire function which satisfies a linear differential equation whose coefficients are rational functions, and if $f(z)$ is bounded on the real axis, then every point of the real axis is a limit point of zeros of the successive derivatives of $f(z)$. (Received March 2, 1955.)

463. B. A. Fleishman: *Periodic solutions of a nonlinear wave equation.* II.

Let $Lu = u_{xx} - u_{tt} - 2\kappa u_t - \alpha u$, with κ and α positive constants. Let $b(x, t)$ be such that $b(0, t) = 0 = b(\pi, t)$ and $b(x, t+p) = b(x, t)$. It was shown by the writer (Bull. Amer. Math. Soc. Abstract 59-4-365) that the initial value problem $Lu = b + \epsilon u^3$ (ϵ constant), $u(0, t) = 0 = u(\pi, t)$, $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$ has a solution $u(f, g; x, t)$, and that f and g can be so chosen that u is p-periodic in t: $u(f, g; x, t+p) = u(f, g; x, t)$, under the conditions that ϵ and $|\alpha - \kappa^2|$ be sufficiently small and, for given p, that κ be sufficiently large (or, for given κ, that p be sufficiently large). By using a more efficient integral operator, which inverts Lu rather than just $u_{xx} - u_{tt}$, and improving the proof in other respects, F. A. Ficken (Bull. Amer. Math. Soc. Abstract 60-2-225) obtained similar results without restriction on $|\alpha - \kappa^2|$. A detailed consideration of Ficken's result as applied to all multiples of the smallest period of $b(x, t)$, which themselves are periods, reveals that no requirements need be imposed on α, κ, and p except that they be positive. This work was supported by the Bureau of Ordnance, Department of the Navy, under Contract NOrd 7386. (Received January 13, 1955.)

464*t*. K. O. Friedrichs: *On differential forms.*

A theory of differential forms and harmonic fields on Riemannian manifolds, compact or with boundary, is developed, the operators d and δ being treated directly by Hilbert space methods. The major tool is the non-invariant "complete" Dirichlet integral $D_e(v)$, essentially the integral of the sum of the squares of all first derivatives. Gaffney's inequality combined with the identity of the weak and strong extensions of d and δ yields: If v admits d and δ weakly, $D_e(v)$ is finite and D_e is finite for the exact, harmonic, and co-exact "components" of v. The complete continuity of $\int v * v$ with respect to $D_e(v)$ leads to the inequality $\int v * v \leq c\int dv * dv + c\int \delta v * \delta v$ and then to the existence of the solution of $d\delta v = df$, $\delta dv = \delta g$. Further it is proved: if dv and δv are bounded, the form v is continuous and so are its three "components." In this approach the metric tensor is only required to have bounded first derivatives. For this reason the reduction by doubling of the theory for manifolds with boundaries to that for compact ones, described by Conner, can be carried out without specializing the metric tensor. Among the results for manifolds with smooth boundaries is the inequality $D_e(v) \leq c\int dv * dv + c\int \delta v * \delta v$ for any form v which is orthogonal to all harmonic fields. The methods differ from those employed by Morrey [Bull. Amer. Math. Soc. Abstracts 61-3-416] in a similar approach. (Received March 23, 1955.)

465. F. W. Gehring: *Maximal theorems for solutions of the heat equation.*

A well known theorem of Hardy and Littlewood (Acta Math. vol. 54) states that if $u(z)$ is harmonic in $|z|<1$, if $p>1$, and if $\int_{-\pi}^{\pi}|u(\rho e^{i\theta})|^p d\theta\leq 1$ for $0\leq\rho<1$, then $\int_{-\pi}^{\pi}\mathrm{LUB}_\rho|u(\rho e^{i\theta})|^p d\theta\leq k(p)$. Similar results were obtained for functions harmonic in the half plane by Tims (J. London Math. Soc. vol. 27). Suppose that $u(x, t)$ is C^2 and that it satisfies the heat equation $u_{xx}=u_t$ in the half-plane $t>0$. Theorems similar to the following are proved. (I) If $p>1$ and if $\int_{-\infty}^{\infty}|u(x, t)|^p dx\leq 1$ for $t>0$, then $\int_{-\infty}^{\infty}\mathrm{LUB}_t\ |u(x, t)|^p dx\leq k(p)$. (II) If $p>1$ and if $(1/2X)\int_{-X}^{X}|u(x, t)|^p dx\leq 1$ for X, $t>0$, then $(1/2X)\int_{-X}^{X}\mathrm{LUB}_t|u(x, t)|^p dx\leq k(p)$ for $X>0$. (Received March 2, 1955.)

466. J. L. Howell: *Nonoscillation and integrability of solutions of a class of fourth-order differential equations.*

Let $b(x)$ be positive and continuous on $(-\infty, \infty)$ and let λ be positive. Then a necessary condition that $(by'')''+\lambda y=0$ have a solution nonoscillatory as $x\rightarrow+\infty$ is that $x^2[b(x)]^{-1}$ be in $L(0, \infty)$. A sufficient condition that every solution be nonoscillatory for large x is that $x^\sigma\int_x^\infty s^2[b(s)]^{-1}ds$ be bounded for some fixed $\sigma>1$. The necessary condition follows from consideration of the possible varieties of behavior of $(by'')'$ and is somewhat stronger than a similar condition deducible from a result of W. M. Whyburn (Amer. J. Math. vol. 52 (1930) pp. 176–178). The sufficient condition is obtained indirectly by iterating estimates of a supposed oscillatory solution. The equation has at most two solutions in $L(0, \infty)$ and at most two solutions in $L(-\infty, 0)$ but no solution in $L(-\infty, \infty)$. The same is proved with L_2 in place of L; then from a result of Glazman (Amer. Math. Soc. Translation No. 96, p. 17) it follows that there are exactly two solutions in $L_2(-\infty, 0)$ and exactly two in $L_2(0, \infty)$. (Received March 2, 1955.)

467. H. K. Hughes and Ralph H. Niemann (p): *A study of Stieltjes integral transforms of a certain class of functions.*

An extension of the well known factorial series $\sum_{n=0}^{\infty}a_n n!\Gamma(s)/\Gamma(s+n+1)$ to the factorial transform $\int_0^\infty B(s, t+1)d\alpha(t)$ was considered by Fraser (Duke Math. J. vol. 11 (1944) pp. 469–486). Carmichael (Trans. Amer. Math. Soc. vol. 17 (1916) pp. 207–232) found that the factorial series was a special case of a more general class of series, namely, $\sum_{n=0}^{\infty}c_n g(s+n)/g(s)$. An extension of this last series to the Stieltjes integral $\int^\infty[g(s+t)/g(s)]dc(t)$ is considered here. The function $g(s)$ is assumed to be analytic and single-valued in a sector of the complex plane and for values of s of large modulus in this sector, $g(s)$ has the asymptotic representation $g(s)\sim s^{P(s)}e^{Q(s)}[1+\sum_{n=1}^{\infty}a_n/s^n]$ where $P(s)$ and $Q(s)$ are polynomials. The function $c(t)$ is assumed to be of bounded variation in every finite interval $(0, T)$. The region of convergence and of absolute convergence of the integral is in general a half-plane. Expressions for the boundary lines of these half-planes are found in terms of the functions $c(t)$ and $g(t)$. The integral converges uniformly in every bounded closed region in the interior of the half-plane of convergence and hence represents an analytic function there. Certain points may have to be excluded. An analytic continuation of the function defined by the integral is obtained when the function $c(t)$ is analytic. (Received March 7, 1955.)

468t. Ernest Ikenberry: *A system of homogeneous spherical harmonics.*

There is given a method for obtaining a basic set of h. s. h.'s: $Y_i = x_i$, $Y_{ij} = x_ix_j - x^2\delta_{ij}/3$, $Y_{ijk} = x_ix_jx_k - 3x^2Y_{(i}\delta_{jk)}/5$, $Y_{ijkl} = x_ix_jx_kx_l - 6x^2Y_{(ij}\delta_{kl)}/7 - x^4\delta_{(ij}\delta_{kl)}/5$, etc., where $x^2 = x_1^2 + x_2^2 + x_3^2$ and where i, j and k may each be assigned any of the values 1, 2 or 3. The parentheses around s subscripts mean to sum over the $s!$ permutations of the subscripts and divide by $s!$. The method gives a complete but not a minimal set of h. s. h.'s of any desired degree. Each h. s. h. of the set has the property that the contracted sum over any pair of indices is zero. The following contraction results are also noted: $x_iY_i = x^2$, $x_iY_{ij} = 2x^2Y_j/3$, $x_iY_{ijk} = 3x^2Y_{jk}/5$, $x_iY_{ijkl} = 4x^2Y_{jkl}/7$, etc. (To be published in the American Mathematical Monthly.) (Received March 3, 1955.)

469t. W. B. Jurkat: *An extension problem for functions with monotonic derivatives.*

Let $F(x)$ be the nth integral of a positive nondecreasing function for all large positive x; the problem to be solved is to find a function $f(x)$, being the nth integral of a nondecreasing function for all x, with the property $f(x) = F(x)$ for all large positive x, $f(x) = 0$ for all large negative x. A complete solution of this and connected boundary value problems is given by means of the theory of moment problems. The main result of the paper consists of determining the behaviour of certain quadratic forms, on which the solution depends. (Received January 12, 1955.)

470. Tosio Kato: *Note on Friedrichs extensions.*

The following theorem will be proved. Let J_1, J_2 be two closed quadratic forms bounded from below with dense domains in a Hilbert space R. Let $J_1 \subset J_2$ and let $m = \dim$ (Domain J_2/Domain J_1) $< \infty$. Let H_1, H_2 be the self-adjoint operators associated with J_1, J_2 respectively. Then the intersection D of the domains of H_1 and H_2 is dense in R and H_1 and H_2 coincide in D. The common restriction H_0 of H_1 and of H_2 to domain D is a symmetric operator bounded from below with deficiency indices (m, m), and the Friedrichs extension of H_0 coincides with H_1. This theorem is often convenient in determining the Friedrichs extension of a given symmetric operator bounded from below. For instance let $R = L^2(0, 1)$ and $H_0 = -d^2/dx^2$ with the boundary condition $u(0) = u'(0) = u(1) = u'(1) = 0$. Then the above theorem is applicable if we take as H_1, H_2 the differential operators $-d^2/dx^2$ with the boundary conditions $u(0) = u(1) = 0$ and $u'(0) = u'(1) = 0$ respectively, showing that H_1 is the Friedrichs extension of H_0. (Received March 1, 1955.)

471. G. L. Krabbe: *Certain abelian rings of operators on l_p.*

Suppose $p > 1$ and let l_p denote the Banach space of all sequences c such that $\sum_{n=-\infty}^{\infty} |c_n|^p < \infty$. To any function A corresponds a transformation A_* defined for any c in l_p by $A_*c = b$, $b_n = \sum_{\nu=-\infty}^{\infty} a_{n-\nu}c_\nu$, $n = 0, \pm 1, \pm 2, \cdots$, with the understanding that a is the sequence of Fourier coefficients of A. If A is in the set $\mathfrak{B}$ of all functions of bounded variation on $[-\pi, \pi]$, then A_* is a bounded operator on l_p [Stečkin, Doklady Akad. Nauk. SSSR, N. S. vol. 71 (1950) pp. 237–240]. Theorem: The sets $\mathfrak{B}$ and $\mathfrak{B}_* = \{A_* | A \in \mathfrak{B}\}$ are isomorphic abelian rings under composition; thus $(AB)_* = A_*B_*$ when $A \in \mathfrak{B}$ and $B \in \mathfrak{B}$. Let $\mathfrak{A}$ be the set of all functions analytic in some circle $\{\lambda \mid |\lambda| < r\}$, $r > \pi$; then $\mathfrak{A}_* = \{A_* | A \in \mathfrak{A}\}$ forms an irreducible subring of $\mathfrak{B}_*$. The spectrum of any A_* in $\mathfrak{A}_*$ is the image under A of the interval $[-\pi, \pi]$; A_* has no eigenvalues when A is one-to-one. This generalizes corresponding results of Toeplitz [F. Riesz, *Les systemes d'équations linéaires a une infinité d'inconnues*, Paris, 1913, pp. 175 and 153]. (Received December 20, 1954.)

472*t*. G. L. Krabbe: *On the spectra of certain operators on l_p.*

Let I denote the identity-function $I(\lambda)=\lambda$. Then I_* is the Hilbert-transformation on l_p (see the preceding abstract for definitions). The ring $\mathfrak{A}_*$ coincides with the ring $\{A(I_*)\mid A\in\mathfrak{A}\}$, in fact $A_*=A(I_*)$ whenever $A\in\mathfrak{A}$ (the symbol $A(I_*)$ is to be interpreted in the sense of the Dunford operational calculus). Let $E(\lambda)=\exp(-i\lambda)$; the spectrum $\sigma(E_*)$ of E_* is exactly the circumference of the unit-circle, and $\sigma(I_*)=[-\pi,\pi]$. The point-spectrum of both operators is empty. Therefore $I_*=i\log E_*$ in the sense specified in a previous article (*On the logarithm of uniformly bounded operators*, Bull. Amer. Math. Soc. Abstract 60-6-708), although $\log E_*$ is not definable by the Dunford calculus. (Received December 20, 1954.)

473*t*. Erwin Kreyszig: *On solutions of certain partial differential equations of elliptic type.*

The Bergman integral operator (1) $\int_{-1}^{1}E(z,z^*,t)f(z(1-t^2)/2)(1-t^2)^{-1/2}dt$, where the generating function E of (1) satisfies (2) $(1-t^2)E_{z^*t}-(E_{z^*}/t)+2ztL(E)=0$, $L(E)=E_{zz^*}+BE_{z^*}+CE$, $E_{z^*}=\partial E/\partial z^*$, $z=x_1+ix_2$, $z^*=x_1-ix_2$, transforms analytic functions $f(z)$ of one complex variable (regular at the origin) into solutions $u(z,z^*)$ of partial differential equations (3) $L(u)=0$ of elliptic type (Bergman, Rec. Math. N.S. vol. 2 (1937) p. 1169, Duke Math. J. vol. 6 (1940) p. 537). (The general equation $u_{zz^*}+au_z+bu_{z^*}+cu=0$ can be always reduced to the form (3).) For every equation (3) various functions E exist. For different purposes different E are advantageous. Of special interest are functions E generating solutions of (3) which satisfy an ordinary differential equation in addition to (3), since then the theory of these equations yields us new tools for investigating solutions of (3). The generating functions "of exponential type" (4) $E=\exp\left(\sum_{\mu=0}^{m}q_\mu(z,z^*)t^\mu\right)$ corresponding to a large class of equations (3) have this property: If $f(z)=z^n$, $n=0,1,\cdots$, the functions (1) generated by (4) satisfy the ordinary linear differential equation (5) $\sum_{\kappa=0}^{k}b_{k-\kappa}u^{(k-\kappa)}=0$, $b_k=1$, with respect to x_1. k is independent of n and $k\leqq m+1$. The b_μ have poles of order $m+1-\mu$ at the point $z=0$; (5) is therefore of Fuchs type at $z=0$. (Received March 17, 1955.)

474*t*. Erwin Kreyszig: *On singularities of certain linear partial differential equations.*

Solutions $u(z,z^*)$ of (1) $u_{zz^*}+Bu_{z^*}+Cu=0$ generated by Bergman's integral operator in the form (2) $u(z,z^*)=\int_{-1}^{1}E[(z/2)^n(1-t^2)^{n-1/2}]dt$ where the generating function (3) $E=\exp\left(\sum_{\mu=0}^{m}q_\mu(z,z^*)t^\mu\right)$ is of exponential type satisfy also an ordinary differential equation (see the previous note). These functions (2) correspond to a large class of equations (1) characterized in the following theorems: I. If the coefficients of (1) are of the form $B=-q_{0,z}-q_2/2z$, $q_{0,z}=\partial q_0/\partial z$, $C=-q_1q_{1,z^*}/2z$ where $q_0=q_0(z)$, $q_1=\sum_{\nu=0}^{\sigma(m)}a_\nu z^{\nu+1/2}$, $a_0=a_0(z^*)$, $a_\nu=\text{const.}$, $\nu=1,2,\cdots,\sigma(m)$, $\sigma(m)=(m/2)-1$ (m even) or $\sigma(m)=(m-1)/2$ (m odd), $q_2=\sum_{\nu=1}^{\tau(m)}d_\nu z^\nu$, $d_\nu=\text{const.}$; $\tau(m)=m/2$ (m even) or $\tau(m)=(m-1)/2$ (m odd), then the solutions of (1) can be generated by (3) and $q_{2\mu+1}=((-2)^\mu/3\cdot5\cdots(2+1))\sum_{\nu=\mu}^{\sigma(m)}\nu(\nu-1)(\nu-2)\cdots(\nu-\mu+1)a_\nu z^{\nu+1/2}$, $\mu=1,2,\cdots,\sigma(m)$, $q_{2\mu}=-((-2)^\mu/2\cdot4\cdots2\mu)\sum_{\nu=\mu}^{\tau(m)}(\nu-1)(\nu-2)\cdots(\nu-\mu+1)d_\nu z^\nu$, $\mu=2,3,\cdots,\tau(m)$. II. If B can be written as in I and $C=-q_{1,z^*}/2z$, then the same holds, but in this case one has $q_1=0$ (consequently $q_3=q_5=\cdots=0$) and d_1 can be a function of s^*. III. If the generating function E is of exponential type no possibilities for B and C exist other than those indicated in I and II. In consequence of these theorems which include results of Bergman (Rec. Math. N.S. vol. 2 (1937) p. 1169), Nielsen and Ramsey (Bull. Amer. Math. Soc. vol. 49 (1943) p. 156) there exist simple relations between singularities of the coefficients of (1) and those of q_0, q_1, q_2. (Received March 17, 1955.)

475t. P. D. Lax: *Differentiability of solutions of partial differential equations*. I. Preliminary report.

The Cauchy problem for symmetric hyperbolic systems $Au=v$ in the sense of Friedrichs (Comm. Pure Applied Math. (7) (1954) pp. 345–392) can be reduced to the problem of finding a periodic solution of $Au=v$. A weak periodic solution can be constructed by projection, about which the following differentiability theorem is proved: If $\|v\| < \infty$, $\|u_s\| < \infty$, $s>0$. Here $\|v\|_s$ denotes the s-fold Dirichlet integral for functions with s strong derivatives. We also introduce the negative norm $|v|_{-s} = \mathrm{Sup}_w\, (v, w)/\|w\|_s$. The tools for proving the differentiability are: (i) the a priori estimate $\|a\|_{-s} \leqq \|b\|_{-s}$ for the adjoint equation $A^*a=b$; (ii) the denseness of the range $A^*a=b$, a smooth, in the norm $\|b\|_{-s}$; (iii) the lemma that if $(u, b) \leqq \|b\|_{-s}$, then $\|u\|_s < \infty$. (i) is proved by constructing a c such that $\Delta^s c = a$ and integrating by parts the left side of $(c, A^*\Delta^s c) = (c, b)$, obtaining $\|c\|_s \leqq \|b\|_{-s}$. (ii) follows from the boundedness of A from below, and (iii) is just a variant of Friedrichs' theorem about the identity of weak and strong differentiability. (Received March 21, 1955.)

476. P. D. Lax: *Differentiability of solutions*. II.

Distribution solutions of $Lu=v$, L an elliptic operation of order $2m$, have this differentiability property: if $\|v\|_{s-m}$ is finite over a subdomain D, so is $\|u\|_{s+m}$ over any subdomain. This generalizes a result of Friedrichs (Comm. Pure Applied Math. (6) (1953) pp. 299–326; see there the differentiability theorems of L. Schwartz, F. Browder, L. Gårding and F. John). The theorem is proved first in the periodic case, by these tools: (i) the a priori estimate $\|a\|_{m+s} \leqq \|b\|_{s-m}$ for the adjoint equation $L^*a=b$, $s \lessgtr 0$; (ii) the denseness of the range of $L^*a=b$, a smooth, in the norm $\|v\|_{s-m}$; (iii) The identity of weak and strong differentiability. (i) follows for $s \geqq 0$ from Gårding's inequality, for $s \leqq 0$ from Gårding's inequality applied to the operator $L^*\Delta^s$ and the function c defined by $\Delta^s c = a$; (ii) follows for $s \leqq 0$ from Gårding's inequality, for $s>0$ from Gårding's inequality applied to $\Delta^s L^*$. The passage from the periodic to the nonperiodic case is accomplished by observing that $u' = \phi u$, ϕ compactly carried over D, can be regarded as periodic solution of $Lu' = \phi u + Nu$, N of degree $2m-1$, and by doing a finite induction. (Received March 21, 1955.)

477. Mark Lotkin: *On the accuracy of the adjoint method of differential corrections.*

When solving systems of ordinary differential equations it is often desirable to obtain also estimates of the effect of small perturbations upon the solutions. For the prediction of such small effects there is frequently employed the "method of adjoints," especially in ballistics and related fields. Now the system of adjoint differential equations involves the calculation of certain partial derivatives, which, especially if empirical functions occur in the original system of differential equations, are not always computable with sufficient accuracy. The problem discussed is thus concerned with the degree of accuracy required of the quantities occurring in the original system in order to guarantee that the differential effects will stay below a prescribed bound. The conclusions are found to lead to quite useful answers. (Received January 13, 1955.)

478. John McCarthy: *Synchronization and the rotation number.*

Let the continuous function $f(x, \lambda)$ satisfy $f(x+1, \lambda)=f(x, \lambda)+1$ and for fixed x increase monotonically from $-\infty$ to $+\infty$ with λ. Letting $x_{n+1}=f(x_n, \lambda)$ and x_0 arbitrary the Poincaré rotation number is $\nu(\lambda)=\lim x_n/n$ and is independent of x_0. The author considers the function $\nu(\lambda)$ and shows that it is continuous and monotonic as a function of λ. It is further shown that in general (for $f(x, \lambda)$ in a second category set in the topology of uniform convergence) for each rational number p/q there are numbers λ_1 and λ_2 such that for $\lambda_1 \leqq \lambda \leqq \lambda_2$, $\nu(\lambda)=p/q$. Thus in general $\nu(\lambda)$ does all its increasing on a set of the first category. The preceding results are applied to get a theory of rotated vector fields on the torus and to indicate that in general a physical system depending on a parameter will have periodic solutions for a set of parameter values of the second category. (Received March 7, 1955.)

479t. E. J. McShane: *On Stieltjes integration.*

For the Stieltjes integral $\int f(x)dg(x)$ in one dimension the following theorems are well known. (I) If the integral exists, the integrand f and the interval function Δ_g can have no common discontinuity. (II) If the integral exists and g is of bounded variation, then f is also integrable with respect to the total variation of g. In this note these two theorems are extended to n-space, both for the ordinary Stieltjes integral and for the modified form of it defined by McShane and Botts. (Received March 1, 1955.)

480. Paul Malliavin: *On the maximum modulus of a function of exponential type in a half-plane.*

Let $f(z)$ be a holomorphic function in $x>0$, satisfying $\lim\sup (\log|f(iy)|)/|y| \leqq d\pi$, L a sequence of positive numbers verifying $|l-l'|>h>0$, $l\neq l'$; denote by $l(r) = \sum_{l<r} 1/l$ $(l\in L)$, by $D_*(L)=\lim\inf_{a\to\infty, b/a\to\infty} (l(b)-l(a))/(\log b-\log a)$, by $D^*(L)$ the superior limit of the same expression. Then a necessary and sufficient condition in order that $\lim_L (\log|f(l)|)/l=-\infty$ imply $\lim (\log|f(x)|)/x=-\infty$ is that $d<D_*(L)$; furthermore if this condition is fulfilled there exists a constant A depending only on L and d such that $A+\lim\sup_L (\log|f(l)|)/l>\lim\sup (\log|f(x)|)/x$. As an application one gets the following gap result on the analytic prolongation of Taylor series: M denoting a sequence of integers there exists a function $h(u)=\sum a_m u^m$ $(m\in M)$ which can be prolonged outside its circle of convergence to infinity inside the angle $|\arg u|<\pi c$ if and only if $D^*(M)>c$. As corollary if one takes two sequences M_1 and M_2 whose reunion is the set of all integers and satisfying $D^*(M_1) = D^*(M_2)=1$, one gets two functions $h_1(u)$ and $h_2(u)$, each of them being holomorphic in the plane cut by the half line $(-1, -\infty)$, their Taylor series in 0 having complementary exponents. (Received March 3, 1955.)

481. Joseph Milkman: *Logarithmic sequences.*

A logarithmic sequence is a set of n real numbers a_i, $i=1, 2, \cdots, n$, such that (1) $0\leqq a_i\leqq 1$; (2) $a_k+a_l<a_r+a_s$ if $kl<rs$ and (3) $a_k+a_l=a_r+a_s$ if $kl=rs$. For $r\leqq n$ and $s\leqq n$ form all the different products rs. The sequence $p_1, p_2, \cdots, p_t$ is the sequence

of all these distinct products arranged in increasing order. Corresponding to $p_i = rs$ define $s_i = a_r + a_s$ and $\Delta_i = s_{i+1} - s_i > 0$. Min $\Delta(a)$ denotes the smallest Δ_i for a given logarithmic sequence a_i. $m_1, m_2, \cdots, m_n$ is a maximizing sequence if it is a logarithmic sequence and min $\Delta(m) \geq$ min $\Delta(a)$ for all logarithmic sequences $a_1, a_2, \cdots, a_n$. If one represents the logarithmic sequences by points $(a_1, \cdots, a_n)$ in n-dimensional Euclidean space, the set L of all points representing logarithmic sequences is convex. A maximizing sequence of order n exists for every natural number n and can be obtained by a finite number of operations. The author has computed the maximizing sequences for $n = 1, 2, \cdots, 7$ and found them to be unique and rational. (Received March 3, 1955.)

482. C. B. Morrey, Jr.: *A variational method in the theory of harmonic integrals*. II.

The variational method introduced in part I is applied to the study of differential forms and boundary value problems on a compact Riemannian manifold with boundary B. The manifold is not assumed to be orientable and parallel theories are developed for even and odd forms. Complete results, analogous to those for a compact manifold, are first obtained for the subspaces $\mathfrak{P}_2^+$ of $\mathfrak{P}_2$ where the normal part $n\omega = 0$ on B, and $\mathfrak{P}_2^-$ where the tangential part $t\omega = 0$ on B. By using the results about special potentials in these spaces, an orthogonal decomposition theorem $\mathfrak{L}_2 = \mathfrak{H} \oplus \mathfrak{B} \oplus \mathfrak{D}$ is probed; $\mathfrak{H}$ consists of all harmonic fields, $\mathfrak{B}$ consists of all elements $\delta\alpha$ for $\alpha \in \mathfrak{P}_2^+$, and $\mathfrak{D}$ consists of all elements $d\beta$ where $\beta \in \mathfrak{P}_2^-$. We may take $\alpha = d\Omega$, $\beta = \delta\Omega$ where Ω is a "free" potential of $\omega - H$ ($\omega =$ given form) in which $nd\Omega = t\Omega = 0$. Using this decomposition, the boundary value results for harmonic fields and forms recently obtained by Duff and Spencer (Ann. of Math. (1952)) and Conner (Proc. Nat. Acad. Sci. U.S.A. (1954) pp. 1151–1155) are obtained immediately. Complete differentiability results are obtained on manifolds of class $C_\mu^* \geq C_1^1$. (Received March 3, 1955.)

483. A. E. Nussbaum: *The Bernstein-Hausdorff-Widder theorem for locally compact abelian semi-groups.*

Let $\{T_x\}_{x \in \gamma}$ be a semi-group of uniformly bounded self-adjoint operators in a Hilbert space. γ is a semi-group contained in a locally compact abelian group $\mathfrak{g}$ and integrable with respect to the Haar measure of $\mathfrak{g}$. A real character of γ is defined to be a continuous homomorphism of γ into the multiplicative semi-group of real numbers. The real characters $\chi(x)$ of γ form a semi-group γ^*, the real character semi-group of γ, topologized by the weakest topology in which the mappings $x: \chi \rightarrow \chi(x)$ are continuous. It is shown that there exists a spectral measure $E(\sigma)$ with respect to the Borel subsets σ of γ^* such that $T_x = \int_{\gamma^*} \chi(x) dE(\sigma)$. The method of proof is that used by R. S. Phillips (Trans. Amer. Math. Soc. vol. 71 (1951)) in his demonstration of Stone's theorem on groups of unitary operators for locally compact abelian parameter groups. Using this theorem and reproducing kernel spaces it is shown how it yields the Bernstein-Hausdorff-Widder theorem on the representation of completely monotonic functions on locally compact abelian semi-groups by generalized Laplace-Stieltjes integrals: A function $f(x)$ on γ is completely monotonic if and only if there exists a non-negative measure $d\alpha(\sigma)$ with respect to the Borel subsets σ of γ^* whose spectrum is contained in the set $\gamma_0 = \{\chi \mid 0 \leq \chi(x) \leq 1\}$ such that $f(x) = \int_{\gamma_0} \chi(x) d\alpha(\sigma)$. (Received March 3, 1955.)

484. Robert Osserman: *A hyperbolic surface in 3-space.*

Using the observation that a simply-connected Riemann surface which contains two intersecting lines of symmetry and a third disjoint from these must be hyperbolic, a hyperbolic surface of the form $z=f(x, y)$ is constructed, where $f(x, y)$ is defined, single-valued, and continuous over the whole (x, y)-plane. (Received March 3, 1955.)

485t. L. B. Robinson: *On a complete system of semitensors obtained by noncommutative symbolic multiplication.*

The author has been able to compute some complete systems of tensors and semitensors with the aid of commutative symbolic multiplication. Now he has computed a complete system of semitensors with the aid of noncommutative symbolic multiplication. He will give the formula for this system. Write $f_{11}=y_1^2I_{22}-y_1y_2I_{12}-y_1y_2I_{21}+y_2^2I_{11}=\Psi_{11}$, $f_{12}=y_1Y_1I_{22}-y_2Y_1I_{12}-y_1Y_2I_{21}+y_2Y_2I_{11}=\Psi_{12}$, $f_{21}=y_1Y_1I_{22}-y_1Y_2I_{12}-y_2Y_1I_{21}+y_2Y_2I_{11}=\Psi_{21}$, $f_{22}=Y_1^2I_{22}-Y_1Y_2I_{12}-Y_1Y_2I_{21}+Y_2^2I_{11}=\Psi_{22}$. The Ψ are arbitrary functions of the covariants calculated by Wilczynski. If one solves this system with respect to I_{ij} one has the complete system of semitensors. (Received March 3, 1955.)

486t. Walter Rudin: *Maximal subalgebras in spaces of continuous functions.*

For any compact Hausdorff space X, let $C(X)$ denote the Banach algebra of all complex continuous functions on X. A subalgebra of $C(X)$ is called maximal if it is not a proper subset of any closed proper subalgebra of $C(X)$. The existence of maximal subalgebras of $C(L)$ (other than maximal ideals), where L is a simple closed curve, has been demonstrated by Wermer (Proc. Amer. Math. Soc. vol. 4 (1953) pp. 866–869; Bull. Amer. Math. Soc. Abstract 61-1-51). The following theorem is now proved. If X is a compact Hausdorff space which contains a subset K homeomorphic to the Cantor set, then $C(X)$ contains a maximal closed subalgebra M_K which separates points on X; every $f\in M_K$ has the property that the boundary of the unbounded component of the complement of $f(K)$ is connected. (Received February 24, 1955.)

487. Walter Rudin: *Uniform approximation by polynomials in two complex variables.*

Let E be an uncountable compact set of real numbers, and let $R(f, g)$ be the uniformly closed algebra (containing the complex constants) generated by the complex continuous functions f and g on E; it is assumed that $R(f, g)$ separates points on E. Of the following three statements, the first is well known, the other two are proved here. (1) If f and g are real, then $R(f, g)$ contains every continuous complex function on E. (2) The same is true if only one of the two generating functions is required to be real. (3) The conclusion is false if neither f nor g are required to be real. If E is an interval, (3) solves a problem posed by Mergelyan (Amer. Math. Soc. Translation No. 101, p. 29). A statement equivalent to (3) is that there exists a set F, homeomorphic to E, in the space of two complex variables z, w, and a complex continuous function h on F which cannot be approximated uniformly on F by polynomials in (z, w). (Received February 22, 1955.)

488t. I. E. Segal: *Tensor algebras over Hilbert spaces.* II.

The system of all skew-symmetric tensors over a complex Hilbert space H, is treated here from the standpoint of an earlier corresponding treatment of the system of symmetric tensors. Either system is canonically unitarily equivalent to the space

$L_2(H)$ relative to a generalized type of probability distribution on the real Hilbert space H of which H_c is the complex extension. In the skew-symmetric case the appropriate distribution is however of a "noncommutative" type, and in fact $L_2(H)$ is the Clifford algebra over H when H is finite-dimensional and is otherwise the L_2-system of a factor of type II_1. In either case the canonical equivalence takes the canonical field Q-operators into simple multiplication operators. It is shown that from a pure probability viewpoint also the Clifford distribution relevant to the skew-symmetric tensors is closely related to the normal distribution relevant to the symmetric tensors. (Received February 24, 1955.)

489*t*. V. L. Shapiro: *Generalized Laplacians and the Abel summation of multiple trigonometric series.*

Letting $x=(x_1, \cdots, x_n)$, $y=(y_1, \cdots, y_n)$, $\alpha x+\beta y=(\alpha x_1+\beta y_1, \cdots, \alpha x_n+\beta y_n)$, $(x, y)=x_1y_1+ \cdots +x_ny_n$, and $|x|=(x, x)^{1/2}$ and denoting the surface of the n-dimensional sphere with center x_0 and radius t by $C_n(x_0, t)$, the author sets $L(F, x_0, t)$ $=\omega_n^{-1}\int_{C_n(0,1)}F(x_0+tx)dS_{n-1}(x)$ where ω_n is the $(n-1)$-dimensional volume of $C_n(0, 1)$ and $dS_{n-1}(x)$ is its $(n-1)$-dimensional volume element. He next sets $\nabla(F, x_0, t)$ $=L(F, x_0, t)-F(x_0)$ and $\psi^*F(x_0)=\limsup_{t\to 0} 2n\nabla(F, x_0, t)t^{-2}$ with a similar definition involving lim inf for $\psi_*F(x_0)$ and calls $\psi^*F(x_0)$ and $\psi_*F(x_0)$ the upper and lower generalized Laplacians of F at the point x_0, respectively. The multiple trigonometric series $S=\sum a_me^{i(m,x)}$ is said to be Abel-summable to $f(x)$ if $\lim_{s\to 0}\sum a_me^{i(m,x)}e^{-|m|s}$ $=f(x)$. S will be called a real-valued multiple trigonometric series if $a_m=\bar{a}_{-m}$. The following theorem is then proved: Let $f^*(x)$ and $f_*(x)$ be the upper and lower Abel sums of the real-valued multiple trigonometric series $S=\sum a_me^{i(m,x)}$. Suppose that $C-\sum_{m\neq 0}a_m|m|^{-2}e^{i(m,x)}$ is the Fourier series of a continuous periodic function $F(x)$. Then for all x the following inequalities hold: (a) $\psi_*F(x)\leq f^*(x)$ and (b) $f_*(x)\leq\psi^*F(x)$. This result enables one to extend many of the previous results on the uniqueness of multiple trigonometric series involving spherical $(C, 1)$ summability to those involving Abel summability. (Received February 23, 1955.)

490. Wilhelm Stoll: *Reconstruction of meromorphic modifications by σ-processes.*

If G and H are complex manifolds of dimension 4, if τ pseudoconformally maps the open subset $A\subseteq G$ onto $B\subseteq H$, if $M=G-A$ and $N=H-B$ lie in analytic sets of dimension $2p\leq 2$, then $\mathfrak{M}=\mathfrak{M}(G, H, M, N, \tau)$ is a modification. $\mathfrak{M}$ is open in both directions if $\tau(U\cap A)\cup N$ and $\tau^{-1}(V\cap B)\cup M$ are open for all open sets $U\supseteq M$ and $V\supseteq N$. $\mathfrak{M}$ is meromorphic if for each 2-dimensional complex submanifold $L\subset G$, with $L\cap M=\{P_0\}$, the set of all accumulation points of all sequences $\tau(P^\nu)$ with $P^\nu\in L\cap A$ and $\lim_{\nu\to\infty} P_\nu=P_0$ consists of at most one point. To each 4-dimensional complex manifold G and each point $P\in G$ there is one and—up to analytic equivalence—only one modification $S_P=\mathfrak{M}(G, H_P, \{P\}, N_P, \sigma_P)$ called a σ-process, where N_P is an analytically imbedded 2-sphere. When $\mathfrak{M}$ is a modification, which is open in both directions and meromorphic, each singularity $P\in G$ of τ is replaced by such a sphere, and then each singularity of $\tau\sigma_P^{-1}$ by such a sphere, and so on. One gets a tree of σ-processes approaching a limit manifold F. Analogously, with respect to H and τ^{-1}, one gets a second tree approaching the same manifold F. The natural projections ω: $F\to G$ and χ: $F\to H$ are maps onto. On A, $\tau=\chi\omega^{-1}$. Hence, the given modification is reconstructed by σ-processes and inverse σ-processes. (Received March 1, 1955.)

491. Seth Warner: *Inductive limits of normed algebras.*

A subset B of a locally multiplicatively-convex algebra E is called ***idempotently bound*** if for some nonzero scalar λ, λB is contained in a bound, idempotent set; E is called ***pointwise idempotently bound*** if every point is an idempotently bound subset. A transformation defined on E is ***idempotently bound*** if it maps idempotently bound subsets into bound subsets. Defining the algebraic inductive limit topology on E with respect to locally m-convex algebras E_α and homomorphisms g_α: $E_\alpha \to E$ in an obvious way, one obtains the following three equivalent conditions on E: (1) E is the algebraic inductive limit of normed algebras; (2) Every convex, equilibrated, absorbing set absorbing all idempotently bound subsets is a neighborhood of zero; (3) Every idempotently bound homomorphism from E is continuous. Algebras satisfying these conditions are called ***ibic algebras***. If E is commutative and metrizable, then E is a pointwise idempotently bound, ibic algebra if and only if $[x \mid x^n \to 0]$ is a neighborhood of zero. If E is a complete, commutative locally m-convex algebra over the complex numbers, every multiplicative linear functional is idempotently bound. Many algebras drawn from integration theory and the theory of distributions, including the space $(\mathcal{D})$ of Schwartz, are ibic algebras. (Received March 1, 1955.)

492*t*. František Wolf: *On selfadjoint differential equations with non-selfadjoint boundary conditions.* Preliminary report.

If A is closed and $\|(A-\lambda_0 I)f\| \geq M\|f\|$ for all $f \in D_A$, def $(A-\lambda_0 I) = m < \infty$ and A_1, A_2 two extensions such that λ_0 is in the resolvent set of both, then $(A_1-\lambda_0 I)^{-1} - (A_2-\lambda_0 I)^{-1}$ is an operator with m-dimensional range. To this finite-dimensional perturbation can be applied the results of D. C. Kleinecke (O.O.R. report 1953, University of California at Berkeley). These perturbed eigenvalues are roots of a meromorphic function given as a determinant whose elements can be computed, if we know the resolvent of the unperturbed operator. In this case the unperturbed problem is a selfadjoint boundary problem, the perturbed problem is of the type mentioned in the title. Knowing the eigenvalues, the eigenfunctions can be obtained by applying the unperturbed resolvent to known functions. (Received March 1, 1955.)

APPLIED MATHEMATICS

493*t*. Milton Abramowitz: *Forced heat convection in laminar flow through a tube.*

The problem of steady state heat transfer in a tube is discussed. The assumption that the mean velocity is so large that axial temperature variations are negligible is discarded and the equation of heat transfer is then solved by the method of Laplace transform. The resulting eigenvalue problem is then discussed and expansions are obtained for the eigenfunctions and eigenvalues. (Received January 24, 1955.)

494*t*. R. G. Blake: *Polynomial solutions in laminated orthotropic plane strain.*

This paper considers exact polynomial solutions of the problem of plane strain in a rectangle consisting of three layers of orthotropic material cemented together, with the two outer layers similar. The lines of separation are taken parallel to the x-axis and in the xy-plane. Using τ_{ij1} for the stresses in the central layer and τ_{ij2} for the stresses in the outer layer, it is shown that the only exact polynomial solutions are (1) $\tau_{xx1} = k_1$, $\tau_{xx2} = k_2$, $\tau_{yy1} = \tau_{yy2} = k_3$, $\tau_{xy1} = \tau_{xy2} = 0$ (k_i constant); (2) $\tau_{xx1} = \tau_{xx2}$

$=\tau_{yy1}=\tau_{yy2}=0$, $\tau_{xy1}=\tau_{xy2}=k$ (k constant); (3) $\tau_{xx1}=k_1xy$, $\tau_{xx2}=k_2xy$, $\tau_{yy1}=\tau_{yy2}=0$, $\tau_{xy1}=k_3y^2+k_4$, $\tau_{xy2}=k_5y^2+k_6$ (k_i constant). This research was supported by a contract with the Office of Naval Research. (Received February 4, 1955.)

495. J. R. Isbell: *Pseudo-main solutions.*

H. Gurk defines (unpublished) a ***main solution*** of an n-person game as a finite solution determined by a family U of minimal winning sets and a system (x_i) of weights for the players; the solution V_U is $\{\alpha^S | S\in U\}$, where α_i^S is x_i for i in S, 0 for i not in S. He defines a ***pseudo-main solution*** as a solution $\{\alpha^S\}\cup W$, where $\{\alpha^S\}$ would be a main solution except that the finite set W is undominated. The author defines (*A class of game solutions*, Proc. Amer. Math. Soc. vol. 6 (1955)) pseudo-main solutions as solutions which fail to be main in that not all S in U are winning sets. These two definitions are equivalent. The solution is determined by the weights (x_i). Let k be the minimum of $\sum_{i\in S} x_i$ for winning sets S. Then U is the family of all sets of weight k which meet all winning sets of weight k; V_U is a solution if and only if no set of weight less than k meets all winning sets of weight k. Theorem. If there exists a system $w=(w_i)$ of weights, all nonzero, in the strong game G, such that every winning set has at least half the total weight, then G has at most one pseudo-main solution, which must be main simple. The theorem applies to all strong simple six-person games. An eleven-person game shows that the hypothesis that all $w_i>0$ is needed. (Received February 14, 1955.)

496. M. S. Klamkin: *On a homogeneous two-dimensional low-pass filter.*

The filter considered consists of an infinite net of squares, the sides of which all contain a self-inductance Lm if they are parallel to the m-direction or a self-inductance Ln if they are parallel to the n-direction, while the vertices (m, n) are connected to a perfectly conductive plate at zero potential via capacitances C. This filter is a generalization of the one considered by van der Pol and Bremmer in their book *Operational calculus*, where $Lm=Ln$. If one now assumes a source of voltage in parallel with the capacitor at the origin (0, 0), delivering a current $I_0(t)$ to the filter, one then has an infinite number of simultaneous partial difference equations for the voltage function at the vertices of the lattice. This voltage function is determined by use of the simultaneous operational calculus. Furthermore, by considering the special case $C=0$, and replacing the inductances Lm and Ln, by resistances Rm and Rn, respectively, the equivalent resistance between the points (0, 0) and (m, n) is calculated. In particular, for $(m, n)=(1, 0)$, $R=Rm/2+(Rm/\pi)\sin^{-1}(Rn-Rm)/(Rn+Rm)$. (Received March 2, 1955.)

497. S. G. Kneale: *Axially symmetric polyharmonic functions.*

Let ∇_p^2 be the $(p+2)$-dimensional Laplacian operator and let $\nabla_p^{2(K+1)}=\nabla_p^2\nabla_p^{2K}$; then a solution of $\nabla_p^{2(K+1)}\Phi=0$ is called a *polyharmonic* function and, in particular, a *K-harmonic p-function*. Such functions are of particular physical interest for $K=0$, 1 and $p=0$, 1, 2, 3. Nicolesco (*Les fonctions polyharmoniques*, Paris, Hermann, 1936) found that harmonic functions are analytic functions satisfying Gauss's theorem of the arithmetic means and its converse. A generalized Kelvin transformation holds for polyharmonic functions and serves to define regularity at infinity. Polyharmonic functions regular in a star-shaped region are expressible as a finite sum involving harmonic functions. Finally, a uniqueness theorem holds for polyharmonic functions.

Attention is restricted to axially symmetric harmonic and biharmonic functions. By application of known facts concerning ultra-spherical polynominals, harmonic functions are shown to possess a Laurent series analogous to the Laurent series for complex analytic functions. In terms of such a series, the author is able to discuss the isolated singularities of harmonic and biharmonic functions. The extension to other polyharmonic functions is evident. (Received March 10, 1955.)

498t. K. S. Miller and L. A. Zadeh: *Signal representation in terms of periodic functions.*

Signal representations of the form $f(y)=\int_0^\infty K(xy)G(x)dx$, where $K(s)$ is a periodic function, are investigated. Formulas for the inverse kernel $K^{-1}(s)$ are developed in integral form and in an infinite trigonometric series. Closed forms for the integral and derivative of $K(s)$ are presented. In the case $K(s)$ is a square wave the reciprocal kernel can be written in terms of an inverse Mellin transform involving the Riemann zeta-function while the coefficients of the infinite series representation are essentially the Möbius mu function. Applications to problems in signal resolution are treated. (Received March 1, 1955.)

499. Morris Morduchow: *An approximate solution, by method of averages, of a differential equation for flow in a tube with normal fluid injection at wall.*

The partial differential equations for the laminar incompressible flow through a rectangular or circular tube with uniform normal velocity along the wall can be transformed into a fourth-order nonlinear ordinary differential equation with mixed boundary conditions. The equation, moreover, contains a parameter R which is usually large in practice, but whose reciprocal is the coefficient of the highest derivative term. Despite the difficulties or inconveniences which might arise in obtaining numerical solutions of such an equation for a complete range of values of R, a simple approximate solution in terms of R, valid for $0 \leq R \leq \infty$, is obtained with relatively little difficulty by a method of averages. This solution is shown to reduce exactly to a small-perturbation solution for small R and approximately to an asymptotic solution for large R. The solution is of particular interest in connection with transpiration or sweat-cooling of aerodynamically heated surfaces. The method of averages serves here as a comparatively simple alternative to the method of least squares. (Received March 2, 1955.)

500. L. E. Payne: *Bounds for the torsional rigidity of a composite beam.*

The cross section of an elastic beam is assumed to be composed of two or more materials possessing different elastic constants. In the St. Venant torsion problem for such a beam the stress function or the normal derivative of the warping function must satisfy certain jump conditions on the separating boundaries. It is shown in this paper that bounds can readily be obtained for the torsional rigidity of the beam. In particular an upper bound is given in terms of continuous functions which approximate the warping function, and a lower bound involves continuous functions which vanish on the outer boundary of the section and approximate the modified stress function. These bounds may be improved by the usual Rayleigh-Ritz procedure. (Received March 3, 1955.)

501. L. A. Rubel: *An estimate of the error due to modified boundary conditions in the Blasius differential equation.*

In the numerical calculation of the solution $f(x)$ of the Blasius differential equation $f'''(x)+f(x)f''(x)=0$ under the boundary conditions of $f(0)=0, f'(0)=0, f'(\infty)=2$, the third boundary condition is customarily modified to $f'(M)=2$, where M is some "large number." Let $f(x; M)$ denote the solution under the modified boundary conditions. It is desirable to justify the customary use of $f(x; M)$ as an approximation to $f(x)$. This paper provides the inequality $|f(x)-f(x; M)| \leqq Mf''(x; M)/f(M; M)$ for all x such that $0 \leqq x \leqq M$. Since the right-hand side of the inequality contains only computable quantities, an estimate of the error may be easily computed. It is also shown that if $M > 1+2(N+1)^{1/2}$ $(N \geqq 3)$, then $|f(x)-f(x; M)| < 10^{-N}$ for all x such that $0 \leqq x \leqq M$, so that $M=8$ yields at least 10 place accuracy, and $M=10$ yields at least 19 place accuracy. (Received February 18, 1955.)

502*t*. K. M. Siegel and T. B. Curtz: *Inequalities involving cylindrical functions.*

The Bessel functions $J_\nu(\nu x)$, $J'_\nu(\nu x)$ of nearly equal argument and order are bounded above by expressions which tend respectively to $J_\nu(\nu)$ and $J'_\nu(\nu)$ as $x \to 1$. The regions in which these results improve previously known inequalities are determined. By an argument similar to that used by Watson for $J_\nu(\nu)$, the well-known Cauchy approximation $J_\nu(\nu) \sim 3^{1/6}\Gamma(2/3)/2^{1/3}\pi\nu^{1/3}$ is shown to be an over-estimate. For the range $1 \leqq x \leqq J'_\nu/\nu$ (j'_ν the smallest positive zero of $J'_\nu(x)$), an easily derived bound for the Neumann function $N_\nu(\nu x)$ is $|N(\nu x)| < 1.73\, J_\nu(\nu)$. (Received January 26, 1955.)

503*t*. I. A. Stegun and Milton Abramowitz: *The generation of Coulomb wave functions by means of recurrence relations.*

A discussion of the computation of Coulomb wave functions from their recurrence relation is given. Specifically, it is demonstrated that the regular solution F_L and the irregular solution G_L may be obtained recursively based on the knowledge of the functions for $L=0$. (Received January 24, 1955.)

504. H. F. Weinberger: *Lower bounds for minima by finite difference methods.*

Let λ be the lowest eigenvalue of the membrane problem for the region D with boundary C. That is, $\Delta u+\lambda u=0$ in D, $u=0$ on C. Let λ^h be the corresponding quantity for the analogous finite difference problem $\Delta_h v+\lambda^h v=0$ in D_h, where D_h is a region consisting of entire squares of a grid with mesh size h, and v is defined at the mesh points of D_h and vanishes at its boundary points. Then, if D_h contains D as well as all its positive translates in the x and y directions of magnitudes at most h, we have $\lambda^h \leqq \lambda$. To each position of D correspond certain values of u at the mesh points of D_h which are zero at the boundary points of D_h. The inequality $\lambda^h \leqq \lambda$ is proved by substituting these values of u in the minimum principle for the finite difference problem and averaging over all admissible positions of D. The same result is obtained for the reciprocal of the torsional rigidity and for the lowest frequency of a clamped plate. The slightly weaker result $\nu^h \leqq \nu/(1-h\nu^{1/2}/6)$ holds for the critical buckling load ν of a clamped plate. This research was supported by the United States Air Force through the Office of Scientific Research. (Received March 8, 1955.)

505. C. B. Tompkins: *A modified Jacobi method for diagonalizing symmetric matrices.*

The Jacobi method for diagonalizing matrices reduces large off-diagonal elements to zero by means of a rotation in the space of the two coordinates involved with the element if the matrix is considered as a matrix of a quadratic form. If the off-diagonal element is k and the diagonal elements of its row and column are a and b, the rotation is θ, where $\tan 2\theta = 2k(a-b)^{-1}$. The present method is designed for use on simple computing machines or for easy coding. By considering the off-diagonal elements in order and by restricting the rotations to be from a tabulated or easily computed set the calculation is lengthened but simplified. The set of rotations might be (for example) 2^{-k} radians $k = 1(1)n$; a single rotation is used as long as it is effective for any off-diagonal element before the next smaller rotation is used; the rotation is to be carried out (in a sense depending on the sign of $k(a-b)^{-1}$) if and only if the rotation considered is less than the Jacobi rotation in the formula above. Convergence to a diagonalized form is proved, and the code does not involve any divisions or roots. (Received March 3, 1955.)

GEOMETRY

506. H. S. M. Coxeter: *The collineation groups of the finite planes $EG(2, 3)$ and $PG(2, 3)$.*

The group of all collineations in the finite projective plane $PG(2, 3)$ is the simple group $LF(3, 3)$ of order 5616. Brahana (Ann. of Math. (2) vol. 31 (1930) p. 531) gave two matrices which generate this group, but did not attempt to find a complete set of defining relations. In terms of a different pair of generators, such relations are now found to be $S^6 = T^2 = (ST)^4 = (S^2T)^4 = (S^3T)^3 = 1$, $(TS^2T)^2S^2 = S^2(TS^2T)^2$. These are established by enumerating the 26 cosets of the subgroup $T^2 = U^3 = (TU)^4 = 1$, $(TUT)^2U = U(TUT)^2$ (where $U = S^2$), which is the Hessian group of order 216, i.e., the group of projective collineations leaving invariant the nine inflections of the general cubic curve in the complex projective plane. This, in turn, has a subgroup of index 9: $U^3 = 1$, $UVU = VUV$ (where $V = T^{-1}UT$), which is the binary tetrahedral group of order 24. (Received March 3, 1955.)

507. Jesse Douglas: *A theorem on skew pentagons.*

Let P_n, $n = 0, 1, 2, 3, 4$, denote the vertices of any (skew) pentagon in euclidean m-dimensional space, while M_n denotes the midpoint of the side $P_{n+2}P_{n-2}$ (indices taken mod 5). The five line segments P_nM_n will be called the *medians* of the pentagon. Theorem: (I) If P_nM_n is extended to Q_n where $M_nQ_n = (1/5^{1/2})P_nM_n$, the five points Q_n are the vertices of an affine-regular pentagon of convex type. (II) If P_nM_n is abbreviated to P_nQ_n' where $M_nQ_n' = (1/5^{1/2})P_nM_n$, the five points Q_n' are the vertices of an affine-regular pentagon of star-shaped type. Definition of affine-regular: the parallel projection on any plane of a regular pentagon π; convex type, $\pi = (1, \omega, \omega^2, \omega^3, \omega^4)$ where $\omega = e^{2\pi i/5}$; star-shaped type, $\pi = (1, \omega^2, \omega^4, \omega, \omega^3)$. Cf. the author's paper *Geometry of polygons in the complex plane* [Journal of Mathematics and Physics (M.I.T.) vol. 19 (1940) pp. 93–130]. $\pm 1/5^{1/2}$ are the only multipliers that will produce a special pentagon of any sort, starting from an arbitrary one; otherwise the transformed pentagon may be an arbitrary one. (Received March 4, 1955.)

508t. Frank Harary: *On local balance and N-balance of a signed graph.*

The results of this note were motivated by their applicability to certain problems in social psychology. A signed graph (s-graph) is obtained from a linear graph when some of its lines are called positive and the remaining lines negative. The sign of a cycle is the product of the signs of its lines. An s-graph is balanced if every cycle is positive. Two characterizations of balanced s-graphs were obtained in *On the notion of balance of a signed graph*, Michigan Math. J. vol. 2 (1953–54) pp. 143–146. An s-graph is locally balanced at point P if every cycle through P is positive. Theorem: If G is balanced at P and Q is a nonarticulation point on a cycle containing P, then G is balanced at Q. An s-graph is N-balanced if all its cycles of length $\leq N$ are positive. A criterion for G to be N-balanced is given and local N-balance is discussed. (Received February 28, 1955.)

509. N. D. Lane: *Differentiable points of arcs in conformal n-space.*

This paper is a generalization to n dimensions of a note on differentiable points in the conformal plane, Canadian Journal of Mathematics vol. 5 (1953) pp. 512–518. Families $\tau_r^{(m)}$ of tangent m-spheres $S_r^{(m)}$, where $0 \leq m \leq n-1$ and $0 \leq r \leq m+1$, having contact of order r with an arc at a point, are defined, and a conformal definition of differentiability, involving the tangent $(n-1)$-spheres, is given. The following properties of the families $\tau_r^{(n-1)} - \tau_{r+1}^{(n-1)}$, $r = 0, 1, \cdots, n-1$, and the natures of the m-spheres $S_{m+1}^{(m)}$, $m = 0, 1, \cdots, n-1$, lead to the classification of the differentiable points p of an arc into $2^{n-1}(3n-1)$ types. For each $r < n$, the $(n-1)$-spheres of $\tau_r^{(n-1)} - \tau_{r+1}^{(n-1)}$ all support, or all intersect the arc at p. If $S_m^{(n-1)} = p$, while $S_{m+1}^{(m)} \neq p$, the spheres of $\tau_m^{(n-1)} - \tau_{m+1}^{(n-1)}$ all support; $m = 1, 2, \cdots, n-1$. If $S_n^{(n-1)} \neq p$, it can support, intersect, or neither support nor intersect the arc at p (i.e., every neighborhood of p on the arc can have points in common with $S_n^{(n-1)}$). (Received February 7, 1955.)

510. J. F. Nash: *Extension of the solution of the imbedding problem to open Riemannian manifolds.*

By a simple device one can reduce the problem of (isometrically) imbedding general Riemannian manifolds to the problem for manifolds which are differentiably homeomorphic to spheres. However, this approach does not give the smallest dimensionality for the Euclidean imbedding space in general. If n-dimensional spheres with any arbitrary smooth Riemannian metric can always be imbedded in N-dimensional space, then any n-manifold with such a metric can be imbedded in $(n+1)N$-dimensional space. Since compact n-manifolds require at most $3n^2/2+11n/2$ dimensions, then open n-manifolds require at most $3n^3/2+7n^2+11n/2$ dimensions. Possibly the work that has been done on the problem of Weyl would fit in with this device to give imbeddings of all 2-manifolds in 9-space. (Received January 31, 1955.)

511. Valdemars Punga: *Indicatrix of calculus of variations and Finsler geometry.*

According to E. Cartan's point of view, Finsler geometry is envisaged as a space of line elements which is locally Euclidean. Using the concept of Carathéodory's indicatrix, Finsler space can be envisaged as point space which is locally Minkowskian. The most fundamental study of Finsler geometry and its generalizations by means of an indicatrix can be found in the papers of the Russian mathematician V. V. Vagner.

In the present paper the author reproduces the main results of Finsler geometry using as a model 3-dimensional Euclidean space with a 2-dimensional surface-indicatrix associated with every point of the space. In this way it is possible to visualize Finsler geometry and to derive its results by the reasoning methods of synthetic geometry. (Received February 24, 1955.)

512t. Moses Richardson: *On finite projective games.*

Let $PG(k, p^n)$ be the finite k-dimensional projective space whose field of coordinates is the Galois field $GF(p^n)$. A simple game is defined by letting the players be the points of this space and letting the minimal winning coalitions be the subspaces of dimension h if $k=2h$, and of dimension $h+1$ if $k=2h+1$. The resulting game is denoted also by $PG(k, p^n)$ and is termed a *finite projective game*. It is shown that $PG(2, p^n)$ is strong if $p^n=2$, but that there exists a blocking coalition of $2p^n$ members if $p^n>2$, and one of $2p^n-1$ members if $p^n>3$. If d is a divisor of n, $1\leq d<n$, then there exists in $PG(2, p^n)$ a blocking coalition of $2p^n-p^d+1$ members. It is also shown that every $PG(2, p^n)$ has a main simple solution. (Received February 7, 1955.)

513. Seymour Schuster: *Pencils of polarities in projective space.* I.

A polarity $(ABC)(Pp)$ in projective space of two dimensions (S_2) is completely determined by a self-polar triangle ABC, and a pair of corresponding elements point P and its polar line p. A *pencil of polarities* is the set of ∞^1 polarities $(ABC)(Pp)$ where A, B, C, and P are fixed while p varies in a pencil of lines. Coxeter has developed this notion and has shown that two types of the classical pencils of conics are, in fact, the conics which arise from the pencils of polarities. They are the general system with four distinct points in common and the double contact system where the conics have in common two distinct points and common tangents at these points. A *pentagonal pencil of polarities* is defined as the ∞^1 polarities which arise from a self-polar pentagon in which one side varies in a pencil of lines. Does this definition yield anything more general than the previous one? Investigation shows the answer to be in the affirmative in that the two triangular systems can both be derived as pentagonal pencils; and, two new systems arise to yield pencils of conics—(1) with two-point contact at a point and two other distinct points in common, and—(2) the four-point contact system. It is also shown that the remaining pencil of conics, with three-point contact cannot be derived from pentagonal pencils. (Received February 9, 1955.)

514t. Seymour Schuster: *Pencils of polarities in projective space.* II.

Definition: *A tetrahedral pencil of polarities* is the set of polarities $(ABCD)(P\pi)$ where the self polar tetrahedron $ABCD$ and point P are fixed, while π, the polar plane of P, varies in an axial pencil. There are four tetrahedral systems P^3_{4k}, where k is 0, 1, 2, or 3, according as the axis of the pencil of planes π meets 0, 1, 2, 3 edges of $ABCD$. If two polarities belong to the same: (1) P^3_{41} system, their product is a general axial homography. (2) P^3_{42} system, their product is a hyperbolic biaxial homography. (3) P^3_{43} system, their product is a spatial homology. The locus of poles of a fixed plane with respect to a tetrahedral pencil is: (a) a twisted cubic passing through the vertices of the fixed self-polar tetrahedron in a P^3_{40} pencil. (b) a conic passing through three vertices of one of the fixed self-polar tetrahedra in a P^3_{41} pencil. (c) a line passing through two of the vertices of one of the fixed self-polar tetrahedra in a P^3_{42} pencil. (d) a line passing through the vertex common to all the self-polar tetrahedra in a P^3_{43} pencil. Every tetrahedral pencil yields a pencil of quadrics. Four of the thirteen nondegenerate pencils of quadrics can so be obtained. Each tetrahedral pencil is associated with the Segre symbol of the corresponding pencil of quadrics as follows: $P^3_{44}-[1111]$; $P^3_{42}-[(11)11]$; $P^3_{4a}-[(11)(11)]$; $P^3_{43}-[(111)1]$. (Received February 9, 1955.)

LOGIC AND FOUNDATIONS

515. R. S. Ledley: *Computational methods in symbolic logic.*

A binary digitalization of the propositional calculus of symbolic logic has been given previously (JORSA 2: 3, Aug. 1954). Based on these results, a general method for systematically generating all antecedence and consequence solutions of given form to a set of Boolean equations has been obtained. Specifically, given the set of equations $\{E_s(A_1, \cdots, A_i, x_1, \cdots, x_k)\}$, and a set of forms $\{F(f_1, \cdots, f_j, x_1, \cdots, x_k)\}$ where E_s and F_s are Boolean functions of the A_r, x_s and f_s, and $f_s = f_s(A_1, \cdots, A_i)$ is unknown, the functions f_s are determined such that the $\{F_s\}$ become antecedent solutions, i.e., $\pi_s F_s \rightarrow E_s$ for all s, or such that the $\{F_s\}$ become consequent solutions, i.e., $\pi_s E_s \rightarrow F_s$ for all s. The method involves the matrices (E_{hi}) and (F_{jh}) which are formed from the designation numbers of $\pi_s E_s$ and $\pi_s F_s$ respectively. The solutions are read from the matrix (R_{ji}) where for antecedent solutions $(F_{jh})\cdot(\overline{E}_{hi}) = (\overline{R}_{ji})$, and for consequent solutions $(\overline{F}_{jh})\cdot(E_{hi}) = (\overline{R}_{ji})$, where the matrix product involves only logical multiplication and addition. If solutions of particular forms do not exist, then the constraint c under which such solutions would exist are given by $(V_j)\cdot(R_{ji}) = (C_i)$ where (V_j) is row vector of all units, and (C_i) is the designation number of c. (Received February 28, 1955.)

STATISTICS AND PROBABILITY

516. D. A. Darling (p) and Paul Erdös: *A limit theorem for the maximum of normalised sums of independent random variables.*

The following theorem is proved: Let $X_1, X_2, \cdots$ be independent random variables with a uniformly bounded third absolute moment, mean 0 and variance 1. Put $S_n = X_1 + X_2 + \cdots + X_n$ and let $U_n = \max_{1 \leq k \leq n} S_k/k^{1/2}$. Then for $-\infty < t - \infty$, $\lim_{n\to\infty} \Pr\{U_n < (2 \log \log n)^{1/2} + 2^{-1}(\log \log \log n)(2 \log \log n)^{-1/2} + t(2 \log \log n)^{-1/2}\} = \exp(-(1/2\pi^{1/2}) \exp(-t))$. A similar theorem is proved for $U_n' = \max_{1 \leq k \leq n} |S_k|/k^{1/2}$ and two strong theorems for the sequence U_n are indicated. The method of proof consists in finding an interpolatory stochastic process for the sequence $S_k/k^{1/2}$, obtaining a limit theorem for its maximum and then an application of the invariance principle of Erdös-Kac. (Received March 3, 1955.)

517t. D. A. Darling and A. J. F. Siegert: *The distribution of certain functionals of Markoff processes.*

Let S be an abstract space of elements $x, y, \cdots$ and $\mathcal{F}$ a Borel field of subsets of S; i.e. $(S, \mathcal{F})$ is a measurable space. (i) Let X_n, $n = 0, 1, \cdots$, be a Markoff chain with values in S having a transition probability $P_{jk}(x; A) = \Pr\{X_k \in A \mid X_j = x\}$, where $k \geq j$, $A \in \mathcal{F}$. For each $j = 0, 1, \cdots$ let $V(x, j)$ be a real-valued $\mathcal{F}$ measurable function such that $E\{\sum_{j=1}^{n} |V(X_j, j)| \mid X_0 = x, X_n \in A\} < \infty$. Then the function $R_{jk}(x; A) = E(\exp\{i\xi \sum_{\nu=j}^{k} V(X_\nu, \nu)\} \mid X_j = x, X_k \in A) P_{jk}(x; A)$ satisfies uniquely either of the integral equations $R_{jk}(x; A) = P_{jk}(x; A) + \sum_{\nu=j}^{k} \int_S (1 - e^{-i\xi V(y,\nu)}) R_{j\nu}(x; dy) P_{\nu k}(y; A)$ or $R_{jk}(x; A) = P_{jk}(x; A) + \sum_{\nu=j}^{k} \int_S (1 - e^{-i\xi V(y,\nu)}) P_{j\nu}(x; dy) R_{\nu k}(y; A)$. (ii) Let $X(t)$, $t \geq 0$, be a Markoff process with transition probability $P(x, s; A, t) = \Pr\{X(t) \in A \mid X(s) = x\}$

and $V(x, t)$ be a real-valued $\mathcal{J}$ measurable function for each t and continuous in t for each x, and such that $V(X(t), t)$ is a measurable process with $\int_s^t E(|V(X(\tau), \tau)| \,|\, X(s) = x, X(t) \in A) d\tau < \infty$. Then the function $R(x, s; A, t) = E(\exp \{i\xi \int_s^t V(X(\tau), \tau) d\tau\} \,|\, X(s) = x, X(t) \in A) P(x, s; A, t)$ satisfies uniquely either of the integral equations $R(x, s; A, t) = P(x, s; A, t) + i\xi \int_s^t \int_S V(y, \tau) P(x, s; dy, \tau) R(y, \tau; A, t) d\tau$ or $R(x, s; A, t) = P(x, s; A, t) + i\xi \int_s^t \int_S V(y, \tau) R(x, s; dy, \tau) P(y, \tau; A, t) d\tau$. If the chain X_n or the process $X(t)$ is temporally homogeneous and V depends only on x, these integral equations reduce to Fredholm equations by using generating functions or Laplace transforms. When S is E_1 in (ii) and the function P satisfies the Fokker-Planck differential equations, R satisfies two equations of the same nature. (Received March 3, 1955.)

518t. Eugene Lukacs: *A characterisation of the gamma distribution.*

The following theorem is proven: Let X and Y be two nondegenerate and positive random variables and suppose that they are independently distributed. The random variables $U = X + Y$ and $V = X/Y$ are independently distributed if, and only if, both X and Y have gamma distributions with the same scale parameter. The theorem is closely related to a recent result of R. G. Laha [Ann. Math. Statist. vol. 25 (1954) pp. 784–787] who characterized the gamma distribution essentially by the independence of the ratio of certain quadratic forms from the mean. However, Laha assumed that the random variables are identically distributed and that their second moments exist; both these assumptions are avoided in the theorem. (Received February 28, 1955.)

TOPOLOGY

519. Eldon Dyer: *A fixed point theorem.*

In this paper it is shown that if M is the Cartesian product of finitely many compact metric chainable continua and f is a continuous mapping of M into itself, then some point of M is its own image under f. Since an interval is a chainable continuum, this is a generalization of the Brouwer fixed point theorem. It is also a generalization of a theorem proved by O. H. Hamilton (*A fixed point theorem for pseudo-arcs and certain other metric continua*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 173–174). For a definition of chainable continua (called by Bing "snake-like continua") see R. H. Bing, *Snake-like continua*, Duke Math. J. vol. 18 (1951) pp. 653–663; Bing also gives a number of examples of chainable continua in this paper. (Received February 22, 1955.)

520t. E. R. Fadell: *On spaces without isolated non cut points.*

Let X denote a topological space and f: $X—X$ a map. A point $x \in X$ is free if $f(x) \neq x$. f is called a ϕ-map if for every $x \in X$ there exists a free point $x' \neq x$ which does not separate $f(x')$ and x. As a matter of convenience the identity map is allowed as a ϕ-map. Also, let $F(f)$ denote the fixed points of f. If g: $X \rightarrow X$ is another map, f and g are said to be F-homotopic provided there exists a homotopy H: $X \times I \rightarrow X$ such that $H_0 = f$, $H_1 = g$, and $F(H_t) = F(f)$ for all t, $0 \leq t \leq 1$. The following theorem holds (Bull. Amer. Math. Soc. Abstract 60-4-450): If X is an ANR (compact metric) and f: $X \rightarrow X$ is a ϕ-map, then there exists a map g: $X \rightarrow X$ which is F-homotopic to f such that $g(X) = X$. In this note it is shown that for a Peano continuum X a necessary and sufficient condition that all maps f: $X \rightarrow X$ be ϕ-maps is that the non cut points of X be dense in X. Also, if X is a connected ANR (compact metric), a necessary and sufficient condition that for every map $f: X \rightarrow X$ there exist a map g, F-homotopic to f, such that $g(X) = X$ is that X fail to possess an isolated non cut point. (Received February 21, 1955.)

521. Mary-Elizabeth Hamstrom: *A note concerning the imbedding of upper semicontinuous collections of continua in continuous collections of continua.*

Let G be an upper semicontinuous collection of arcs and points in the plane such that G is an arc with respect to its elements. The collection G is said to be imbedded in a collection W provided that each element of G is contained in some element of W and each element of W contains one and only one element of G. If n is a natural number and H is a closed subcollection of arcs of G, let $H^{(n)}$ denote the collection of all elements g of H such that if ϵ is a positive number every open subcollection of G containing g contains an element of H which has a subarc the distance between whose end points is less than ϵ but whose diameter is not less than $1/n$. It is shown that if each arc g of G is a continuum of condensation of G^* and is contained in a simple closed curve J such that $J \cdot G^* = g$, and if $G^* - g$ has two components they lie in different complementary domains of J, then if H is a closed subcollection of arcs of G, for no natural number n is it true that $H^{(n)} = H$. It follows from results in the author's recent paper (*Concerning the imbedding of upper semicontinuous collections of continua in continuous collections of continua*, Amer. J. Math. vol. 76 (1954) pp. 793–810) that there is a continuous collection W of simple closed curves in the plane such that G is imbedded in W. (Received March 1, 1955.)

522t. Erwin Kreyszig: *Continuation problems in the theory of functions of several complex variables.*

By omitting the nonuniformizable points from the "Riemann domain" (complex n-dimensional analogue of the Riemann surface, see Behnke and Stein, Math. Ann. vol. 124 (1951) p. 1) one obtains a complex manifold. Continuation of complex manifolds is therefore a problem analogous to the continuation of Riemann surfaces. Let M^{n*} and M^n be complex manifolds of complex dimension n. Let $N^* \subset M^{n*}$ and $N \subset M^n$ be closed. Then if there exists a proper analytic mapping T of M^{n*} into M^n which maps $M^{n*} - N^*$ one to one onto $M^n - N$, then M^{n*} is called a proper analytic modification of M^n with respect to N. Theorems: I. If M^{n*} is a proper analytic modification of M^n with respect to N, then the rings of holomorphic functions and the fields of meromorphic functions in M^{n*} and M^n are isomorphic, and the canonical isomorphism of the lattices of elements of analytic varieties in $M^{n*} - N^*$ and $M^n - N$ can be continued to a homomorphism of the lattice of elements of analytic varieties of M^{n*} into that of M^n. II. If in addition to the hypotheses of I, N^* and N are irreducible varieties of the same dimension, then T is one to one and maps M^{n*} onto M^n; the two manifolds are therefore analytically equivalent. (Received April 6, 1955.)

523t. Erwin Kreyszig: *Monoidal transformations in topology.*

Monoidal transformations of algebraic geometry can be generalized and are then useful for solving problems in topology and theory of functions of several complex variables. Let M^n be a complex manifold of complex dimension n and $N \subset M^n$ an irreducible analytic variety of complex dimension k. Then N can be replaced by a fibre bundle N^* with the base N and a complex $n-k-1$-dimensional projective space as fibre. Let us call this a $\sigma^{n,\lambda}$-process; it is a generalization of Hopf's σ-process (Rend. Mat. Roma vol. 10 (1951) p. 169) and is independent of the choice of the local coordinates in M^n. $M^{n*}=(M^n-N)\cup N^*$ is again a complex manifold. Every projective space $S^{a_1}\times S^{a_2}\times\cdots\times S^{a_m}$ (where the index a_μ indicates the complex dimension) can be obtained from another (of the same complex dimension) by using certain $\sigma^{n,\lambda}$-processes. Because $\sigma^{n,\lambda}$-processes are special modifications of complex manifolds, all theorems of the previous note can be applied to these processes for example in order to investigate the behavior of holomorphic and meromorphic functions with respect to different closures of the complex affine space. (Received April 6, 1955.)

524. W. T. Kyner: *A generalization of the Borsuk and Borsuk-Ulam theorems.*

Let E be a real, infinite-dimensional, linear topological space with a bounded convex neighborhood of the origin. Let V be a closed convex body, symmetric with respect to the origin and let S be its boundary. A map f of S into E is said to be antipodal if $f(x)=-f(-x)$ for all x in S. If F is a compact map, the map $f(x)=x-F(x)$ is said to be a compact displacement of S. Using Rothe's theory of topological order, the author has obtained the following Borsuk type theorem: *The order of the origin with respect to a nonzero, antipodal, compact displacement of the set S is odd.* This implies that if F is a compact map defined on V, and if F is antipodal on S (or homotopic to an antipodal map), then F has a fixed point in V. If one adds the additional restriction that $F(S)$ be closed, one obtains a generalization of the Borsuk-Ulam theorem: *If F is a strongly compact, antipodal map of S into E, then there is at least one point in S that maps onto the origin of E.* (Received March 3, 1955.)

525. A. B. Novikoff: *The intersection of convex sets.*

Let C be a convex set in E_n containing the origin in its exterior, and let λC denote its homothetic enlargement from the origin in the ratio $\lambda:1$. In a paper on conformal mapping, M. Shiffman used the lemma that for $n=2$ the intersection of the boundaries of C and λC consists of at most 2 connected components. He conjectured that for $n\geq 3$, the intersection consists of at most one connected component. This is established. Under certain restrictions on the topology, the same result holds for topological vector spaces in general. (Received January 24, 1955.)

526t. C. D. Papakyriakopoulos: *On the ends of knot groups.*

The main result is the following: If K is a polygonal knot in the 3-dimensional sphere S such that $S-K$ is aspherical, then its group $\pi_1(S-K)$ has two ends or one end according as $\pi_1(S-K)$ is free cyclic or not. This clarifies a result of E. Specker (*Die erste Cohomologiegruppe von Ueberlagerungen*, Comment. Math. Helv. vol. 23 (1949), see p. 329). The main result is a consequence of the following two theorems: (1) Let M be an aspherical finite 3-dimensional manifold whose fundamental group $\pi_1(M)$ is infinite. Suppose that the injection $\pi_1(N_i)\rightarrow\pi_1(M)$ is an isomorphism for each component N_i of the boundary N. Then $\pi_1(M)$ has one end. (2) Let M be an aspherical 3-dimensional manifold whose boundary N is an orientable surface of genus one, and suppose that the injection $\pi_1(N)\rightarrow\pi_1(M)$ is not an isomorphism. Then M is finite and orientable, and $\pi_1(M)$ is free cyclic. So $\pi_1(M)$ has two ends. (Received February 7, 1955.)

527*t*. R. L. Plunkett: *A fixed point theorem for continuous multi-valued transformations.*

Using one definition of continuity for multi-valued transformations (see Strother, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 988–993), it is proved in this paper that a nondegenerate continuous curve (compact, locally connected, metric, continuum) has the fixed point property for continuous multi-valued transformations if and only if it is a dendrite. This is an extension of a theorem in the above paper which states that a bounded closed interval of real numbers has the fixed point property. A corollary to the theorem proved in this paper is: *If X and Y are nondegenerate continuous curves, then $X \times Y$ does not have the fixed point property for continuous multi-valued transformations.* (Received February 28, 1955.)

528*t*. A. H. Stone: *Metrisability of decomposition spaces.*

Let f be a continuous mapping of a metric space S onto a topological space E. If f is closed, the following statements are equivalent: (1) E satisfies the first countability axiom, (2) for each $p \in E$, the frontier of $f^{-1}(p)$ (in S) is compact, (3) E is metrisable. (This completes results of Vainstein, Whyburn, and Hanai.) Sufficient conditions are given for E to be metrisable when f is open, or when f is quasi-compact and S is locally compact. A simple characterization is obtained of the case in which f is simultaneously open and closed; in this case E is always metrisable. (Received February 23, 1955.)

529. Fred Supnick: *On a class of 3-manifolds and the bioduplication problem.*

Bioduplicants (e.g. proteins, nucleic acids, nucleoproteins) are here represented by 3-manifolds $K(c_1, \cdots, c_n)$ partitioned into 3-cells c_i (the building blocks) where c_i and c_j $(i \neq j)$ either have no points in common, or have a finite number of disjoint 2-cells in common (the bonding loci). The c_i and their junction "caps" are presumed to be finite-polyhedral. Let F and G be noninterlinking entities resulting from a bioduplication process; it is presumed that neither ruptures during separation. This is formulated topologically as follows: Let $F(f_1, \cdots, f_n)$ and $G(g_1, \cdots, g_n)$ be partitioned 3-manifolds with $I(F) \cdot I(G) = 0$ which may be isotopically deformed without interiors overlapping to F', G' and F'', G'' such that (i) $(F'+G')$ is a partitioned 3-manifold whose 3-cells are $(f_i'+g_i')$ $(i=1, \cdots, n)$, $(F'+G')$ being isotopically deformable to F and to G so that $(f_i'+g_i')$ coincides with f_i and g_i respectively, and (ii) F'' and G'' fall on opposite sides of a surface topologically equivalent to a plane. It is shown that it is possible to isotopically deform F to K so that any line perpendicular to a plane P and intersecting K, intersects it only in a line segment all of whose points belong to the same 3-cells. That is, F has a 2-dimensional structure. (Received March 2, 1955.)

R. D. SCHAFER,
Associate Secretary
L. A. MACCOLL,
Editor, Proceedings of the Symposium

THE APRIL MEETING IN CHICAGO

The five hundred thirteenth meeting of the American Mathematical Society was held at the University of Chicago, Chicago, Illinois, on Friday and Saturday, April 22–23, 1955. Attendance was approximately 220, including 203 members of the Society.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor A. H. Taub addressed the Society on the topic *Determination of flows behind shock waves* at 2:00 P.M. on Friday, April 22.

There were five sessions for contributed papers, two on Friday morning, two on Friday afternoon, and a final session which included late papers on Saturday morning. Presiding officers for the various sessions were Professors Tibor Radó, M. E. Shanks, Ralph Hull, G. B. Price, and L. M. Graves.

The files of the Mathematical Sciences Employment Register were available to interested parties during the course of the meeting.

The Society was entertained by a tea in the Common Room of Eckhart Hall on Friday afternoon, an occasion at which the ladies of the local Department of Mathematics acted as hostesses.

Abstracts of the papers presented follow. The name of a paper presented by title is followed by "*t*." In the case of joint authorship, the name of the person presenting the paper is followed by (p).

ALGEBRA AND THEORY OF NUMBERS

530. A. A. Albert (p) and Mrs. M. S. Frank: *New classes of simple Lie algebras.*

Let $B_n = F[x_1, \cdots, x_n]$ be the algebra of all polynomials in $x_1, \cdots, x_n$ with coefficients in a field F of characteristic p subject only to the condition that $x_1^p = \cdots = x_n^p = 0$. Then it is shown that the set of all derivations $A = (a_1, \cdots, a_n)$ with divergence equal to the sum of coordinates is a simple Lie algebra of dimension $(n-1)p^n$ over F. The set of all derivations $D(\phi)$ defined by $a_i = \partial\phi/\partial x_{i+m}$, $a_{i+m} = -\partial\phi/\partial x_i$, where $n = 2m$ and the leading coefficient of ϕ is zero, forms a Lie algebra of dimension $p^{2m}-2$. This algebra is simple but has the same dimension as a classical matric algebra. It is proved that it is isomorphic to the classical algebra if and only if $m = 1$ and $p = 3$. The algebra suggests the definition of a class of Lie algebras defined by the multiplication table $u_\alpha u_\beta = f(\alpha, \beta)u_{\alpha+\beta}$ where the subscripts range over the elements of an elementary p-group which is abelian and of order p^n, and where f is a functional. Using this definition we construct algebras which yield simple Lie algebras of dimensions p^m, $p^m - 1$, and $p^m - 2$ for every integer $m > 1$ and every odd prime p. (Received March 3, 1955.)

531. J. R. Büchi (p) and J. B. Wright: *Invariant theory in groups.* Preliminary report.

A number of examples are presented illustrating different types of theories derived from group-theory. The examples suggest a key-concept, which is important in organizing these algebraic theories into a hierarchy, corresponding to Klein's hierarchy of geometries. (Received April 4, 1955.)

532t. A. A. Grau: *Relative irreducibility of polynomials in a skew-field.* Preliminary report.

Let a^α be the image of an element a under an automorphism α of a skewfield S. The set of formal polynomials $f(x) = \sum a_i x^i$ with coefficients a_i in S form a Euclidean ring $S_\alpha[x]$ if $a^\alpha x = xa$. The polynomial $f(x)$ is α-irreducible if it cannot be expressed as a product of nontrivial factors in $S_\alpha[x]$. An α-irreducible polynomial may determine a two-sided ideal $\mathfrak{a}$ in $S_\alpha[x]$, which is maximal as a left ideal, so that $S_\alpha[x]/\mathfrak{a}$ is a skewfield in which $f(x)$ has a solution. The case where such an ideal is not determined remains to be considered. (Received March 8, 1955.)

533. Franklin Haimo: *Some extensions of normal subgroups in a class of semi-direct products.*

Let H be an abelian group, let K be a group, and let G be a semi-direct product of H and K where H is normal in G. If V is a normal, n-nilpotent subgroup of K, conditions are found which allow one to extend V to a normal, n-nilpotent subgroup of G. In particular, if the circle composition (Jacobson, *Lectures in abstract algebra*, I) of the automorphisms of H generated by any two elements of K is the identity, then the members of the ascending central series of the normal subgroup V extend easily to the corresponding members of the ascending central series of the extension of V to a normal subgroup of G. Conditions are found on a single element k of K so that the normal subgroup of K generated by k will have the above extension property. (Received March 9, 1955.)

534. D. R. Hughes: *Planar division neo-rings.*

The axioms of a division ring can be weakened by discarding the demand that addition be associative and commutative, and demanding instead that the addition form a loop and that the resulting system retain the property of coordinatizing a projective plane (see Paige, *Neofields*, Duke Math. J. vol. 16 (1949)). Such a system will be called a planar division neo-ring (PDNR); examples of infinite PDNRs which are not division rings are known. If a PDNR is finite and has power-associative multiplication, then its addition is shown to be commutative and to have the inverse property. The chief result is the following: if p is a prime divisor of the order of a finite PDNR whose multiplication is associative and commutative, then the mapping $x \to x^p$ is an automorphism. Mainly by means of this result, it can be shown that all PDNRs with associative multiplication and order $\leqq 300$ actually have prime-power order. Non-prime-power integers which cannot be rejected by any known means exist however; the smallest known to the author is 2501. (Received March 7, 1955.)

535. Arno Jaeger: *On the replacibility of partial differential equations by ordinary ones in fields of prime characteristic.*

Let F be a separably generated algebraic function field of n independent indeterminates of prime number characteristic $p > 0$ with a separating transcendence basis $(x_1, x_2, \cdots, x_n)$ over its ground field K. Let D_i $(i = 1, 2, \cdots, n)$ be the partial derivations of F over K uniquely defined by $D_i(x_j)=\delta_{ij}$ (Kronecker delta), and let C be the field of all elements of F which are constant under each partial derivation. Then it is shown that there exist derivations d over C, for instance $d=\sum_{i=1}^{n} x_i^{p+1}D_i$, such that $(d, d^p, d^{p^2}, \cdots, d^{p^{n-1}})$ is a basis for the F-module of all derivations of F over C. Hence all partial differential equations in the D_i can be expressed as ordinary differential equations in d. (Received March 4, 1955.)

536t. J. A. Kalman: *A common abstraction of Boolean algebras and l-groups*. Preliminary report.

A distributive lattice with an involution $x\rightarrow x'$ such that $x\cap x'\leq y\cup y'$ identically will be called a *normal i-lattice*. *Results*. The operation $x\triangle y=(x\cup y)\cap(x'\cup y')$ is associative in any normal i-lattice. Every normal i-lattice is an i-sublattice (sublattice which always contains x' with x) of a normal i-lattice with a zero (element 0 such that $0'=0$). A normal i-lattice with a zero 0 may be recovered from its sublattice of "positive" elements if and only if 0 is meet-irreducible. The only subdirectly irreducible normal i-lattices are the chains with one, two, or three elements. Each of the following conditions is necessary and sufficient that a normal i-lattice L be a Boolean algebra: (a) $x\cup x'=y\cup y'$ identically; (b) if $a\triangle x=b\triangle x$ for some x, then $a=b$ (L need not be assumed distributive); (c) $x\cap(y\triangle z)=(x\cap y)\triangle(x\cap z)$ identically. An additive l-group G (e.g. vector lattice) is a normal i-lattice with $x'=-x$: the l-ideals in G are then just the normal subgroups of G which contain with any x also all $x\triangle y$ for y in G. Every normal i-lattice is an i-sublattice of a vector lattice: two proofs are given, both requiring the axiom of choice. (Received March 7, 1955.)

537t. J. A. Kalman: *Some results in lattice theory*. Preliminary report.

I. A lattice may be defined as a set closed under associative operations $\cup$ and $\cap$ which satisfy the four absorption laws: $a\cap(a\cup b)=a\cup(b\cap a)=(b\cup a)\cap a=(a\cap b)\cup a=a$. II. $|a\cup c-b\cup c|+|a\cap c-b\cap c|=|a-b|$ is an identity in l-groups. G. Birkhoff (*Lattice theory*, 1st ed., Theorem 7.8) discovered this identity for vector lattices, but (Ann. of Math. vol. 43 (1942) p. 309) he was unable to generalize it to noncommutative l-groups. III. A number of identities for abelian l-groups are proved. Samples: (i) $2(x\triangle y)=|x-y|-|x+y|$ ($\triangle$ as in the previous paper); (ii) $2(x\triangle y)^+=|x|+|y|-|x+y|$; (iii) $|x\triangle y|=|x|\cap|y|$ (valid in all normal i-lattices: $|x|=x\cup x'$); (iv) $|x+y|+|x-y|=|x+|y||+|x-|y||$; (v) $||x+y|-|x-y|+2y|=|x+|y||-|x-|y||+2|y|$. IV. A vector lattice may be defined as a real linear space R together with a mapping $x\rightarrow|x|$ of R into R which satisfies (III (iv), III (v), (a) $|kx|=|k||x|$ for all real k, and (b) $|x|=0$ implies $x=0$. If the postulate (b) is omitted, $R\cong S\times T$ (direct union) with S a vector lattice and $|t|=0$ for all t in T. Other definitions of vector lattices in terms of a postulated absolute are given. (Received March 7, 1955.)

538. L. A. Kokoris: *Simple power-associative algebras of degree two*.

A gap in the structure theory of power-associative algebras is closed by proving the following theorem. Let A be a simple commutative power-associative algebra of degree two over a center F of characteristic zero. Then A is a Jordan algebra. (Received January 6, 1955.)

539. P. J. McCarthy: *Sufficient conditions for a genus of indefinite ternary quadratic forms to contain only one class*.

This paper continues the work of A. Meyer (see B. W. Jones, Canadian Journal of Mathematics vol. 4 (1952) pp. 120–128, vol. 5 (1953) pp. 271–272; B. W. Jones and D. B. Marsh, Bull. Amer. Math. Soc. Abstract 61-3-393). Let f be a primitive, indefinite, ternary quadratic form with integral matrix A. Let Ω be the g.c.d. of the two-rowed minor determinants of A, and let the integer Δ be determined by $|A| = \Omega^2\Delta$. Then f is in a genus of one class provided it satisfies one of the following sets of conditions. (1) (Ω, Δ) divides 6, $\Omega \not\equiv 0 \not\equiv \Delta \pmod 4$, $|A| \not\equiv 0 \pmod{81}$, and $c_3(f) = 1$ (where $c_3(f)$ is the Hasse-Pall invariant of f with respect to 3). (2) $\Omega \equiv 8 \pmod{16}$, $\Delta \equiv 2 \pmod 4$, and $(\Omega, \Delta) = 2$. (3) $\Omega \equiv 4 \pmod 8$, $\Delta \equiv 2 \pmod 4$, and $(\Omega, \Delta) = 2$. (4) $\Omega \equiv 8 \pmod{16}$, Δ is odd, $(\Omega, \Delta) = 1$, and the generic character of f with respect to 4 is equal to $(-1)^{(\Delta-1)/2}$. The proofs of these theorems make no use of the theory of automorphs of ternary quadratic forms. (Received March 7, 1955.)

540. R. B. Reisel: *A generalisation of the Wedderburn-Malcev Theorem to infinite-dimensional algebras.*

A is an associative algebra over the field F having (Jacobson) radical N, such that $\bigcup_{k=1}^{\infty} N^k = 0$, and such that A/N is locally separable. A is assumed to be complete in the natural topology determined by the powers of N. It is then proved that if A/N has countable dimension over F, there exists a subalgebra S of A such that $A = S \oplus N$ (vector space direct sum). For the uniqueness result, this dimensionality restriction is not needed, but it is assumed that for every positive integer n, the A/N-module N^n/N^{n+1} is complete with respect to a topology in which a fundamental system of neighborhoods of zero is composed of the centralizers of finite-dimensional separable subalgebras of A/N. Then it is shown that if there exist two subalgebras S and S^* of A such that $A = S \oplus N = S^* \oplus N$, there exists $r \in N$, with quasi-inverse r', such that the mapping $x \to x - xr - r'x + r'xr$ is an isomorphism of S onto S^*. An example is given to show the necessity of the additional hypothesis on the radical in the uniqueness result when A/N is infinite-dimensional. (Received March 4, 1955.)

541t. W. E. Roth: *On the characteristic polynomial of the product of several matrices.*

If A_i, $i = 1, 2, \cdots, r$, are $n \times n$ matrices with elements in the field F, and whose characteristic polynomials are $m_i(x) = m_{i,0} + m_{i,1}x + m_{i,2}x^2 + \cdots + m_{i,r-1}x^{r-1}$, where $m_{i,j}$, $j = 0, 1, \cdots, r-1$, are polynomials in x^r with coefficients in F, and if for every k, $h = 1, 2, \cdots, r$, $A_k - A_h = \alpha_{kh}D$, where the rank of D does not exceed unity and α_{ij} is a constant in F, then the characteristic polynomial of the product, $A_1A_2A_3 \cdots A_r$, is given by $m(x)$, where $m(x^r)$ is the determinant, $|\mu_{i,j}|$, and $\mu_{i,j} = m_{i,j-i}x^{j-i}$ if $j \geqq i$ and $\mu_{i,j} = m_{i,r+j-i}x^{r+j-i}$ if $j < i$. (Received February 28, 1955.)

542. H. A. Simmons: *A class of maximum numbers associated with a symmetric equation in n reciprocals.*

Our first paper on this subject appeared in Trans. Amer. Math. Soc. vol. 34, pp. 876–907. Equation (21) of that article was beyond the scope of the procedure there and in three later papers. However, in the present paper, the terms "Kellogg solution" and "E-solution" are used with the meanings they had in the first paper, and Theorems 2 and 3 on page 887 of that article are extended to cover the case of the equation (21) just mentioned; that is, in previous language, "the remarkable properties" of the Kellogg solution of equation (21) have been established. Our main new developments are use of a special form in which to represent any E-solution of (21), definition of the pseudo-Kellogg solution relative to any given E-solution of (21) or to certain associated derived sets, and use of essentially our previous transformations to carry sets toward or actually into pseudo-Kellogg solutions. (Received March 9, 1955.)

543. Leonard Tornheim: *Inessential discriminant divisors of normal quartic fields.*

Let K be an algebraic number field generated by a root of the irreducible equation $x^4+Ax^2+B=0$ where A and B are rational integers and no square dividing A has its square dividing B. All normal quartic fields are included. Necessary and sufficient conditions on A and B are given for 3 to be an inessential discriminant divisor. Also the highest power of 3 possible is the first. (Received March 9, 1955.)

544. L. M. Weiner: *Note on S-matrices.*

Let $u=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ be an n-dimensional vector over a field F subject to the condition $\sum_{i=1}^{n} \alpha_i \neq 0$, and let $A=(a_{ij})$ be an n by n matrix over F. Then A is called an S-matrix for the vector u if $\sum_{i=1}^{n} \alpha_i a_{ij} = \sum_{i=1}^{n} \alpha_i a_{ji} = S(A)$ for $j=1, 2, \cdots, n$. If one redefines multiplication of matrices such that when $A=(a_{ij})$ and $B=(b_{ij})$, then $C=AB=(c_{ij})$ with $c_{ij}=\sum_{k=1}^{n} \alpha_k a_{ik} b_{kj}$, it is seen that the set of all S-matrices for a fixed vector u forms a subalgebra R of the total matric algebra of degree n for which $S(A+B)=S(A)+S(B)$, $S(\alpha A)=\alpha S(A)$ for α in F, and $S(AB)=S(A)S(B)$. The algebra R is the direct sum of the one-dimensional ideal M consisting of all matrices A for which $a_{ij}=a$, a constant, and the $(n-1)^2$-dimensional ideal N consisting of all matrices A for which $S(A)=0$. If $\alpha_k \neq 0$, then a basis for the ideal N is given by the matrices A_{ij} $(i, j=1, 2, \cdots, k-1, k+1, \cdots, n)$ where A_{ij} is the matrix which has $\alpha_i\alpha_j$ in the kth row and kth column, $-\alpha_k\alpha_i$ in the kth row and jth column, $-\alpha_k\alpha_j$ in the ith row and kth column, and α_k^2 in the ith row and jth column. (Received February 18, 1955.)

545. Oswald Wyler: *On ranks in regular rings.* Preliminary report.

In an earlier paper (Compositio Math. vol. 9 (1951) pp. 193–208), the author has defined a general notion of rank for elements of rings, especially regular rings. In the present paper, an ordering of ranks in a regular ring $\mathfrak{R}$ is studied. If Ra denotes the rank of an element a of $\mathfrak{R}$, and if a and b are in $\mathfrak{R}$, we say that $Ra \leqq Rb$ if there are elements p, q in $\mathfrak{R}$ such that $a=pbq$. Then $Ra \leqq Rb$ if and only if there is c in $\mathfrak{R}$ such that $Rb=Ra+Rc$. This definition leads to a quasi-ordering of ranks in $\mathfrak{R}$. If the lattice of principal left ideals of $\mathfrak{R}$ satisfies weak continuity conditions, the relation $Ra \leqq Rb$ defines a (partial) ordering of ranks in $\mathfrak{R}$. (Received March 9, 1955.)

ANALYSIS

546. W. A. Beck. *Some commutation relations in a Hilbert space.*

Let N, M, and B denote (bounded) operators in a Hilbert space $\mathfrak{H}$. Consider the operator $C=BN-MB$. Suppose N is a normal operator. An extension of a method of proof used by Fuglede (*A commutativity theorem for normal operators*, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 35–40) shows that the commutation relations $CN=NC$ and $MN=NM$ imply that $BN=NB$ if certain restrictions are imposed on the spectra of N and M, namely, (i) $\lambda \in \text{ptsp } N$ implies $\lambda \notin \text{ptsp } M$, and (ii) $\lambda \in \text{Closure } (\text{sp } N - \text{ptsp } N)$ implies $\lambda \notin \text{sp } M$. (This work was supported in part by the National Science Foundation research grant NSF-G481.) (Received March 10, 1955.)

547. J. S. Frame: *A hypergeometric solution of the equation $x^n = px - q$.*

If $p, q > 0$ and $u = q^{n-1}p^{-n} < (n-1)^{n-1}n^{-n}$, the smaller positive root x of the equation $x^n = px - q$ can be expressed explicitly in terms of the generalized hypergeometric series $f_a = f_a(n, q) = \sum_{k=0}^{\infty} C_k^{nk+a}u^k$ by each of the three formulas $px/q = nf_0/[1+(n-1)\cdot f_0] = f_1/f_0 = q^{-1}\int_0^q f_0 dq$. The proof of this theorem is based on the lemma $\sum C_k^{nk+a} \cdot t^k(1-t)^{nk-k+a} = (1-nt)^{-1}$ for integral n, $a \geqq 0$, which is proved by using a contour integral in the complex plane. For the cubic case $n = 3$, the functions f_0 and f_1 are the hypergeometric functions $F(1/3, 2/3, 1/2, q^3/4)$ and $F(4/3, 2/3, 3/2, q^3/4)$ whose ratio can be expressed as a hypergeometric continued fraction. All three roots of the irreducible cubic are then expressed to at least five significant figure accuracy by using certain simple remainder estimates for the fourth convergents of two hypergeometric continued fractions. (Received March 9, 1955.)

548. Jesus Gil de Lamadrid: *On mappings and their derivatives in locally convex topological vector spaces.*

Let E and F be two such spaces, $\mathcal{G}(E, F)$ the space of all mappings of E into F, let $\mathcal{L}(E, F) \subset \mathcal{G}(E, F)$ consist of all linear continuous mappings, and τ be a topology of $\mathcal{G}(E, F)$. For $f \in \mathcal{G}(E, F)$, $\Delta_t(x_0)f \in \mathcal{G}(E, F)$ is defined as $\Delta_t(x_0)f(h) = (f(x_0+th) - f(x_0))/t$, $t \neq 0$, and $f'(x_0) = \lim_{t\to 0} \Delta_t(x_0)f \in \mathcal{G}(E, F)$. With Michal [Bull. Amer. Math. Soc. vol. 43 (1937) pp. 394–401], $f'(x_0)$ is called the (τ) derivative of f at x_0. For normed E, and F the field R of reals, f' becomes the gradient of f of E. H. Rothe [Duke Math. J. vol. 15 (1948) pp. 421–431]. It is shown that $f': E \rightarrow \mathcal{G}(E, F)$ generalizes the ordinary idea of derivative ($E = R$) and shares with it many important properties. For instance, f' determines f within a constant. These ideas are applied to generalizing certain results of Rothe [op. cit.], relating the continuity properties of $f: E \rightarrow R$ to those of its gradient f'. The main result states that if $f'(E) \subset \mathcal{L}(E, F)$, τ is the bounded-open topology and f' is compact, then f (restricted to bounded sets) is continuous in a topology, in strength (fineness) between the weak and the strong topologies of E—weak and strong in the sense of Dieudonné [Bull. Amer. Math. Soc. vol. 59 (1953) pp. 495–512]. These results extend to mappings $f: K \rightarrow F$, for $K \subset E$ and K having some suitable properties. (Received March 9, 1955.)

549t. Casper Goffman and G. M. Petersen: *A class of consistent summability methods.*

A set of regular summability methods which are mutually consistent and together sum all bounded real sequences is given. Indeed, the methods in the set may all be taken to be stronger than any given regular method. (Received February 14, 1955.)

550. Casper Goffman and G. M. Petersen: *Submethods of a matrix summability method.*

A submethod of a regular matrix method is one obtained by deleting a set of rows from the matrix. A one-one correspondence is established between the submethods of a method A and the interval (0, 1]. For every bounded sequence not summed by A, the set of submethods which sum it is of the first category but may be of measure 0 or 1. The set of submethods equivalent to A is of the first category, but A has equivalent methods B and C such that the set of submethods of B equivalent to B is of measure

0 and the set of submethods of C equivalent to C is of measure 1. (Received February 14, 1955.)

551. E. L. Griffin, Jr.: *Isomorphism of substantial rings of operators.*

Let M, $\tilde{M}$ be substantial rings of operators on Hilbert spaces H, H of arbitrary dimension, with commutants M', $\tilde{M}'$ respectively (substantial refers to the property of having no type III part). As in a previous paper, *Some contributions to the theory of rings of operators*, Trans. Amer. Math. Soc. vol. 75 (1953) pp. 471–504, one considers essentially bounded coupling operators C, $\tilde{C}$ of the above rings. If ϕ is an isomorphism (not necessarily adjoint preserving) of M onto $\tilde{M}$ carrying C into $\tilde{C}$, then there exists a linear mapping W which takes a dense linear subset of H in a one-to-one manner onto a dense linear subset of $\tilde{H}$ and which induces ϕ. This implies that such isomorphisms preserve weak and σ-weak convergence of uniformly bounded directed sets in arbitrary substantial rings. The method of proof is similar to that of the paper quoted above. (Received March 9, 1955.)

552t. Peter Henrici: *On generating functions of the Jacobi polynomials.*

Every solution of $(D)\,\partial^2u/\partial x^2+\partial^2u/\partial y^2+(2\mu+1)x^{-1}\partial u/\partial x+(2\nu+1)y^{-1}\partial u/\partial y=0$ which is regular at (0, 0) and even in x and in y admits a series expansion of the form $\sum a_n\rho^nP_n^{(\nu,\mu)}(\tau)$, where $\rho=x^2+y^2$, $\rho\tau=x^2-y^2$, and may therefore be regarded as a generating function of the Jacobi polynomials. The special solution $u(x, y)=\omega^{-1}(\omega+1+\rho)^{-\mu}(\omega+1-\rho)^{-\nu}F(-\kappa,\kappa+1;\ \mu+1;\ 2^{-1}\omega^{-1}(\omega-1-\rho))F(\kappa,-\kappa+1;\ \nu+1;\ 2^{-1}\omega^{-1}(\omega-1+\rho))$, where $\omega=(1-2\rho\tau+\rho^2)^{1/2}$, contains several of the known generating functions as special cases. Its coefficients a_n can be represented by generalized hypergeometric sums. Letting $\tau=2\xi\nu^{-1}-1$, $\kappa^2=-\nu\eta$, $\nu\rightarrow\infty$ yields the Hardy-Hille series of products of Laguerre polynomials. By specializing τ one obtains identities of Cayley-Orr type (see Bailey, Cambridge tract No. 32, p. 84) between the coefficients of the Taylor expansions of certain products of hypergeometric functions. (Received March 11, 1955.)

553t. Peter Henrici: *Addition theorems for general Legendre and Gegenbauer functions.*

The classical addition theorems of Heine, Hobson and R. Lagrange for the Legendre functions with upper index zero are extended to general Legendre functions. A typical result is the following: $\rho_1'^{-\nu}P_\mu^{-\nu}(\rho_1)=2^\nu\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)(\mu+\nu+1)_n(\nu-\mu)_n\cdot P_\mu^{-\nu-n}(\rho)P_\mu^{-\nu-n}(\tau)C_n^\nu(\eta)$, where μ, ν are arbitrary, $\rho_1=\rho\tau-\rho'\tau'\eta$, $\rho'=(\rho^2-1)^{1/2}$, $\tau'=(\tau^2-1)^{1/2}$, $\rho_1'=(\rho_1'^2-1)^{1/2}$, the expansion being valid provided $\Re\rho\geqq 0$, $\Re\tau>0$, ρ not real and $\leqq 1$, and η such that ρ_1 lies in the interior of the ellipse with foci at $\rho\tau\pm\rho'\tau'$ and passing through -1. Similar expansions are proved for the Legendre functions of the second kind and for the general Gegenbauer function. Numerous known results, among them Gegenbauer's addition theorems for the Bessel functions, are obtainable from these formulae as special or limiting cases. The proofs make use of a one-to-one correspondence, given by $f(w)=(1-w)^{-\mu-\nu-1}u(c(1-w)^{-1},\ -icw(1-w)^{-1})$, between the solutions $u(x, y)$ of (D) (see preceding abstract) regular at $(c, 0)$ $(c\neq 0)$ and the analytic functions $f(w)$ regular at 0. (Received March 11, 1955.)

554. R. D. James: *Generalized primitives and n-convex functions.*

A generalized integral of Perron type which, starting with a function $f(x)$, goes directly to a primitive $F(x)$ of order n, was defined in a previous paper [Trans. Amer. Math. Soc. vol. 76 (1954) pp. 149–176]. In the present paper it is shown that $F(x)$ has generalized derivatives $D^{n-2k}F(x)$ for $1 \leq k \leq [n/2]$ and that $D^nF(x) = f(x)$ almost everywhere. These generalized derivatives may be regarded as primitives of order $n-2k$. They are not, in general, continuous but they cannot have ordinary discontinuities. The proof requires some differentiability properties of convex functions of order n, that is, functions whose divided differences of order n are all non-negative. (Received March 7, 1955.)

555. J. H. B. Kemperman: *Bounds on polynomials.*

Let $f(w) = \sum_0^n (a_j \cos jw + b_j \sin jw)$, a_j and b_j real, $|f(u)| \leq 1$ for u real. Then for u, v real: (a) $|f(u+iv)|^2 \leq \cosh^2 nv - (1-f(u)^2)$. Equality only for $v=0$ or $f(w) = \cos(nw-\alpha)$, α real. Let $P(z) = \sum_0^n a_j z^j$, a_j real, (b) $|P(z)| \leq 1$ for $-1 \leq z \leq +1$; N. G. de Bruijn conjectured that for $k=0, 1, \cdots, n$ and any complex z, (c) $|P^{(k)}(z)| \leq T_n^{(k)}(z^*)$. Here, $z^* = (|z-1|+|z+1|)/2$, $T_n(z) = \cos(n \arccos z)$. It is shown that (c) holds for $k=0$, $k=1$ and (obviously) $k=n$. Equality for $k=1$ only if $P(z) = \pm T_n(z)$ and $z = \pm z^*$. The case $k=0$ follows from (a) putting $z = \cos w$. A. C. Schaeffer and G. Szegö, Trans. Amer. Math. Soc. (1941), proved a.o. (c) for z real when (b) is replaced by the weaker assumption (d) $|P(\cos m\pi/n)| \leq 1$ ($m=0, 1, \cdots, n$). Let β_k be the largest zero of $T_n^{(k-1)}(z)$ ($\beta_1 = \cos \pi/2n$, $\beta_2 = \cos \pi/n$, $0 < \beta_{k+1} < \beta_k$). It is shown that even when (b) is replaced by (d) one has $|P^{(k)}(z)| \leq |T_n^{(k)}(z)| \leq T_n^{(k)}(z^*)$ for $|z| \geq \beta_k$ ($k=0, 1, \cdots, n$; $\beta_0 = 1$). For $k=0$ this was already proved by P. Erdös, Bull. Amer. Math. Soc. (1947). (Received March 9, 1955.)

556t. M. S. Webster: *A recurrence relation for ultraspherical polynomials.*

The ultraspherical polynomials are uniquely determined by the nonlinear recurrence relation given by Thiruvenkatachar and Nanjundiah [Proc. Indian Acad. Sci. Sect. A vol. 33 (1951) pp. 373–384].

557. F. M. Wright: *Some sufficient conditions for a determinate Hamburger moment sequence.*

Let $\{\mu_n\}$ ($n=0, 1, 2, \cdots$) be a given positive definite Hamburger moment sequence. Using certain relationships between the moments and the coefficients in the J-fraction expansion of the formal power series $\sum_{n=0}^{\infty} \mu_n/z^{n+1}$, together with Carleman's inequality, Schwarz's inequality, and a sufficient condition in terms of the coefficients in the J-fraction expansion of $\sum_{n=0}^{\infty} \mu_n/z^{n+1}$ for $\{\mu_n\}$ to be a determinate Hamburger moment sequence, the author obtains an infinite sequence of sufficient conditions in terms of the moments themselves for $\{\mu_n\}$ to be a determinate Hamburger moment sequence. Another infinite sequence of sufficient conditions, each simpler but weaker than the corresponding condition of the above sequence, is then given. It is shown how the famous Carleman criterion for a determinate Hamburger moment sequence follows from the preceding and hence is weaker than any of the sufficient conditions proved thus far in this paper. Finally, another sufficient condition in terms of the moments themselves for $\{\mu_n\}$ to be a determinate Hamburger moment sequence is given, but the author does not know how this condition compares with those proved earlier in this paper. (Received March 7, 1955.)

558. R. E. Zindler: *The qualitative behavior of integral curves of systems of three differential equations near a singular point.*

Integral curves of differential equations of the form $dx_i/dt = U_i(x_i)$ are studied for $i = 1, 2, 3$ and where the U_i are continuous and either all vanish or one or more are infinite at the origin or the limit as $r = (x^2+y^2+z^2)^{1/2} \to 0$ of at least one of them is not independent of the direction of approach. It is assumed that no other singularities exist for $r \leq r_0$. The concept of an exceptional direction is defined and tests are provided to determine whether a direction is exceptional. The equations are rewritten in spherical coordinates $dr/dt = P_1(r, \phi, \psi)$, $r\, d\phi/dt = P_2(r, \phi, \psi)$, $r \sin \phi\, d\psi/dt = P_3(r, \phi, \psi)$ and it is further assumed that the $\lim_{r \to 0} P_i\Delta^{-1}$ where $\Delta = (\sum U_i^2)^{1/2}$ are independent of the direction of approach. Numerous theorems are stated in regard to the behavior of curves in the neighborhood of an exceptional direction with various conditions placed on the qualitative behavior functions $K = P_1\Delta^{-1}$ and $K_{ij} = P_iP_j^{-1}$. It is shown that in the neighborhood of each nonexceptional direction there exists a cone about the direction from which no curve can enter the origin. (Received March 8, 1955.)

559*t*. R. E. Zindler: *A note on L. E. Reizin's paper "Behavior near to a singular point of integral curves of a system of three differential equations.*

An examination of L. E. Reizin's paper (Math. Reviews vol. 15, p. 311) showed that in applying the conditions of Theorem 2, Reizin concluded that $m = 1$. A study of the conditions of the theorem revealed that m could be any positive integer. The proof of Theorem 2 is made in the same manner as Reizin's when the integral condition on the function $B(r)$ is modified to be $\int_0^a B(r)r^{-m-1}dr < +\infty$. In a manner suggested by Dr. J. L. Brown, Jr., the existence of a suitable function $B(r)$ is shown if functions f and g are Holder continuous in the radial variable r. (Received March 8, 1955.)

APPLIED MATHEMATICS

560. P. C. Hammer (p) and A. W. Wymore: *Numerical evaluation of multiple integrals.* Preliminary report.

There are established formulas for numerical integrations over adequately symmetrical regions in n-space for $n \geq 2$. For example, one formula, based on fourteen points in the solid sphere, gives exact integrals for the general fifth degree polynomial in three variables. This formula being valid under affine transformations then applies to integrals over arbitrary ellipsoids. Generation of formulas for cartesian product regions such as cylinders, squares, and cubes is shown to be readily done provided formulas for the factor spaces are known. Hitherto the full use of affine transformations and cartesian products seems not to have been made. Tests of formulas at the University of Wisconsin Numerical Analysis Laboratory have indicated that many will be satisfactory for machine calculations and some will be useful for desk calculation. (Received March 7, 1955.)

561. Jacob Korevaar: *Distributions defined by fundamental sequences.* II. *The definite integral.*

Define $f^{(-1)}(t) = \int_0^t f(u)du$, $f^{(-2)}(t) = \int_0^t f^{(-1)}(u)du$, etc. A sequence of functions $f_n(t)$, integrable for $t \geq 0$, $n=1, 2, \cdots$, is called fundamental when to every $A>0$ there is a positive integer p such that $\{f_n^{(-p)}\}$ converges uniformly on $[0, A]$. A fundamental sequence $\{f_n\}$ defines a distribution ϕ on the half-line $t \geq 0$. The derivative of a distribution was defined in I (*Distributions and their Laplace transforms* $\cdots$, Bull. Amer. Math. Soc. Abstract 61-3-428). To every distribution ϕ and every $A>0$ there are an integrable function $F(t)$ and an integer $r \geq 0$ such that $\phi = [F]^{(r)}$ on $[0, A)$; $[F]^{(r)}$ denotes the rth derivative of the distribution $[F]$ associated with the function F (see I). A distribution ϕ is said to be integrable over $[a, b]$ $(a>0)$ when it may be given by a fundamental sequence $\{f_n\}$ such that $\{f_n^{(-1)}\}$ is uniformly convergent in some neighborhood of a and in some neighborhood of b. One defines $\int_a^b \phi(t)dt = \lim_{n\to\infty} \int_a^b f_n(t)dt$. The Dirac distribution δ_τ is given by $f_n(t)=n$ on the interval $(\tau, \tau+1/n)$, $f_n(t)=0$ outside that interval. Hence $\int_a^b \delta_\tau(t)dt=1$ whenever $a<\tau<b$. A distribution ϕ is shown to be integrable over $[a, b]$ if and only if there is a continuous function $G(t)$ such that in some neighborhood of a and in some neighborhood of b one has $\phi=[G]'$; $\int_a^b \phi(t)dt$ is equal to $G(b)-G(a)$. (Received March 10, 1955.)

562*t*. Jacob Korevaar: *Distributions defined by fundamental sequences.* III. *Convergence, multiplication and division.*

A sequence of functions $f_n(t)$ integrable for $t \geq 0$ is said to be convergent to the distribution ϕ on (a, b) $(a \geq 0)$ when there are an integer $p \geq 0$, a function $F(t)$ integrable for $t \geq 0$, and a sequence of polynomials $P_n(t)$ of degree $\leq p-1$ such that $f_n^{(-p)}(t) - P_n(t) \to F(t)$ as $n \to \infty$, uniformly or dominatedly on $[a, b]$, while $\phi = [F]^{(p)}$ on (a, b). When the $f_n(t)$ are continuous and possess a piece-wise continuous derivative $f_n'(t)$ on $[a, b]$, then $f_n \to \phi$ on (a, b) implies $f_n' \to \phi'$ on (a, b). It follows that every trigonometric series $a_0/2 + \sum_{n=1}^\infty (a_n \cos nt + b_n \sin nt)$ with $|a_n|+|b_n|<Mn^k$, $n=1, 2, \cdots$, is convergent on every interval (a, b). One defines the product ϕg of a distribution ϕ given by a convergent sequence $\{f_n\}$ as above and a function $g(t) \in C^k[a, b]$ with $k \geq p$ by the sequence $\{f_n g\}$ which is convergent on (a, b). When ϕ is given as $[H]^{(r)}$ on (a, b), $H(t)$ integrable for $t \geq 0$, $r \leq k$, then an equivalent definition is $\phi g = [Hg]^{(r)} - C_{r,1}[Hg']^{(r-1)} + \cdots + (-1)^r[Hg^{(r)}]$. Example: $\delta_\tau = [U_\tau]'$ where $U_\tau(t)$ is the unit step function, hence $\delta_\tau(t)g(t) = g(\tau)\delta_\tau(t)$ whenever $g \in C^1$ or even $g \in C^0$. Let $\phi g = 0$ on (a, b). When $g(t)>0$ on (a, b), then $\phi=0$ on (a, b); when ϕ is not equal to 0 on any subinterval of (a, b), then $g=0$ on (a, b). Division is studied. Example: the equation $\phi(t)\cdot(t-\tau) = \omega(t)$ $(a<\tau<b)$ always has infinitely many solutions $\phi(t)$ on (a, b). Any two solutions differ by $c\delta_\tau(t)$. (Received March 10, 1955.)

GEOMETRY

563*t*. John DeCicco: *Some theorems on the projective geometry of surfaces.*

A set of six functions 1, m, n, λ, μ, ω, of class C^3 in a two-dimensional region represent the Christoffel symbols of the second kind of a surface S if and only if they obey a certain system of five partial differential equations, one of which is of the first order, and the remaining ones are of the second order. Necessary conditions are obtained that a second order differential equation of the cubic type represents that of the ∞^2 geodesics of a surface S. In particular, a new generalization of Beltrami's theorem on geodesic mappings of surfaces is obtained. Thus, in projective maps, we have the two independent projective invariants: $U = -G\partial K/\partial x + F\partial K/\partial y$, $V = F\partial K/\partial x - E\partial K/\partial y$, where E, F, G, are the fundamental magnitudes of first order and K is the Gaussian curvature of a surface S. (Received January 4, 1955.)

LOGIC AND FOUNDATIONS

564. B. A. Galler: *Cylindric and polyadic algebras.* Preliminary report.

Halmos (Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 296–301) developed the theory of polyadic algebras as an algebraic analogue of the first-order predicate calculus. Tarski (Bull. Amer. Math. Soc. Abstracts 58-1-85, 58-1-86) developed the cylindric algebras for the same purpose. It is shown in this paper that the cylindric algebras are essentially the polyadic algebras which contain a "binary predicate" analogous to the identity predicate of the first-order predicate calculus. (Received March 9, 1955.)

565*t*. P. C. Hammer: *The connectedness concept in general topology.*

A set-valued set function g defined for all subsets of a space M is an *inclusion preserving enlargement* (i.p.e.) *function* if gX contains X and if X contains Y implies gX contains gY universally. Let g_1 and g_2 be two i.p.e. functions. A pair X, Y of proper subsets of M are said to be *g_1g_2-separated* if the intersection of g_1X and g_2Y and the intersection of g_2X and g_1Y are empty. A set X is *g_1g_2-connected* provided no proper dichotomy of the set is g_1g_2-separated. Many of the usual theorems concerning connected sets as well as certain stronger ones are shown to hold for this generalization of connectedness. Let S be a class of pairs (U, V) of sets such that (U, V) in S implies (V, U) in S and the intersection of U and V is empty. A pair X, Y of sets is *S-separated* if there exists (U, V) in S such that U contains X and V contains Y. The connectedness based on this concept is appropriate in many theorems on connected sets. (Received March 7, 1955.)

566*t*. A. R. Schweitzer: *A bond between the foundations of geometry and transformation group theory.* I.

Kinematics is interpreted as a bond connecting the foundations of geometry and Lie's theory of transformation groups. In the foundations of geometry kinematics is represented by metrical properties. The latter have been axiomatically expressed by Hilbert in terms of equality (congruence) of linear segments (*Grundlagen der Geometrie*, 7th ed., Leipzig and Berlin, 1930, p. 11) by F. Schur in terms of motion as a defined concept on the basis of descriptive-projective properties (*Grundlagen der Geometrie*, Leipzig and Berlin, 1909) by Pieri in terms of motion as an undefined concept (Turin, 1899) and by others. (Received March 8, 1955.)

567*t*. A. R. Schweitzer: *A bond between the foundations of geometry and transformation group theory.* II.

In continuation of the preceding paper the author first discusses transition from kinematics as represented by metrical properties in the foundations of geometry to Lie's theory of transformation groups with references to Hilbert (loc. cit. p. 178) F. Schur (loc. cit. pp. iii–iv, 40) H. Weyl, *Mathematische Analyse des Raumproblems* (Berlin, 1923, p. 30), G. Kowalewski, *Einführung in die Theorie der kontinuierlichen Gruppen* (New York, 1950, pp. 20, 21–30; 97, 145). Secondly, the author considers kinematics as encompassed by Lie's theory with reference to S. Lie, *Theorie der Transformationsgruppen*, Leipzig, 1893, Part III. In his treatise Kowalewski finds it convenient to refer to R. Lipschitz's essay, *Untersuchungen über die Summen von*

Quadralen (loc. cit. p. 20) and to the quaternion (loc. cit. pp. 21–30). The author discusses these references in relation to his article in Math. Ann. vol. 69 (1910). (Received March 8, 1955.)

568*t*. A. R. Schweitzer: *On the foundations of metrical geometry.*

The author constructs a complete set of axioms for the foundations of geometry in terms of two undefined relations between n-simplexes symbolized by $\alpha_1\alpha_2 \cdots \alpha_{n+1}K\beta_1\beta_2 \cdots \beta_{n+1}$, $\alpha_1\alpha_2 \cdots \alpha_{n+1}E\beta_1\beta_2 \cdots \beta_{n+1}$ $(n=1, 2, \cdots)$ and respectively expressing sameness of orientation and equality (congruence). In the present paper metrical geometry in terms of the latter relation is considered. The n-simplex $\alpha_1\alpha_2 \cdots \alpha_{n+1}$ is isosceles relatively to a given linear segment, say $\alpha_1\alpha_2$, if and only if $\alpha_1\alpha_2\alpha_3 \cdots \alpha_{n+1}E\alpha_2\alpha_1\alpha_3 \cdots \alpha_{n+1}$. The linear segment $\alpha_1\alpha_2$ is the "base" of the isosceles n-simplex $\alpha_1\alpha_2 \cdots \alpha_{n+1}$; the equal $(n-1)$-simplexes $\alpha_1\alpha_3\alpha_4 \cdots \alpha_{n+1}$, $\alpha_2\alpha_3\alpha_4 \cdots \alpha_{n+1}$ are "sides" and the $(n-2)$-simplex $\alpha_3\alpha_4 \cdots \alpha_{n+1}$ is the "top." From an isosceles n-simplex a set of "n-simplicial dividers" is obtained by omitting the interior points of the base; the dividers are said to lie on the associated n-simplex and two sets of dividers are equal if and only if the associated isosceles n-simplexes are equal. The author develops foundations of metrical geometry in terms of an undefined relation of equality between two sets of n-simplicial dividers. Two linear segments are equal if and only if there exist equal simplicial dividers which they subtend. (Received March 8, 1955.)

STATISTICS AND PROBABILITY

569*t*. H. S. Konijn: *On some tests for treatment effects in paired replicates.*

Several methods are considered for testing whether or not a certain treatment causes a shift in the distribution of some measured characteristic. Among the tests the distributions of which are studied in detail, Wilcoxon's is most suitable for detecting small shifts, if few a priori assumptions regarding the distribution of the measurements are justified. When the members of each pair of objects have been drawn from a finite collection of objects, the sign test, even though possessing strong minimax properties, can be exceedingly inefficient in detecting small shifts; in other cases the asymptotic efficiency is more than 1/2. In passing, tests for the two-sample location problem and some of their properties are reviewed. (Received March 9, 1955.)

570. J. F. Nash: *Definability, randomness, and indistinguishable random processes.*

One can define a concept of definability via the class of ordinals for which a Gödel constructible enumeration of the preceding ordinals exists. From this a concept of a random sequence (of 1's and 0's) can be defined. If this class of ordinals is assumed (nonconstructively) enumerable, then random sequences exist (in the idealized sense). Another idea comes from regarding a random process as a machine for generating a random sequence of 1's and 0's. Two such machines are indistinguishable if distinction cannot be made from comparison of output sequences, one from each. Or, formally, if they correspond to two measures on the space of sequences, each continuous with respect to the other. If the machines have the probability of making the nth sequence digit 1 independent of past choices a simple criterion for indistinguishability obtains.

The series formed by the squared difference of the probabilities of 1 at the nth digit for the two processes, divided by the minimum of these probabilities and the two complements, should diverge. From this idea one can obtain results of the large numbers type by comparison of the coin process with an indistinguishable one. (Received March 15, 1955.)

571. J. M. Shapiro: *A condition for existence of moments of infinitely divisible distributions.*

Let $F(x)$ be an infinitely divisible distribution and let $G(u)$ be the bounded nondecreasing function given by the Lévy-Khintchine formula for the representation of the characteristic function of $F(x)$. It is shown that $\int_{-\infty}^{\infty} x^{2k} dF(x) < \infty$ is equivalent to $\int_{-\infty}^{\infty} u^{2k} dG(u) < \infty$. A simple relation is given between the moments of $G(u)$ and those of $F(x)$ in terms of the semi-invariants of $F(x)$. (Received March 3, 1955.)

TOPOLOGY

572t. R. H. Bing: *An upper semicontinuous decomposition of E^3 into points and tame arcs whose decomposition space is not E^3.*

It is found that one of the useful theorems for E^2 does not generalize to E^3. An example is given of an upper semicontinuous decomposition G of E^3 such that each element of G is either a point or a tame arc but the decomposition space is not topologically equivalent to E^3. The elements of G are so tame that no horizontal plane intersects any one of them in two points. In terms of continuous transformations this says that there is a map f of E^3 onto a metric space X topologically different from E^3 such that for each point x of X, $f^{-1}(x)$ is either a point or a tame arc. (Received March 7, 1955.)

573. L. E. Pursell: *Fixed ideals and automorphisms of certain function rings.*

Let $S(X)$ be any ring of functions from a topological space X to a division ring D satisfying the following: (i) $Z(f)$ is closed for $f \in S$, (ii) if $x \notin$ a closed set F, $\exists f \in S \ni Z(f)$ contains a neighborhood of F but not x, (iii) if $f(x) \neq 0$ for all x in a closed set F, $\exists g \in S \ni (fg)(x) = 1$ for $x \in F$, and (iv) for $x \in X$, $\exists f, g \in S \ni x = Z(f) - Z(g)$. Define $A(f) = \{g \in S \mid fg = 0\}$ and $H(f) = \{g \in S \mid g \neq 0$ and $f/A(g)$ is a unit in $S/A(g)\}$. An ideal I in S is fixed if and only if $\exists g, h \in S \ni H(g) \not\supset H(h)$ but $H(gf) \supset H(hf)$ for each f in I. Under an isomorphism i: $S(X) \rightarrow S'(X')$ the image of a (maximal) fixed ideal is a (maximal) fixed ideal. This one-to-one correspondence between maximal fixed ideals determines a one-to-one correspondence from X onto X' which is a homeomorphism $\phi(i)$. If $A(S(X))$ is the automorphism group of $S(X)$ and $H(X)$ is the homeomorphism group of X, then ϕ is a homomorphism of $A(S(X))$ into $H(X)$. (Received March 7, 1955.)

574t. L. E. Pursell: *Fixed ideals and automorphisms of certain rings of real-valued continuous functions.*

Properties (i)–(iv) of the preceding abstract are satisfied by the ring $C(X, R)$ of all real-valued continuous functions on a normal space X with points which are G-delta sets, by the ring $C_0(X, R)$ of all real-valued continuous functions on X with compact supports if X is a locally-compact space with points which are G-delta sets, and by the ring $C^r(M)$ of all r-continuously differentiable functions on a differentiable manifold M with a neighborhood-finite coordinate covering. For $C(X, R)$ and $C_0(X, R)$, the homomorphism ϕ is an isomorphism from the automorphism group onto $H(X)$. For any subring S of $C(X, R)$ satisfying (i)–(iv) and (v) (for each $x \in X$, $\{f(x) \mid f \in S\} = R$) the homomorphism ϕ is an isomorphism into $H(X)$; any automorphism α of S can be extended to $C(X, R)$; and $f(x) = (\alpha f)(\phi(\alpha)x)$. (Received March 7, 1955.)

575. E. H. Rothe: *Mapping degree in Banach spaces and spectral theory.*

An attempt is made to define the mapping degree in Banach spaces directly and not by approximation from finite dimensional spaces as is done in the theory of Leray and Schauder as well as in that of Nagumo. Due to the lack of a definition of an orientation in Banach spaces, already the case of a linear map $f(x)$ requires consideration. For such maps, the definition of the degree is given in terms of that part of the spectrum of $F(x) = x - f(x)$ which is real and >1 (essentially using a theorem by Leray-Schauder as definition), and the homotopy theorem for the degree becomes a theorem in spectral perturbation theory. To be able to make full use of this latter theory, the paper deals with complex Banach spaces, and the results are carried over to real Banach spaces which can be imbedded in the complex spaces considered. Finally, the degree is defined and its homotopy property proved for nonlinear maps of a very smooth type. (Received February 14, 1955.)

576. D. E. Sanderson: *An extension of a deformation theorem of Alexander's.*

J. W. Alexander proved that any homeomorphism of an n-cell onto itself which is the identity on the boundary can be deformed onto the identity by an isotopy which is fixed on the boundary (Proc. Nat. Acad. Sci. U.S.A. vol. 9 (1923) pp. 406–407). An extension of this theorem for $n \leqq 3$ can be given as follows: *Let h be any orientation-preserving homeomorphism of E^3 onto itself which is the identity on a closed subset K of E^3 and maps each component of $E^3 - K$ onto itself. Then h can be deformed onto the identity by an isotopy which is fixed on K.* Using a recent result by M. K. Fort (Proc. Amer. Math. Soc. vol. 5 (1954) pp. 456–459), a theorem announced by the author (Bull. Amer. Math. Soc. Abstract 59-6-567), and Alexander's theorem, an isotopy is obtained which is fixed on K and deforms h onto a homeomorphism f which is piecewise linear on $E^3 - K$. Next, a shellable triangulation (for definitions, see abstract referred to above) of each component of $E^3 - K$ is obtained with the aid of Fort's result and, finally, f is deformed onto the identity on each simplex in the order of shelling. The desired isotopy results. (Received February 25, 1955.)

577. Jerome Spanier: *Generalised almost complex structures.* Preliminary report.

An m-dimensional differentiable manifold M is *almost complex* if the structural group of its tangent bundle is reducible to $U(n)$ ($m = 2n$ or $2n+1$). If $m = 2n+1$, a reduction to $O(2n)$ can always be made. Such an almost complex structure may be regarded as a cross section in a suitable bundle associated to the tangent bundle of M. Examples of odd-dimensional almost complex manifolds are the odd-dimensional spheres and the tangent sphere bundle of any manifold. Two almost complex structures are *equivalent* if the cross sections they determine are homotopic by a vertical homotopy. The following are proved: (1). S^{2n+1} admits infinitely many inequivalent almost complex structures if and only if $2n+1\equiv 3 \pmod 4$. (2). If $p:E\rightarrow B$ is a fiber space with almost complex base B and fiber F admitting an almost complex structure invariant under the structural group G, then E is almost complex. (2) is used to prove: (3). The Stiefel manifold $V_{n,k}=O(n)/O(n-k)$ is almost complex if either n or k is even. (1) depends on the lemma: If $p:E\rightarrow S^n$ is any fiber space admitting a cross section, then two cross sections are homotopic if and only if they are homotopic by a vertical homotopy. (Received March 7, 1955.)

578t. A. W. Wymore: *On weak compactness in functional analysis.*

Let $C(S)$ be the set of continuous functions on a compact Hausdorff space S. Three types of compactness (countable, sequential and bi-) for subsets of $C(S)$ in two topologies (point-open and weak) are analyzed. The fundamental equivalences in this context are obtained by intrinsic methods. In particular, the Osgood theorem is studied. (Theorem (Osgood, 1897): Let $\{x_n\}$ be a norm-bounded sequence in $C(S)$ such that $\lim_n x=0$ for every $t\in S$. Then $\lim_n x=0$ weakly in $C(S)$.) This theorem is shown to be equivalent to important assertions in the theory of weak compactness for which it had formerly been used as a tool (*e.g.*, The Osgood Theorem: The point-open closed convex hull of a point-open compact set is point-open compact). Corresponding theorems in Banach spaces and locally convex spaces are obtained as corollaries. (Received March 7, 1955.)

J. W. T. Youngs,
Associate Secretary

THE APRIL MEETING IN STANFORD

The five hundred fourteenth meeting of the American Mathematical Society was held at Stanford University, California, on Saturday, April 30. Attendance was approximately 100 including 86 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Irving Kaplansky of the University of Chicago and the University of California, Los Angeles delivered an address entitled *Operator algebras*. Professor Kaplansky was introduced by Professor J. L. Kelley. There were two sessions for contributed papers, over which Professors M. M. Schiffer and Randolph Church presided.

After the meetings, those attending were entertained at tea by the Stanford Department of Mathematics at the Stanford Alumni Hall.

Abstracts of papers presented at the meeting follow. The name of a paper presented by title is followed by "*t*." In the case of joint authorship, the name of the person presenting the paper is followed by (p).

ALGEBRA AND THEORY OF NUMBERS

579. Chen-Chung Chang: *A necessary and sufficient condition for an α-complete Boolean algebra to be an α-homomorphic image of an α-complete field of sets.*

Let α be any infinite cardinal. An α-complete Boolean algebra A is *α-representable* if it is an α-homomorphic image of an α-complete field of sets. For every $x \in A$, x has the property P_α, $P_\alpha(x)$, iff there exists a doubly indexed system of elements $\{a_{i,j}\}$, $i \in I, j \in J$, $\bar{I} \leq \alpha$, $\bar{J} \leq \alpha$, such that (1) $\prod_{j \in J} a_{i,j} = 0$ for each $i \in I$, and (2) for every function f on I to J the set of elements $\{a_{i,f(i)}; i \in I\}$ either contains x or contains a complementary pair. Let $I(\alpha, A) = \{x; x \in A \text{ and } P_\alpha(x)\}$. Theorems I–III below hold for α-complete Boolean algebras A. I. $I(\alpha, A)$ is an α-complete ideal in A and $A/I(\alpha, A)$ is α-representable. II. A is α-representable if, and only if, $I(\alpha, A) = \{0\}$. III. If J is any α-complete ideal in A, then A/J is α-representable if, and only if, $I(\alpha, A) \subseteq J$. IV. $I(\aleph_0, A) = \{0\}$ for any Boolean algebra A. II and IV immediately imply Loomis' result on representation of σ-complete Boolean algebras (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 757–760). It follows from I–IV that the ideal $I(\alpha, A)$ may be regarded as the α-radical of the Boolean algebra A with respect to α-representation. It is known that there exist α-complete Boolean algebras A which are not α-representable (cf. Sikorski, Fund. Math. vol. 43 (1948) p. 247). Consequently, for these special algebras $I(\alpha, A) \neq \{0\}$. (Received March 7, 1955.)

580*t*. L. A. Kokoris: *Power-associative rings of characteristic two.*

It is proved that a commutative ring A whose characteristic is 2 is power-associative if $(x^2y)y = (y^2x)x$ and $x^{2^{n-1}}x^{2^{n-1}} = x^{2^n}$ for every x, y in A and every positive integer

n. When A is a commutative algebra over a field F of characteristic 2, the identity $(x^2y)y=(y^2x)x$ is automatically satisfied if F contains at least four elements. Examples are given to show that the hypotheses are necessary. (Received February 2, 1955.)

581t. P. J. McCarthy: *Some conditions under which genera of indefinite ternary quadratic forms exist which contain more than one class.*

In the course of research to determine necessary conditions for a genus of indefinite ternary quadratic forms to contain only one class, the following results have been obtained. For similar results see B. W. Jones and E. H. Hadlock, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 539–543. Let f be a properly primitive, indefinite, ternary quadratic form with integral matrix A. Let the reciprocal form of f be properly primitive. Let Ω be the g.c.d. of the two-rowed minor determinants of A, and let the integer Δ be determined by $|A|=\Omega^2\Delta$. Let $\Omega=2^a\Omega_1^2p$ and $\Delta=-2^b\Delta_1^2p$, where Ω_1 and Δ_1 are odd, and p is an odd prime. Then the order of f contains a genus containing more than one class provided a and b are both even, $a+b\geqq 2$ when $p\equiv 5 \pmod 8$, and $a+b\geqq 4$ when $p\equiv 3 \pmod 4$. The same result holds true if $\Omega=2^a\Omega_1^2p$ and $\Delta=-2^b\Delta_1^2$, where Ω_1 and Δ_1 are odd, p is an odd prime, and p divides Δ_1. (Received March 7, 1955.)

582t. T. S. Motzkin: *Elimination theory for algebraic inequalities.*

Let $f(x)$, $x=(x_1, \cdots, x_n)$ be a polynomial with coefficients in a field K, and let $a(f)$, $b(f)$, and $c(f)$ be, respectively, the set of all x with $f=0$, $f\neq 0$ and (if K is the field R of reals) $f\geqq 0$. Let A, B, C and D be, respectively, a set obtainable by union and intersection from finitely many sets a; a and b; c; c and b. Let A_m, in n-space, be the projection of an A in $(m+n)$-space defined by polynomials $f(x, t)$, $t=(t_1, \cdots, t_m)$, and $A_{(m)}$ the same, using only polynomials homogeneous in t and excluding $t=0$. Let $\{A_m\}$ be the set of all A_m. Similarly define $\{A_{(m)}\}$, B_m, etc. Then for closed K, $\{A_{(m)}\}=\{A\}\neq\{A_m\}=\{B_{(m)}\}=\{B_m\}=B$; for $K=R$, $\{A\}$, $\{B\}$, $\{C\}$, $\{D\}$ are all different and $\{A_{(1)}\}=\{A\}$, $\{A_{(m+1)}\}=\{C_{(m)}\}=\{C\}$, $\{B_1\}=\{B\}$, $\{A_m\}=\{B_m\}=\{B_{(m+1)}\}=\{C_m\}=\{D_{(m)}\}=\{D_m\}=D$. (Received March 9, 1955.)

583. Morgan Ward: *Second order recurrences over p-adic fields.*

Let (w) be a linear recurrence of order two whose associated monic polynomial has coefficients in the rational p-adic field. Only the case when the coefficients are both units and the initial values w_0, w_1 integers is of interest. An index n for which w_n has a positive value is called a zero of (w). (w) may have no zeros; if it has, it is essentially determined by them. For each recurrence (w), there exists an essentially unique polar recurrence (v). The relationships between the zeros of (v) and (w) include as a special case Lucas' laws of apparitions and repetition for his numerical functions of the second order over the rational field. (Received March 7, 1955.)

ANALYSIS

584. J. O. Carter: *Some coefficient inequalities for schlicht functions.*

Preliminary report.

This paper presents a straightforward method of obtaining a new upper bound on the absolute value of the fifth coefficient in the power series expansion of a schlicht analytic function. The method is based on the fact that if $f(z)=z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+\cdots$ is a schlicht function, then $[f(z^{-3})]^{-1/3}=z+b_2z^{-2}+b_5z^{-5}+b_8z^{-8}+\cdots$ is also a schlicht function and $2|b_2|^2+5|b_5|^2+8|b_8|^2\leqq 1$, the classical area principle

[Bernardi, Duke Math. J. vol. 19 (1952) pp. 263–287]. The fifth coefficient a_5 is expressed as a linear combination of b_2, b_3, and b_4. The application of the above inequality leads to the upper bound $|a_5| \leq 5.4$. (Received April 28, 1955.)

585. J. M. G. Fell: *Separable representations of purely infinite algebras*. Preliminary report.

A representation of an algebra A by bounded operators on a Hilbert space K is called separable if K is separable. This work is part of a program of finding what qualitative restrictions are imposed on the *-representations of a *-algebra by the assumption that the representation is separable. Let A be a purely infinite W^*-algebra (for terminology, see Kaplansky, Ann. of Math. vol. 53 (1951) pp. 235–249), and let R be a separable *-representation of A. It is shown that R is necessarily countably additive on the central projections of A. If in addition A is of type I, a complete description of R can be given: To each cardinal $n = 0, 1, 2, \cdots, \aleph_0$ there is a unique central projection e_n in A such that (1) the e_n are pairwise orthogonal and have the unit element as their sum, and (2) R is the direct sum, over all n, of n copies of the projection of A into Ae_n. (Received February 25, 1955.)

586. R. D. Gordon: *A general solution of $dz = pdx + qdy$ with application to partial differential equations*.

A general solution of $dz = pdx + qdy$ in terms of a certain class of variable transformations is described. By means of the resulting representations of x, y, z, p, q, the general solution of a general partial differential equation of second order (in two independent variables) is determined in terms of solutions of first order equations only. The solution readily fits standard conditions on an open boundary, subject to two differential inequalities associated with the boundary, the relation of which to the theory of characteristics has not been established. (Received February 2, 1955.)

587t. R. T. Prosser: *A general dimension function for W^* algebras*.

Let A be a W^* algebra, P the projections of A, Z the center of A, Γ the spectrum of Z, and σ the canonical representation of Z onto $C(\Gamma)$. For each nonzero $p \in P$, let $\mu(p)$ be the least upper bound of the cardinalities of the orthogonal families of equivalent nonzero projections $q \leq p$, and put $\mu(0) = 0$. Denote by L the positive reals, suitably extended by the adjunction of infinite cardinals, and topologized with the interval topology. A general dimension function for A is a mapping d which assigns to each $p \in P$ a continuous function on Γ to L, such that for all p, $q \in P$, (1) $pq = 0$ implies $d(p+q) = d(p) + d(q)$, (2) $p \in Z$ implies $d(pq) = \sigma(p)d(q)$, (3) $|d(p)| \leq \mu(p)$, (4) $d(p) = d(q)$ if and only if $p \sim q$. The existence and essential uniqueness of a general dimension function for any W^* algebra is established, and it is shown that much of the known structure of the algebra is carried by its dimension function. Similar results have recently been obtained by I. E. Segal. (Received March 9, 1955.)

588. R. T. Prosser: *On the structure of C^* algebras*.

Let A be a C^* algebra, A^* the dual space, and Ω the associated state space. By a left ideal of A is meant a norm closed left ideal; by a regular support of Ω is meant the intersection of Ω with a weak-*-closed supporting hyperplane in A^*. If J is a left ideal of A, then the set $\Xi = \{\omega : \omega \in \Omega$ and $\omega(a^*a) = 0$ for all $a \in J\}$ is a regular support of Ω, and conversely, if Ξ is a regular support of Ω, then the set $J = \{a : a \in A$ and

$\omega(a^*a)=0$ for all $\omega\in\Xi\}$ is a left ideal of A. It is shown that these relations generate a one-one inclusion-reversing correspondence between the left ideals of A and the regular supports of Ω. This correspondence relates the algebraic structure of A to the geometric structure of Ω, and has a number of interesting consequences. In particular, the maximal left ideals of A correspond to the extreme points of Ω. Since every support of Ω is generated by the extreme points it contains, it follows that every left ideal of A is the intersection of the maximal left ideals containing it. These results are easily verified when A is commutative. (Received March 9, 1955.)

APPLIED MATHEMATICS

589. W. L. Hart (p) and T. S. Motzkin: *A Newton-Raphson method for systems of equations.*

Consider a system of k equations $f(x)=0$, where $f=(f_1,\cdots,f_k)$ are real functions of the n real variables $x=(x_1,\cdots,x_n)$, with continuous derivatives $f_{ij}=\partial f_i/\partial x_j$. For an approximation $x^{(h)}$, $h=0,1,\cdots$, to a solution obtain the next approximation $x^{(h+1)}=x^{(h)}+\rho_h\Delta x^{(h)}$, $\rho_h>0$, by setting $\Delta x^{(h)}=\sum_{i=1}^{k}\Delta_i x^{(h)}$, $\Delta_i x_j^{(h)}=-f_i(x^{(h)})f_{ij}(x^{(h)})\cdot\left(\sum_{j=1}^{n}f_{ij}^2(x^{(h)})\right)^{-1}$. If $f(x)=0$ is a homogeneous linear system of rank $r>1$ and $x^{(0)}$ is arbitrary, then $x^{(h)}$ tends geometrically to the solution nearest to $x^{(0)}$ provided the $\rho_h<2r/k$ are properly chosen, e.g., $\rho_h=2/k$. A best constant $\rho_h=\rho'$, in a certain sense, exists and $2/k\leq\rho'\leq 2(r-1)/k$. The same results hold for a general system $f(x)=0$ if the rank of the matrix (f_{ij}) is n, and if none of its rows is zero, at a solution $x=\alpha$, provided $x^{(0)}$ is sufficiently near α. For a linear system $f(x)=0$, consistent or not, with (f_{ij}) of rank r, under similar conditions for ρ, $x^{(h)}$ tends to x^*, where x^* is the point nearest $x^{(0)}$ of those points which minimize $\sum_{i=1}^{k}f_i^2(x)$, under the assumption that $\sum_{j=1}^{n}f_{ij}^2=1$. (Received March 9, 1955.)

590*t*. J. R. Jackson: *Notes on some scheduling problems.*

A theorem is established whereby the columns of a 3-row real matrix can be permuted in such a way as to minimize the maximum of certain "walks" through the matrix, over all permutations of the columns. The theorem is a slight generalization of some of the work of Selmer Johnson (*Optimal two- and three-stage production schedules with setup times included*, Rand Corp., 1953), and implies his results. It also yields solutions to the following scheduling problems: (1) To choose a schedule to minimize the overall production time for N items, the nth of which must be processed first by Machine A, then by Machine B_n, and finally by Machine C, with the requirement that Machines A and C process the items in the same order (Machine B_n is used only for Item n). (2) The two-machine problem of Johnson, with the possibility added that some items may start their processing on the second machine before they are completed on the first (under certain minor restrictions); as when the setting up of the second machine can be begun before the item to be processed is actually available. (Research supported by the Office of Naval Research). (Received February 14, 1955.)

GEOMETRY

591. W. J. Firey: *On the densest packing of congruent, convex, material bodies.*

Material bodies are those which cannot have common interior points. Let $f(P)$ be the characteristic function of a distribution or packing of congruent convex material bodies. $K(r)$ is an admissible family of sets if: (a) $K(r)$ is convex for each r; (b) $K(r) \subset K(r')$ when $r<r'$; (c) given any point P, P is in $K(r)$ for r sufficiently large; (d) if $D(r)$ and $d(r)$ are the diameter and breadth of $K(r)$, there is a number M such that $D(r)/d(r) \leqq M$. Set $J(r) = \int_{K(r)} f(P)dP/V(r)$ ($V(r)$ is the volume of $K(r)$). The upper and lower limits of $J(r)$ are the upper and lower densities of the packing with respect to $K(r)$. A packing is absolutely densest with respect to $K(r)$ if no other packing of the same kind of bodies has a greater upper density with respect to $K(r)$, and if further the upper and lower densities of the packing with respect to $K(r)$ are equal. Theorem: There exists an absolutely densest packing of any particular kind of convex material body and its density is independent of the admissible family used to compute the density. (Received March 14, 1955.)

592*t*. I. M. Singer: *Transitive holonomy groups.*

Let $\mathcal{B} = [B, p, M, R_{2m}, R_{2m}]$ be a principal fiber bundle whose base space M is a manifold of dimension $2m$ and whose fiber R_{2m} is the rotation group acting on S^{2m-1}, the sphere of dimension $2m-1$. Let f be a map of M into the base space of the universal bundle relative to M which induces $\mathcal{B}$. Different conditions are given in terms of f insuring that the holonomy group of any connection on $\mathcal{B}$ acts transitively on S^{2m-1}. For example, if $2m=4k+2$ and if $f_*[\pi_{2m}(M)]$ is of infinite order, all holonomy groups will be transitive. The methods yield, among other things, a geometric proof that for $M=S^{2m}$, the holonomy group of any Riemannian connection is always R_{2m}. If $\mathcal{B}$ is the bundle of orthogonal frames of M relative to a Riemannian geometry on M, the theorems give a topological proof of only a part of M. Berger's extensive results [C. R. Acad. Sci. Paris vol. 237 (1953) pp. 472–473 and 1306–1308; vol. 238 (1954) pp. 985–986], namely when M is never reducible or symmetric. (Received March 9, 1955.)

STATISTICS AND PROBABILITY

593. Z. W. Birnbaum (p) and O. M. Klose: *Bounds for the variance of the U-statistic in terms of* $p = \text{Prob}\,\{Y<X\}$.

Let X_i, $i=1, 2, \cdots, m$, and Y_j, $j=1, 2, \cdots, n$, be samples of independent random variables X and Y, with distribution functions $F(x)$ and $G(y)$, and let $U =$ number of pairs (X_i, Y_j) such that $X_i > Y_j$. Then $mn\{p(1-p)+[2(2p)^{3/2}/3-2p^2] \cdot [\min(m, n)-1]\} \leqq \sigma^2(U) \leqq mnp(1-p) \operatorname{Max}(m, n)$ and both bounds are sharp for $m=n$ (upper bound obtained by D. Van Dantzig). If $F(s) \gtrless G(s)$, i.e. X is "stochastically smaller" than Y, then the upper bound can be replaced by $mn\{(m+n+1) +6(n-m)k^2+4(2m-n-1)k^3-3(m+n-1)k^4\}/12$ where $k=(1-2p)^{1/2}$. This bound is sharp for any $m \leqq n$. (Received March 10, 1955.)

594. Edwin Hewitt (p) and H. S. Zuckerman: *Arithmetic and limit theorems for a class of random variables.*

Let G be a finite commutative semigroup for which there exists an integer $m>1$ with the property that $x^{m+1}=x$ for all $x \in G$. Let $\mathfrak{P}(G)$ be the set of all non-negative functions f on G with $\sum_{x \in G} f(x) = 1$. $\mathfrak{P}(G)$ is closed under the convolution $f^*g(x) = \sum_{uv=x} f(u)g(v)$, and also under convex addition. $\mathfrak{P}(G)$ consists of all probability distributions of random variables with values in G. $\mathfrak{P}(G)$ has a unit iff G has a unit. $\mathfrak{P}(G)$ always has a zero. An element $f \in \mathfrak{P}(G)$ is idempotent iff f is uniformly distributed on a subgroup of G. An element $f \in \mathfrak{P}(G)$ is the limit of a product

$g_1 * g_2 * \cdots * g_n$ where the g_i's are nontrivial iff f is positive exactly on a union of cosets of a subsemigroup D of G, where $o(D) > 1$ and D has a unit. Several examples are worked out in detail. (Received February 28, 1955.)

TOPOLOGY

595. James Dugundji and E. A. Michael (p): *Local and uniformly local topological properties.*

It is shown that certain local properties of a metrizable space, such as local connectedness, local contractibility, or LC^n, hold uniformly with a properly chosen metric. An analogous result is proved for properties such as being an ANR. (Received March 11, 1955.)

J. W. GREEN,
Associate Secretary

LEONARD EUGENE DICKSON
1874–1954

Leonard Eugene Dickson was born in Independence, Iowa on January 22, 1874. He was a brilliant undergraduate at the University of Texas receiving his B.S. degree as valedictorian of his class in 1893. He was a chemist with the Texas Biological Survey from 1892–1893. He served as a teaching fellow at the University of Texas receiving the M.A. degree in 1894. He held a fellowship at the University of Chicago from 1894 to 1896 and was awarded its first Ph.D. in Mathematics in 1896. He spent the year 1896–1897 in Leipzig and Paris, was instructor in mathematics at the University of California 1897–1899, Associate Professor at Texas 1899–1900, Assistant Professor at Chicago 1900–1907, Associate Professor 1907–1910, and Professor in 1910. He was appointed to the Eliakim Hastings Moore Distinguished Professorship in 1928, and became Professor Emeritus in 1939. He served as Visiting Professor at the University of California in 1914, 1918, and 1922.

Professor Dickson was awarded the $1,000 A.A.A.S. Prize in 1924 for his work on the arithmetics of algebras. He was awarded the Cole Prize of the American Mathematical Society in 1928 for his book *ALGEBREN UND IHRE ZAHLENTHEORIE.* He served as Editor of the Monthly 1902–1908, and the Transactions from 1911 to 1916, and he was President of the American Mathematical Society from 1916–1918. He was elected to membership in the National Academy of Sciences in 1913 and was a member of the American Philosophical Society, the American Academy of Arts and Sciences, and the London Mathematical Society. He was also a foreign member of the Academy of the Institute of France, and an honorary member of the Czechoslovakian Union of Mathematics and Physics. He was awarded the honorary Sc.D. degree by Harvard in 1936 and Princeton in 1941.

Professor Dickson died in Texas on January 17, 1954.

Dickson was one of our most prolific mathematicians. His bibliography (prepared by Mr. Richard Block, a student at the University of Chicago) contains 285 titles. Of these 18 are books, one a joint book with Miller and Blichfeldt. One of the books is his major three-volume *History of the theory of numbers* which would be a life's work by itself for a more ordinary man.

Dickson was an inspiring teacher. He supervised the doctorate dissertation of at least 55 Chicago Ph.D's. He helped his students to get started in research after the Ph.D. and his books had a worldwide influence in stimulating research.

Attention should be called to the attached bibliography. It includes Dickson's books with titles listed in capitals. It does not include Dickson's portion of the report of the Committee on Algebraic Number Theory, nor does it include Dickson's monograph on ruler and compass constructions which appeared in Monographs on Modern Mathematics.

We now pass on to a brief discussion of Dickson's research.

1. *Linear groups.* Dickson's first major research effort was a study of finite linear groups. All but seven of his first forty-three papers were on that subject and this portion of his work led to his famous first book, [44]. The linear groups which had been investigated by Galois, Jordan, and Serret were all groups over the field of p elements. Dickson generalized their results to linear groups over an arbitrary finite field. He obtained many new systems of simple groups, and he closed his book with a still valuable summary of the known systems of simple groups.

Dickson's work on linear groups continued until 1908 and he wrote about 44 additional papers on the subject. In these later papers he studied the isomorphism of certain simple groups and questions about the existence of certain types of subgroups. He also derived a number of theorems on infinite linear groups.

2. *Finite fields and Chevalley's Theorem.* In [44] Dickson gave the first extensive exposition of the theory of finite fields. He applied his deep knowledge of that subject not only to linear groups but to other problems which we shall discuss later. He studied irreducibility questions over a finite field in [113], the Galois theory in [114], and forms whose values are squares in [139]. His knowledge of the role of the non-null form was shown in [155]. In [142] Dickson made the following statement: "For a finite field it seems to be true that every form of degree m in $m+1$ variables vanishes for values not all zero in the field." This result was first proved by C. Chevalley in his paper *Démonstration d'une hypothèse de M. Artin*, Hamb. Abh. vol. 11 (1935) pp. 73–75. At least the conjecture should have been attributed to Dickson who actually proved the theorem for $m=2$, 3.

3. *Invariants.* Several of Dickson's early papers were concerned with the problems of the algebraic geometry of his time. For example, see [4], [48], [54]. This work led naturally to his study of algebraic invariants and his interest in finite fields to modular invariants. He wrote a basic paper on the latter subject in [141], and many other papers on the subject. In these papers he devoted a great deal of

space to the details of a number of special cases. His book, [172], on the classical theory of algebraic invariants, was published in 1914, the year after the appearance of his colloquium lectures. His amazing productivity is attested to by the fact that he also published his book, [173], on linear algebras in 1914.

4. *Algebras*. Dickson played a major role in research on linear algebras. He began with a study of finite division algebras in [105], [115], [116], and [117]. In these papers he determined all three and four-dimensional finite (non-associative) division algebras over a field of characteristic not two, a set of algebras of dimension six, and a method for constructing algebras of dimension mk with a subfield of dimension m. In [126] he related the theory of ternary cubic forms to the theory of three-dimensional division algebras. His last paper on non-associative algebras, [268], appeared in 1937 and contained basic results on algebras of degree two.

Reference has already been made to Dickson's first book on linear algebras. In that text he gave a proof of his result that a real Cayley division algebra is actually a division algebra. He presented the Cartan theory of linear associative algebras rather than the Wedderburn theory but stated the results of the latter theory in his closing chapter without proofs. The present value of this book is enhanced by numerous bibliographical references.

Dickson defined cyclic algebras in a Bulletin abstract of vol. 12 (1905–1906). His paper, [160], on the subject did not appear until 1912 where he presented a study of algebras of dimension 16.

Dickson's work on the arithmetics of algebras first appeared in [204]. His major work on the subject of arithmetics was presented in [213] where he also gave an exposition of the Wedderburn theory. See also [237] and [238].

The text [231] is a German version of [213]. However, the new version also contains the results on crossed product algebras which had been published in [223], and contains many other items of importance.

5. *Theory of numbers*. Dickson always said that mathematics is the queen of the sciences, and that the theory of numbers is the queen of mathematics. He also stated that he had always wished to work in the theory of numbers and that he wrote his monumental three-volume *History of the theory of numbers* so that he could know all of the work which had been done in the subject. His first paper, [28], contained a generalization of the elementary Fermat theorem which arose in connection with finite field theory. He was interested in the existence of perfect numbers and wrote [166], and [167] on the re-

lated topic of abundant numbers. His interest in Fermat's last theorem appears in [190], [136], [137], [138], and [144]. During 1926–1930 he spent most of his energy on research in the arithmetic theory of quadratic forms, in particular on universal forms.

Dickson's interest in additive number theory began in 1927 with [229]. He wrote many papers on the subject during the remainder of his life. The analytic results of Vinogradov gave Dickson the hope of proving the so-called ideal Waring theorem. This he did in a long series of papers. His final result is an almost complete verification of the conjecture made by J. A. Euler in 1772. That conjecture stated that every positive integer is a sum of J nth powers where we write $3^n = 2^n q + r$, $2^n > r > 0$, and $J = 2^n + q - 2$. Dickson showed that if $n > 6$ this value is correct unless $q + r + 3 > 2^n$. It is still not known whether or not this last inequality is possible but if it does occur the number $g(n)$ of such nth powers required to represent all integers is $J + f$, or $J + f - 1$, according as $fq + f + q = 2^n$, or $fq + f + q > 2^n$, where f is the greatest integer in $(4/3)^n$.

6. *Miscellaneous.* We close by mentioning Dickson's interest in the theory of matrices which is best illustrated by his text, *Modern algebraic theories.* His geometric work in [179], [181], [182], [183], [184], [185], and [186] must also be mentioned, as well as his interesting monograph [219] on differential equations from the Lie group standpoint.

BIBLIOGRAPHY

1. *On the inscription of regular polygons*, Ann. of Math. vol. 9 (1895) pp. 73–84.

2. *Gergonne's pile problem*, Bull. Amer. Math. Soc. vol. 1 (1895) pp. 184–186.

3. *Cyclic numbers*, Quarterly Journal of Pure and Applied Mathematics vol. 27 (1895) pp. 366–377.

4. *A quadratic Cremona transformation defined by a conic*, Rend. Circ. Mat. Palermo vol. 9 (1895) pp. 256–257.

5. *Analytic functions suitable to represent substitutions*, Amer. J. Math. vol. 18 (1896) pp. 210–218.

6. *The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group*, Ann. of Math. vol. 11 (1897), Part I, pp. 65–102; Part II, pp. 161–183.

7. *A triply infinite system of simple groups*, Quarterly Journal of Pure and Applied Mathematics vol. 29 (1897) pp. 169–178.

8. *Systems of continuous and discontinuous simple groups*, Bull. Amer. Math. Soc. vol. 3 (1897) pp. 265–273.

9. *Higher irreducible congruences*, Bull. Amer. Math. Soc. vol. 3 (1897) pp. 381–389.

10. *Orthogonal group in a Galois field*, Bull. Amer. Math. Soc. vol. 4 (1898) pp. 196–200.

11. *Systems of simple groups derived from the orthogonal group*, Bull. Amer. Math. Soc. vol. 4 (1898) pp. 382–389.

12. *The structure of the hypoabelian groups*, Bull. Amer. Math. Soc. vol. 4 (1898) pp. 495–510.

13. *The first hypoabelian group generalized*, Quarterly Journal of Pure and Applied Mathematics vol. 30 (1898) pp. 1–16.

14. *A new triply-infinite system of simple groups obtained by a twofold generalization of Jordan's first hypoabelian group*, Bull. Amer. Math. Soc. vol. 5 (1898) pp. 10–12.

15. *Concerning a linear homogeneous group in $C_{m,q}$ variables isomorphic to the general linear homogeneous group in m variables*, Bull. Amer. Math. Soc. vol. 5 (1898) pp. 120–135.

16. *Simplicity of the Abelian group on two pairs of indices in the Galois field of order* 2^n, $n>1$, Quarterly Journal of Pure and Applied Mathematics vol. 30 (1898) pp. 383–384.

17. *The quadratic Cremona transformation*, Proceedings of the California Academy of Sciences vol. 1 (1898) pp. 13–23.

18. *Systems of simple groups derived from the orthogonal group*, Proceedings of the California Academy of Sciences vol. 1 (1898) pp. 29–40.

19. *Systems of simple groups derived from the orthogonal groups*, Proceedings of the California Academy of Sciences vol. 1 (1899) pp. 47–57.

20. *The structure of certain linear groups with quadratic invariants*, Proc. London Math. Soc. vol. 30 (1899) pp. 70–98.

21. *A class of linear groups including the Abelian one*, Quarterly Journal of Pure and Applied Mathematics vol. 31 (1899) pp. 60–66.

22. *The group of linear homogeneous substitutions on mq variables which is defined by the invariant* $\Phi=\sum_{i=1}^{i=m}\xi_{i1}\xi_{i2}\cdots\xi_{iq}$, Proc. London Math. Soc. vol. 30 (1899) pp. 200–208.

23. *Determination of the structure of all linear homogeneous groups in a Galois field which are defined by a quadratic invariant*, Amer. J. Math. vol. 21 (1899) pp. 193–256.

24. *Sur plusieurs groupes linéaires isomorphes au groups simple d'ordre* 25920, C. R. Acad. Sci. Paris vol. 128 (1899) pp. 873–875.

25. *The largest linear homogeneous group with an invariant Pfaffian*, Bull. Amer. Math. Soc. vol. 5 (1899) pp. 338–342.

26. *The known finite simple groups*, Bull. Amer. Math. Soc. vol. 5 (1899) pp. 470–475.

27. *Sur une généralisation du théorème de Fermat*, C. R. Acad. Sci. Paris vol. 128 (1899) pp. 1083–1085.

28. *A generalization of Fermat's theorem*, Ann. of Math. vol. 1 (1899) pp. 31–36.

29. *Report on the recent progress in the theory of linear groups*, Bull. Amer. Math. Soc. vol. 6 (1899) pp. 13–27.

30. *Concerning the four known simple linear groups of order* 25920, *with an introduction to the hyper-abelian linear groups*, Proc. London Math. Soc. vol. 31 (1899) pp. 30–68.

31. *The abstract group isomorphic with the symmetric group of k letters*, Proc. London Math. Soc. vol. 31 (1899) pp. 351–353.

32. *The structure of the linear homogeneous groups defined by the invariant* $\lambda_1\xi_1^r+\lambda_2\xi_2^r+\cdots+\lambda_m\xi_m^r$, Math. Ann. vol. 52 (1899) pp. 561–581.

33. *Definition of the Abelian, the two hypoabelian, and related linear groups as quotient groups of the groups of isomorphisms of certain elementary groups*, Trans. Amer. Math. Soc. vol. 1 (1900) pp. 30–38.

34. *Certain subgroups of the Betti-Mathieu group*, Amer. J. Math. vol. 22 (1900) pp. 49–54.

35. *A new definition of the general Abelian linear group*, Trans. Amer. Math. Soc vol. 1 (1900) pp. 91–96.

36. *Canonical form of a linear homogeneous substitution in a Galois field*, Amer. J. Math. vol. 22 (1900) pp. 121–137.

37. *Proof of the existence of the Galois field of order p^r for every integer r and prime number p*, Bull. Amer. Math. Soc. vol. 6 (1900) pp. 203–204.

38. *Concerning the cyclic subgroups of the simple group G of all linear fractional substitutions of determinant unity in two non-homogeneous variables with coefficients in an arbitrary Galois field*, Amer. J. Math. vol. 22 (1900) pp. 231–252.

39. *Isomorphism between certain systems of simple linear groups*, Bull. Amer. Math Soc. vol. 6 (1900) pp. 323–328.

40. *An abstract simple group of order* 25920, Proc. London Math. Soc. vol. 32 (1900) pp. 3–10

41. *Determination of an abstract simple group of order $2^7\cdot 3^4\cdot 5\cdot 7$ holoedrically isomorphic with a certain orthogonal group and with a certain hypoabelian group*, Trans. Amer. Math. Soc. vol. 1 (1900) pp. 353–370.

42. *Proof of the non-isomorphism of the simple Abelian group on $2m$ indices and the orthogonal group on $2m+1$ indices for $m>2$*, Quarterly Journal of Pure and Applied Mathematics vol. 32 (1900) pp. 42–63.

43. *Linear substitutions commutative with a given substitution*, Proc. London Math. Soc. vol. 32 (1900) pp. 165–170.

44. *LINEAR GROUPS WITH AN EXPOSITION OF THE GALOIS FIELD THEORY*, Leipzig, Teubner, 1901, 10+312 pp.

45. *Distribution of the ternary linear homogeneous substitutions in a Galois field into complete sets of conjugate substitutions*, Amer. J. Math. vol. 23 (1901) pp. 37–40.

46. *Canonical forms of quaternary Abelian substitutions in an arbitrary Galois field*, Trans. Amer. Math. Soc. vol. 2 (1901) pp. 103–138.

47. *Concerning real and complex continuous groups*, Bull. Amer. Math. Soc. vol. 7 (1901) pp. 340–350.

48. *The configurations of the 27 lines on a cubic surface and the 28 bitangents to a quartic curve*, Bull. Amer. Math. Soc. vol. 8 (1901) pp. 63–70.

49. *Concerning the Abelian and related linear groups*, Proc. London Math. Soc. vol. 33 (1901) pp. 313–325.

50. *The alternating group on eight letters and the quaternary linear congruence group modulo two*, Math. Ann. vol. 54 (1901) pp. 564–569.

51. *Théorie des groupes linéaires dans un domaine arbitraire de rationalité*, C. R. Acad. Sci. Paris, vol. 132 (1901) pp. 1547–1548.

52. *Theory of linear groups in an arbitrary field*, Trans. Amer. Math. Soc. vol. 2 (1901) pp. 363–394.

53. *Representation of linear groups as transitive substitution groups*, Amer. J. Math. vol. 23 (1901) pp. 337–377.

54. *A class of groups in an arbitrary realm connected with the configuration of the 27 lines on a cubic surface*, Quarterly Journal of Pure and Applied Mathematics vol. 33 (1901) pp. 145–173.

55. *On systems of isothermal curves*, Amer. Math. Monthly vol. 8 (1901) pp. 187–192.

56. *The known systems of simple groups and their inter-isomorphisms*, Compte Rendu du Deuxième Congrès International des Mathématiciens, Paris, 1900, Paris, Gauthier-Villars, 1902, pp. 225–229.

57. *Cyclic subgroups of the simple ternary linear fractional group in a Galois field*, Amer. J. Math. vol. 24 (1902) pp. 1–12.

58. *The groups of Steiner in problems of contact*, Trans. Amer. Math. Soc. vol. 3 (1902) pp. 38–45.

59. *The groups of Steiner in problems of contact (Second Paper)*, Trans. Amer. Math. Soc. vol. 3 (1902) pp. 377–382.

60. *Canonical form of a linear homogeneous transformation in an arbitrary realm of rationality*, Amer. J. Math. vol. 24 (1902) pp. 101–108.

61. *Linear groups in an infinite field*, Proc. London Math. Soc. vol. 34 (1902) pp. 185–205.

62. *Theorems on the residues of multinomial coefficients with respect to a prime modulus*, Quarterly Journal of Pure and Applied Mathematics vol. 33 (1902) pp. 378–384.

63. *A class of simply transitive linear groups*, Bull. Amer. Math. Soc. vol. 8 (1902) pp. 394–401.

64. *The hyperorthogonal groups*, Math. Ann. vol. 55 (1902) pp. 521–572.

65. *An elementary exposition of Frobenius's theory of group-characters and group-determinants*, Ann. of Math. vol. 4 (1902) pp. 25–49.

66. *On the group defined for any given field by the multiplication table of any given finite group*, Trans. Amer. Math. Soc. vol. 3 (1902) pp. 285–301.

67. *A matrix defined by the quaternion group*, Amer. Math. Monthly vol. 9 (1902) pp. 243–248.

68. *COLLEGE ALGEBRA*, New York, Wiley, 1902, 7+214 pp.

69. *Ternary orthogonal group in a general field*, University of Chicago Press, 1903, 8 pp.

70. *Groups defined for a general field by the rotation groups*, University of Chicago Press, 1903, 17 pp.

71. *INTRODUCTION TO THE THEORY OF ALGEBRAIC EQUATIONS*, New York, Wiley, 1903, 5+104 pp.

72. *Definitions of a field by independent postulates*, Trans. Amer. Math. Soc. vol. 4 (1903) pp. 13–20.

73. *Definitions of a linear associative algebra by independent postulates*, Trans. Amer. Math. Soc. vol. 4 (1903) pp. 21–26.

74. *On the groups defined for an arbitrary field by the multiplication tables of certain finite groups*, Proc. London Math. Soc. vol. 35 (1903) pp. 68–80.

75. *Three sets of generational relations defining the abstract simple group of order* 504, Bull. Amer. Math. Soc. vol. 9 (1903) pp. 194–204.

76. *Generational relations defining the abstract simple group of order* 660, Bull. Amer. Math. Soc. vol. 9 (1903) pp. 204–206.

77. *The abstract group G simply isomorphic with the alternating group on six letters*, Bull. Amer. Math. Soc. vol. 9 (1903) pp. 303–306.

78. *Fields whose elements are linear differential expressions*, Bull. Amer. Math. Soc. vol. 10 (1903) pp. 30–31.

79. *On the subgroups of order a power of p in the quaternary Abelian group in the Galois field of order* p^n, Trans. Amer. Math. Soc. vol. 4 (1903) pp. 371–386.

80. *On the reducibility of linear groups*, Trans. Amer. Math. Soc. vol. 4 (1903) pp. 434–436.

81. *The abstract group simply isomorphic with the group of linear fractional transformations in a Galois field*, Proc. London Math. Soc. vol. 35 (1903) pp. 292–305.

82. *Generational relations of an abstract simple group of order* 4080, Proc. London Math. Soc. vol. 35 (1903) pp. 306–319.

83. *Generational relations for the abstract group simply isomorphic with the linear fractional group in the* $GF[2^n]$, Proc. London Math. Soc. vol. 35 (1903) pp. 443–454.

84. 1. *Ternary orthogonal groups in a general field.* 2. *The groups defined for a general field by the rotation group*, University of Chicago Press, 1904, 26 pp.

85. *The subgroups of order a power of 2 of the simple quinary orthogonal group in the Galois field of order $p^n = 8l \pm 3$*, Trans. Amer. Math. Soc. vol. 5 (1904) pp. 1–38.

86. *Determination of all groups of binary linear substitutions with integral coefficients taken modulo 3 and of determinant unity*, Ann. of Math. vol. 5 (1904) pp. 140–144.

87. *Determination of all subgroups of the known simple group of order* 25920, Trans. Amer. Math. Soc. vol. 5 (1904) pp. 126–166.

88. *Two systems of subgroups of the quaternary Abelian group in a general Galois field*, Bull. Amer. Math. Soc. vol. 10 (1904) pp. 178–184.

89. *A new extension of Dirichlet's theorem on prime numbers*, Messenger of Mathematics, vol. 33 (1904) pp. 155–161.

90. *Addition to the paper on the four known simple linear groups of order* 25920, Proc. London Math. Soc. vol. 1 (1904) pp. 283–284.

91. *On the subgroups of order a power of p in the linear homogeneous and fractional groups in the $GF[p^n]$*, Bull. Amer. Math. Soc. vol. 10 (1904) pp. 385–397.

92. *Memoir on Abelian transformations*, Amer. J. Math. vol. 26 (1904) pp. 243–318.

93. *Application of groups to a complex problem in arrangements*, Ann. of Math. vol. 6 (1904) pp. 31–34.

94. *On the minimum degree of resolvents for the p-section of the periods of hyperelliptic functions of four periods*, Jber. Deutschen Math. Verein. vol. 13 (1904) pp. 559–560.

95. *A property of the group G_2^{m} all of whose operators except identity are of period* 2, Amer. Math. Monthly vol. 11 (1904) pp. 203–206.

96. *The groups of a tactical configuration*, Bull. Amer. Math. Soc. vol. 11 (1905) pp. 177–179.

97. *The minimum degree τ of resolvents for the p-section of the periods of hyperelliptic functions of four periods*, Trans. Amer. Math. Soc. vol. 6 (1905) pp. 48–57.

98. *On the cyclotomic functions*, Amer. Math. Monthly vol. 12 (1905) pp. 86–89.

99. *On the real elements of certain classes of geometrical configurations*, Ann. of Math. vol. 6 (1905) pp. 141–150.

100. *A new system of simple groups*, Math. Ann. vol. 60 (1905) pp. 137–150.

101. *Determination of the ternary modular groups*, Amer. J. Math. vol. 27 (1905) pp. 189–202.

102. *Definitions of a group and a field by independent postulates*, Trans. Amer. Math. Soc. vol. 6 (1905) pp. 198–204.

103. *On semi-groups and the general isomorphism between finite groups*, Trans. Amer. Math. Soc. vol. 6 (1905) pp. 205–208.

104. *Determination of all the subgroups of the three highest powers of p in the group G of all m-ary linear homogeneous transformations modulo p*, Quarterly Journal of Pure and Applied Mathematics vol. 36 (1905) pp. 373–384.

105. *On finite algebras*, Nachr. Ges. Wiss. Göttingen (1905) pp. 358–393.

106. *On the class of the substitutions of various linear groups*, Bull. Amer. Math. Soc. vol. 11 (1905) pp. 426–432.

107. *Subgroups of order a power of p, in the general and special m-ary linear homogeneous groups in the $GF[p^n]$*, Amer. J. Math. vol. 27 (1905) pp. 280–302.

108. *A general theorem on algebraic numbers*, Bull. Amer. Math. Soc. vol. 11 (1905) pp. 482–486.

109. *On hypercomplex number systems*, Trans. Amer. Math. Soc. vol. 6 (1905) pp. 344–348.

110. *Expressions for the elements of a determinant of an array*, Amer. Math. Monthly vol. 12 (1905) pp. 217–221.

111. *On the quaternary linear homogeneous groups modulo p of order a multiple of p*, Amer. J. Math. vol. 28 (1906) pp. 1–16.

112. *On quadratic, hermitian and bilinear forms*, Trans. Amer. Math. Soc. vol. 7 (1906) pp. 275–292.

113. *Criteria for the irreducibility of functions in a finite field*, Bull. Amer. Math. Soc. vol. 13 (1906) pp. 1–8.

114. *On the theory of equations in a Galois field*, Bull. Amer. Math. Soc. vol. 13 (1906) pp. 8–10.

115. *Linear algebras in which division is always uniquely possible*, Trans. Amer. Math. Soc. vol. 7 (1906) pp. 370–390.

116. *On linear algebras*, Amer. Math. Monthly vol. 13 (1906) pp. 201–205.

117. *On commutative linear algebras in which division is always uniquely possible*, Trans. Amer. Math. Soc. vol. 7 (1906) pp. 514–522.

118. *The abstract form of the special linear homogeneous group in an arbitrary field*, Quarterly Journal of Pure and Applied Mathematics vol. 38 (1907) pp. 141–145.

119. *The abstract form of the Abelian linear groups*, Quarterly Journal of Pure and Applied Mathematics vol. 38 (1907) pp. 145–158.

120. *The symmetric group on eight letters and the senary first hypoabelian group*, Bull. Amer. Math. Soc. vol. 13 (1907) pp. 386–389.

121. *Modular theory of group characters*, Bull. Amer. Math. Soc. vol. 13 (1907) pp. 477–488.

122. *Invariants of binary forms under modular transformations*, Trans. Amer. Math. Soc. vol. 8 (1907) pp. 205–232.

123. *Invariants of the general quadratic form modulo* 2, Proc. London Math. Soc. vol. 5 (1907) pp. 301–324.

124. *Modular theory of group-matrices*, Trans. Amer. Math. Soc. vol. 8 (1907) pp. 389–398.

125. *On quadratic forms in a general field*, Bull. Amer. Math. Soc. vol. 14 (1907) pp. 108–115.

126. *On triple algebras and ternary cubic forms*, Bull. Amer. Math. Soc. vol. 14 (1908) pp. 160–169.

127. *The Galois group of a reciprocal quartic equation*, Amer. Math. Monthly vol. 15 (1908) pp. 71–78.

128. *A class of groups in an arbitrary realm connected with the configuration of the 27 lines on a cubic surface. Second paper*, Quarterly Journal of Pure and Applied Mathematics vol. 39 (1908) pp. 205–209.

129. *On the canonical forms and automorphs of ternary cubic forms*, Amer. J. Math. vol. 30 (1908) pp. 117–128.

130. *Representations of the general symmetric group as linear groups in finite and infinite fields*, Trans. Amer. Math. Soc. vol. 9 (1908) pp. 121–148.

131. *On families of quadratic forms in a general field*, Quarterly Journal of Pure and Applied Mathematics vol. 39 (1908) pp. 316–333.

132. *On higher congruences and modular invariants*, Bull. Amer. Math. Soc. vol. 14 (1908) pp. 313–318.

133. *On the factorization of large numbers*, Amer. Math. Monthly vol. 15 (1908) pp. 217–222.

134. *Criteria for the irreducibility of a reciprocal equation*, Bull. Amer. Math. Soc. vol. 14 (1908) pp. 426–430.

135. *Invariantive reduction of quadratic forms in the* $GF[2^n]$, Amer. J. Math. vol. 30 (1908) pp. 263–281.

136. *On the last theorem of Fermat*, Messenger of Mathematics vol. 38 (1908) pp. 14–32.

137. *On the last theorem of Fermat*, II, Quarterly Journal of Pure and Applied Mathematics vol. 40 (1908) pp. 27–45.

138. *On the congruence* $x^n+y^n+z^n\equiv 0 \pmod{p}$, J. Reine Angew. Math. vol. 135 (1908) pp. 134–141.

139. *Definite forms in a finite field*, Trans. Amer. Math. Soc. vol. 10 (1909) pp. 109–122.

140. *Rational reduction of a pair of binary quadratic forms; their modular invariants*, Amer. J. Math. vol. 31 (1909) pp. 103–146.

141. *General theory of modular invariants*, Trans. Amer. Math. Soc. vol. 10 (1909) pp. 123–158.

142. *On the representation of numbers by modular forms*, Bull. Amer. Math. Soc. vol. 15 (1909) pp. 338–347.

143. (with M. Kaba), *On the representation of numbers as the sum of two squares*, Amer. Math. Monthly vol. 16 (1909) pp. 85–87.

144. *Lower limit for the number of sets of solutions of* $x^e+y^e+z^e\equiv 0$ (mod. p), J. Reine Angew. Math. vol. 135 (1909) pp. 181–188.

145. *On commutative linear groups*, Quarterly Journal of Pure and Applied Mathematics vol. 40 (1909) pp. 167–196.

146. *A theory of invariants*, Amer. J. Math. vol. 31 (1909) pp. 337–354.

147. *Equivalence of pairs of bilinear or quadratic forms under rational transformation*, Trans. Amer. Math. Soc. vol. 10 (1909) pp. 347–360.

148. *Combinants*, Quarterly Journal of Pure and Applied Mathematics vol. 40 (1909) pp. 349–365.

149. *On certain diophantine equations*, Messenger of Mathematics vol. 39 (1909) pp. 86–87.

150. *Modular invariants of a general system of linear forms*, Proc. London Math. Soc. vol. 7 (1909) pp. 430–444.

151. *On the factorization of integral functions with p-adic coefficients*, Bull. Amer. Math. Soc. vol. 17 (1910) pp. 19–23.

152. *An invariantive investigation of irreducible binary modular forms*, Trans. Amer. Math. Soc. vol. 12 (1911) pp. 1–18.

153. *A fundamental system of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. vol. 12 (1911) pp. 75–98.

154. *Note on modular invariants*, Quarterly Journal of Pure and Applied Mathematics vol. 42 (1911) pp. 158–161.

155. *On non-vanishing forms*, Quarterly Journal of Pure and Applied Mathematics vol. 42 (1911) pp. 162–171.

156. *Binary modular groups and their invariants*, Amer. J. Math. vol. 33 (1911) pp. 175–192.

157. *Notes on the theory of numbers*, Amer. Math. Monthly vol. 18 (1911) pp. 109–110.

158. *On the negative discriminants for which there is a single class of positive binary quadratic forms*, Bull. Amer. Math. Soc. vol. 17 (1911) pp. 534–547.

159. *Note on cubic equations and congruences*, Ann. of Math. vol. 12 (1911) pp. 149–152.

160. *Linear algebras*, Trans. Amer. Math. Soc. vol. 13 (1912) pp. 59–73.

161. *Amicable number triples*, Amer. Math. Monthly vol. 20 (1913) pp. 84–92.

162. *On the rank of a symmetrical matrix*, Ann. of Math. vol. 15 (1913) pp. 27–28.

163. *Theorems and tables on the sum of the divisors of a number*, Quarterly Journal of Pure and Applied Mathematics vol. 44 (1913) pp. 264–296.

164. *Proof of the finiteness of the modular covariants*, Trans. Amer. Math. Soc. vol. 14 (1913) pp. 299–310.

165. *On binary modular groups and their invariants*, Bull. Amer. Math. Soc. vol. 20 (1913) pp. 132–134.

166. *Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors*, Amer. J. Math. vol. 35 (1913) pp. 413–422.

167. *Even abundant numbers*, Amer. J. Math. vol. 35 (1913) pp. 423–426.

168. *The invariants, seminvariants and linear covariants of the binary quartic form modulo 2*, Ann. of Math. vol. 15 (1914) pp. 114–117.

169. *Linear associative algebras and Abelian equations*, Trans. Amer. Math. Soc. vol. 15 (1914) pp. 31–46.

170. *ON INVARIANTS AND THE THEORY OF NUMBERS*, Amer. Math. Soc. Colloquium Publications, vol. 4, 1914, pp. 1–110.

171. *ELEMENTARY THEORY OF EQUATIONS*, New York, Wiley, 1914, 5+184 pp.

172. *ALGEBRAIC INVARIANTS*, New York, Wiley, 1914, 10+100 pp. [Mathematical monographs, edited by M. Merriman and R. S. Woodward, No. 14.]

173. *LINEAR ALGEBRAS*, Cambridge University Press, 1914, 8+73 pp. [Cambridge Tracts in Math. and Math. Physics, No. 16.]

174. *The points of inflection of a plane cubic curve*, Ann. of Math. (2) vol. 16 (1914) pp. 50–66.

175. *Invariants in the theory of numbers*, Trans. Amer. Math. Soc. vol. 15 (1914) pp. 497–503.

176. *On the trisection of an angle and the construction of regular polygons of 7 and 9 sides*, Amer. Math. Monthly vol. 21 (1914) pp. 259–262.

177. *Modular invariants of the system of a binary cubic, quadratic and linear form*, Quarterly Journal of Pure and Applied Mathematics vol. 45 (1914) pp. 373–384.

178. *Recent progress in the theories of modular and formal invariants and in modular geometry*, Proc. Nat. Acad. Sci. U.S.A. vol. 1 (1915) pp. 1–4.

179. *Projective classification of cubic surfaces modulo 2*, Ann. of Math. (2) vol. 16 (1915) pp. 139–157.

180. *Invariants, seminvariants, and covariants of the ternary and quaternary quadratic form modulo 2*, Bull. Amer. Math. Soc. vol. 21 (1915) pp. 174–179.

181. *Invariantive theory of plane cubic curves modulo 2*, Amer. J. Math. vol. 37 (1915) pp. 107–116.

182. *Quartic curves modulo 2*, Trans. Amer. Math. Soc. vol. 16 (1915) pp. 111–120.

183. *Classification of quartic curves, modulo 2*, Messenger of Mathematics vol. 44 (1915) pp. 189–192.

184. *The straight lines on modular cubic surfaces*, Proc. Nat. Acad. Sci. U.S.A. vol. 1 (1915) pp. 248–253.

185. *Geometrical and invariantive theory of quartic curves modulo 2*, Amer. J. Math. vol. 37 (1915) pp. 337–354.

186. *Invariantive classification of pairs of conics modulo 2*, Amer. J. Math. vol. 37 (1915) pp. 355–358.

187. *On the relation between linear algebras and continuous groups*, Bull. Amer. Math. Soc. vol. 22 (1915) pp. 53–61.

188. (with G. A. Miller and H. F. Blichfeldt), *THEORY AND APPLICATIONS OF FINITE GROUPS*, New York, Wiley, 1916, 17+390 pp.

189. *An extension of the theory of numbers by means of correspondences between fields*, Bull. Amer. Math. Soc. vol. 23 (1916) pp. 109–111.

190. *Fermat's last theorem and the origin and nature of the theory of algebraic numbers*, Ann. of Math. (2) vol. 18 (1917) pp. 161–187.

191. *On quaternions and their generalization and the history of the eight square theorem*, Ann. of Math. (2) vol. 20 (1919) pp. 155–171, 297.

192. *Mathematics in war perspective*, Bull. Amer. Math. Soc. vol. 25 (1919) pp. 289–311.

193. *Applications of the geometry of numbers to algebraic numbers*, Bull. Amer. Math. Soc. vol. 25 (1919) pp. 453–455.

194. *HISTORY OF THE THEORY OF NUMBERS*, vol. I, *DIVISIBILITY AND PRIMALITY*, Carnegie Institution of Washington, Publication No. 256, 1919, 12+486 pp.

195. *HISTORY OF THE THEORY OF NUMBERS*, Vol. II. *DIOPHANTINE ANALYSIS*, Carnegie Institution of Washington, Publication No. 256, 1920, 12+803 pp.

196. *Les polynomes égaux à des determinants*, C. R. Acad. Sci. Paris vol. 171 (1920) pp. 1360–1362.

197. *Some relations between the theory of numbers and other branches of mathematics*, Conférence générale, Comptes Rendus du Congrés Internation[illegible] Mathématiciens, Strasbourg, 1920, Toulouse, H. Villat, 1921, pp. 41–56.

198. *Homogeneous polynomials with a multiplication theorem*, Comptes Rendus du Congrés International des Mathématiciens, Strasbourg, 1920, Toulouse, H. Villat, 1921, pp. 215–230.

199. *Algebraic theory of the expressibility of cubic forms as determinants, with application to diophantine analysis*, Amer. J. Math. vol. 43 (1921) pp. 102–125.

200. *Quaternions and their generalizations*, Proc. Nat. Acad. Sci. U.S.A. vol. 7 (1921) pp. 109–114.

201. *Fallacies and misconceptions in diophantine analysis*, Bull. Amer. Math. Soc. vol. 27 (1921) pp. 312–319.

202. *Determination of all general homogeneous polynomials expressible as determinants with linear elements*, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 167–179.

203. *A new method in diophantine analysis*, Bull. Amer. Math. Soc. vol. 27 (1921) pp. 353–365.

204. *Arithmetic of quaternions*, Proc. London Math. Soc. vol. 20 (1921) pp. 225–232.

205. *Rational triangles and quadrilaterals*, Amer. Math. Monthly vol. 28 (1921) pp. 244–250.

206. *Reducible cubic forms expressible rationally as determinants*, Ann. of Math. vol. 23 (1921) pp. 70–74.

207. *A fundamental system of covariants of the ternary cubic form*, Ann. of Math. vol. 23 (1921) pp. 78–82.

208. *La composition des polynomes*, C. R. Acad. Sci. Paris vol. 172 (1921) pp. 636–640.

209. *PLANE TRIGONOMETRY WITH PRACTICAL APPLICATIONS*, Chicago, Sanborn, 1922, 12+176+35 pp.

210. *A FIRST COURSE IN THE THEORY OF EQUATIONS*, New York, Wiley, 1922, 6+168 pp.

211. *Impossibility of restoring unique factorization in a hypercomplex arithmetic*, Bull. Amer. Math. Soc. vol. 28 (1922) pp. 438–442.

212. *The rational linear algebras of maximum and minimum ranks*, Proc. London Math. Soc. (2) vol. 22 (1923) pp. 143–162.

213. *ALGEBRAS AND THEIR ARITHMETICS*, Chicago, University of Chicago Press, 1923, 12+241 pp.

214. *HISTORY OF THE THEORY OF NUMBERS*, Vol. III. *QUADRATIC AND HIGHER FORMS* (With a chapter on the class number by G. H. Cresse), Carnegie Institution of Washington, Publication No. 256, 1923, 4+313 pp.

215. *A new simple theory of hypercomplex integers*, J. Math. Pures Appl. (9) vol. 2 (1923) pp. 281–326.

216. *Integral solutions of* $x^2-my^2=zw$, Bull. Amer. Math. Soc. vol. 29 (1923) pp. 464–467.

217. *On the theory of numbers and generalized quaternions*, Amer. J. Math. vol. 46 (1924) pp. 1–16.

218. *Algebras and their arithmetics*, Bull. Amer. Math. Soc. vol. 30 (1924) pp. 247–257.

219. *Differential equations from the group standpoint*, Ann. of Math. (2) vol. 25 (1924) pp. 287–378.

220. *Quadratic fields in which factorization is always unique*, Bull. Amer. Math. Soc. vol. 30 (1924) pp. 328–334.

221. *Algèbres nouvelles de division*, C. R. Acad. Sci. Paris vol. 181 (1925) pp. 836–838.

222. *Resolvent sextics of quintic equations*, Bull. Amer. Math. Soc. vol. 31 (1925) pp. 515–523.

223. *New division algebras*, Trans. Amer. Math. Soc. vol. 28 (1926) pp. 207–234.

224. *MODERN ALGEBRAIC THEORIES*, Chicago, New York, Boston; Sanborn, 1926, 9+276 pp.

225. *All integral solutions of* $ax^2+bxy+cy^2=w_1w_2\cdots w_n$, Bull. Amer. Math. Soc. vol. 32 (1926) pp. 644–648.

226. *Quadratic forms which represent all integers*, Proc. Nat. Acad. Sci. U.S.A. vol. 12 (1926) pp. 756–757.

227. *Quaternary quadratic forms representing all integers*, Amer. J. Math. vol. 49 (1927) pp. 39–56.

228. *Integers represented by positive ternary quadratic forms*, Bull. Amer. Math. Soc. vol. 33 (1927) pp. 63–70.

229. *Extensions of Waring's theorem on nine cubes*, Amer. Math. Monthly vol. 34 (1927) pp. 177–183.

230. *A generalization of Waring's theorem on nine cubes*, Bull. Amer. Math. Soc. vol. 33 (1927) pp. 299–300.

231. *ALGEBREN UND IHRE ZAHLENTHEORIE*, Zurich, Orell Füssli, 1927, 8+308 pp. (translation of completely revised and extended manuscript).

232. *Extensions of Waring's theorem on fourth powers*, Bull. Amer. Math. Soc. vol. 33 (1927) pp. 319–327.

233. *Singular case of pairs of bilinear, quadratic, or Hermitian forms*, Trans. Amer. Math. Soc. vol. 29 (1927) pp. 239–253.

234. *Generalization of Waring's theorem on fourth, sixth, and eighth powers*, Amer. J. Math. vol. 49 (1927) pp. 241–250.

235. *Ternary quadratic forms and congruences*, Ann. of Math. (2) vol. 28 (1927) pp. 333–341.

236. *All positive integers are sums of values of a quadratic function of* x, Bull. Amer. Math. Soc. vol. 33 (1927) pp. 713–720.

237. *Outline of the theory to date of the arithmetics of algebras*, Proceedings of the International Mathematical Congress held in Toronto, 1924, Toronto, The University of Toronto Press, 1928, vol. 1, pp. 95–102.

238. *Further development of the theory of arithmetics of algebras*, Proceedings of the International Mathematical Congress held in Toronto, 1924, Toronto, The University of Toronto Press, 1928, vol. 1, pp. 173–184.

239. *A new theory of linear transformations and pairs of bilinear forms*, Proceedings of the International Mathematical Congress, held in Toronto, 1924, Toronto, The University of Toronto Press, 1928, vol. 1, pp. 361–363.

240. *Simpler proofs of Waring's theorem on cubes with various generalizations*, Trans. Amer. Math. Soc. vol. 30 (1928) pp. 1–18.

241. *Additive number theory for all quadratic functions*, Amer. J. Math. vol. 50 (1928) pp. 1–48.

242. *Generalizations of the theorem of Fermat and Cauchy on polygonal numbers*, Bull. Amer. Math. Soc. vol. 34 (1928) pp. 63–72.

243. *Extended polygonal numbers*, Bull. Amer. Math. Soc. vol. 34 (1928) pp. 205–217.

244. *Quadratic functions or forms, sums of whose values give all positive integers*, J. Math. Pures Appl. vol. 7 (1928) pp. 319–336.

245. *New division algebras*, Bull. Amer. Math. Soc. vol. 34 (1928) p[illegible]–560.

246. *Universal quadratic forms*, Trans. Amer. Math. Soc. vol. 31 (1929) pp. 164–189.

247. *The forms* $ax^2+by^2+cz^2$ *which represent all integers*, Bull. Amer. Math. Soc. vol. 35 (1929) pp. 55–59.

248. *INTRODUCTION TO THE THEORY OF NUMBERS*, Chicago, The University of Chicago Press, 1929, 8+183 pp.

249. *Construction of division algebras*, Trans. Amer. Math. Soc. vol. 32 (1930) pp. 319–334.

250. *STUDIES IN THE THEORY OF NUMBERS*, Chicago, The University of Chicago Press (The University of Chicago Science Series), 1930, 10+230 pp.

251. *Proof of a Waring theorem on fifth powers*, Bull. Amer. Math. Soc. vol. 37 (1931) pp. 549–553.

252. *MINIMUM DECOMPOSITIONS INTO FIFTH POWERS*, Vol. III, Mathematical Tables, British Assoc. for the Advancement of Science (Committee for the Calculation of Mathematical Tables), London, Office of the British Assoc., 1933, 368 pp.

253. *Eliakim Hastings Moore*, Science (2) vol. 77 (1933) pp. 79–80.

254. *Minimum decompositions into n-th powers*, Amer. J. Math. vol. 55 (1933) pp. 593–602.

255. *Recent progress on Waring's theorem and its generalizations*, Bull. Amer. Math. Soc. vol. 39 (1933) pp. 701–727.

256. *Waring's problem for cubic functions*, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 1–12.

257. *Waring's problem for ninth powers*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 487–493.

258. *Universal Waring theorem for eleventh powers*, J. London Math. Soc. vol. 9 (1934) pp. 201–206.

259. *Two-fold generalizations of Cauchy's lemma*, Amer. J. Math. vol. 56 (1934) pp. 513–528.

260. *The converse of Waring's problem*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 711–714.

261. *A new method for universal Waring theorems with details for seventh powers*, Amer. Math. Monthly vol. 41 (1934) pp. 547–555.

262. *A new method for Waring theorems with polynomial summands*, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 731–748.

263. *Polygonal numbers and related Waring problems*, Quart. J. Math. Oxford Ser. vol. 5 (1934) pp. 283–290.

264. *Congruences involving only e-th powers*, Acta Arithmetica vol. 1 (1935) pp. 161–167.

265. *Cyclotomy and trinomial congruences*, Trans. Amer. Math. Soc. vol. 37 (1935) pp. 363–380.

266. *Cyclotomy, higher congruences, and Waring's problem*, Amer. J. Math. vol. 57 (1935) pp. 391–424.

267. *Cyclotomy, higher congruences, and Waring's problem*, II, Amer. J. Math. vol. 57 (1935) pp. 463–474.

268. *Linear algebras with associativity not assumed*, Duke Math. J. vol. 1 (1935) pp. 113–125.

269. *Cyclotomy when e is composite*, Trans. Amer. Math. Soc. vol. 38 (1935) pp. 187–200.

270. *Universal Waring theorems with cubic summands*, Acta Arithmetica vol. 1 (1935) pp. 184–196 and Prace Matematyczno-Fizyczne vol. 43 (1936) pp. 223–235.

271. *Waring theorems of new type*, Amer. J. Math. vol. 58 (1936) pp. 241–248.

272. *On Waring's problem and its generalization*, Ann. of Math. (2) vol. 37 (1936) pp. 293–316.

273. *A new method for Waring theorems with polynomial summands*. II, Trans. Amer. Math. Soc. vol. 39 (1936) pp. 205–208.

274. *Universal Waring theorems*, Mh. Math. Phys. vol. 43 (1936) pp. 391–400.

275. *The ideal Waring theorem for twelfth powers*, Duke Math. J. vol. 2 (1936) pp. 192–204.

276. *Proof of the ideal Waring theorem for exponents* 7–180, Amer. J. Math. vol. 58 (1936) pp. 521–529.

277. *Solution of Waring's problem*, Amer. J. Math. vol. 58 (1936) pp. 530–535.

278. *A generalization of Waring's problem*, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 525–529.

279. *The Waring problem and its generalizations*, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 833–842.

280. *New Waring theorems for polygonal numbers*, Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 62–65.

281. *Universal forms* $\sum a_i x_i^n$ *and Waring's problem*, Acta Arithmetica vol. 2 (1937) pp. 177–196.

282. *NEW FIRST COURSE IN THE THEORY OF EQUATIONS*, New York, Wiley, 1939, 9+185 pp.

283. *All integers except* 23 *and* 239 *are sums of eight cubes*, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 588–591.

284. *MODERN ELEMENTARY THEORY OF NUMBERS*, University of Chicago Press, 1939, 7+309 pp.

285. *Obituary: Hans Frederik Blichfeldt*, 1873–1945, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 882–883.

A. A. ALBERT

BOOK REVIEWS

Theorie der Funktionen einer reellen Veränderlichen. By I. P. Natanson. Berlin, Akademie-Verlag, 1954. 12+478 pp. 26 DM.

This is a textbook in real function theory, first published in 1941 in Russian. A revised and expanded Russian edition appeared in 1950, and it is this that has now been translated into German.

Briefly, the contents are as follows. (These phrases are the reviewer's summaries, not necessarily the author's chapter titles.) I–II: countable sets, open and closed sets on the real line. III–VI: Lebesgue measure on the real line, measurable functions, the Lebesgue integral and its properties. VII: orthogonality in Hilbert space, Fourier series. VIII: functions of bounded variation, and Stieltjes integrals. IX: absolute continuity and the differentiation of the Lebesgue integral. X: singular integrals and trigonometric series. XI–XIII: measure and integration in two dimensions, Fubini theorem, differentiation of set functions. XIV: ordinal numbers. XV: Baire classification of functions. XVI: topological properties of certain function spaces. XVII: the role played by the Russians in the development of real function theory.

The assumed prerequisite for a course based on this book is differential and integral calculus—in the European, not the American, manner. Thus in most schools in this country this would have to be a second course in real function theory. Specifically, the theory of limits is assumed understood—lim, lim sup, lim inf, and the notion of uniform convergence are not even defined. Furthermore, though certain properties of the real number system are investigated, this is not done in any systematic way. For example, there is a proof of the Bolzano-Weierstrass theorem based on the nonempty intersection of a nested set of closed intervals, this latter being referred to as a "well known theorem from the theory of limits."

The reviewer would like to suggest that regardless of what background is assumed, there are two essentially independent criteria on which a real function theory book may be classed as "advanced" or "elementary." (1) The degree of abstraction and generality in the subject matter considered. (2) The thoroughness with which investigations are prosecuted—the extent to which all facets of a problem are considered and "best possible" theorems are ferreted out. Natanson's book must be rated "elementary" on the first of these counts and "advanced" on the second.

Specifically, the only measure that appears in the book is Lebesgue measure, and this is only for bounded sets. In one sense this is to be commended rather than criticized. It gives the student something concrete to work with, yet the theory for this restricted case is fairly representative. On the other hand, many applications are ruled out, and the student is given no clue as to the way in which a more general theory would differ from this restricted one.

The book exudes a spirit of, "Let's see if we can't find out some more about this," and in general Natanson has done an excellent job of rounding out each discussion on which he embarks. Examples: The discussion of topology on the real line includes decomposition theorems for open and closed sets—decomposition of open sets into components and of closed sets into perfect and countable sets. The discussion of orthogonality in Hilbert space includes an investigation of completeness of orthogonal systems and the proof that L_2 and l_2 are equivalent.

Of course, any author who tries to be thorough is going to be heckled by people who can think of theorems he left out. The reviewer is really well pleased with Natanson's thoroughness but will join the hecklers with one example. In giving criteria for passage to the limit under the integral sign, Natanson in each case states his conclusion $\lim \int_E f_n(x)dx = \int_E f(x)dx$, whereas what he actually proves is $\lim \int_E |f_n(x) - f(x)| dx = 0$. Indeed, there is the slight curiosity that "convergence in the mean" is defined to mean strong convergence in L_2; the concept in L_1 is not discussed.

The book is written with the student in mind. Concrete examples accompany the introduction of new ideas. Proofs are given in sufficient, but not burdensome, detail. There is one feature, which is probably a deliberate part of Natanson's pedagogical plan, with which the reviewer would take issue. In several places there appear repetitions of essentially the same proof to give corollaries first and the master theorem at the end. For example, it is proved, in the order indicated, that convergence in measure justifies passage to the limit under the integral sign when accompanied by (a) uniform boundedness, (b) uniform domination by an integrable function, (c) uniform absolute continuity of the integrals. Admittedly, this is only one man's opinion, but the reviewer feels that one of the great thrills in the study of real function theory comes from proving a big theorem and then seeing how many corollaries drop out easily.

Through Chapter X there are exercises collected at the ends of the chapters. For the most part these seem to be fairly substantial problems, though not unreasonably difficult. There are no exercises in Chapters XI–XVII.

There is a slip in the transfinite induction theorem. It seems to permit one to prove a proposition true for "all ordinals." Incidentally, the Burali-Forti paradox on the set of all ordinals is explained two pages before this theorem. Aside from this, the reviewer found only typographical errors—and not very many of these. The format of the book is quite pleasing.

M. E. MUNROE

Differential line geometry. By V. Hlavatý. Trans. by H. Levy. Groningen, Noordhoff, 1953. 6+10+495 pp. 22.50 Dutch florins; cloth, 25 Dutch florins.

This book was first published in Czech in 1941 and was later translated into German and published by Noordhoff in 1945. The present edition is a translation from the German edition with the author's collaboration. The translator himself has added to the text by suggesting changes in some theorems and by adding a few new ones.

The book is meant to be a definitive work on three-dimensional differential line geometry, where line geometry in 3-space is studied as point geometry on a 4-dimensional quadric in a projective point space of 5 dimensions. Klein discovered the mapping of line geometry onto the 4-dimensional quadric, and for this reason the quadric is called the Klein quadric (or K-quadric) and the 5-dimensional space the Klein- (or K-) space. This viewpoint of three-dimensional line geometry has been used by authors before, but never as extensively as in this text. Both the classical material on the subject and new contributions by the author have been included.

The text is necessarily quite long and the author has tried to overcome some of the difficulties of length by dividing his work into "books," each of which can be read without the others. This naturally leads to some repetition of material. The books are divided into chapters which are numbered consecutively throughout the text. There are five books: the first (Chapter I) an introduction to line space; the second (Chapter II) on ruled surfaces; the third (Chapters III, IV, and V) on congruences; the fourth (Chapters VI–IX) on complexes, and the last (Chapter X) on line-space. Tensor calculus is used at all times, and simplifies the notation. For those readers who are unfamiliar with the tensor calculus the author has included in an appendix a straightforward, well written account of that part of the subject necessary for a reading of the text.

In the first book the author defines Plücker coordinates, Klein points, and Klein 5-dimensional space. He states that all topics, as far as possible, will be discussed from the viewpoints of projective, affine, and metric geometry. Unless explicitly stated otherwise, all theorems and definitions are given for projective geometry.

The standard material on ruled surfaces in projective differential geometry is given in the second book, ruled surfaces appearing in *K*-space as *K*-curves on the *K*-quadric. Also five particular linear complexes are introduced, as are three projective curvatures which determine a *K*-curve (hence the corresponding ruled surface) uniquely to within initial conditions. By use of one of the five special complexes, many of the familiar metric properties of ruled surfaces are derived.

More or less well known material on congruences, which are represented by *K*-surfaces in *K*-space, is presented in Chapter III. Chapter IV deals with congruences in affine and metric geometry, from both the algebraic and analytic viewpoints. Two uniquely defined metric tensors are introduced and used in the study of the congruences. Scalars representing the first and second mean curvatures of a congruence are defined in terms of these tensors. Normal congruences are studied, as are principal surfaces of congruences, distribution curvature, central points, and central planes. Pseudo-parallelism, autoparallelism, and teleparallelism are discussed and Frenet equations are developed for surfaces on congruences.

In Chapter V the relations between the *K*-surface and *K*-space are studied. For example, the fundamental equations of Weingarten, of Mainardi-Codazzi, and of Gauss are derived for congruences whose second osculating *K*-spaces are 5-dimensional (4-dimensional, 3-dimensional).

The complexes studied in the fourth book are not necessarily linear. Chapter VI deals with foundations and with fundamental equations concerning 3-dimensional manifolds in *K*-space (the images in *K*-space of complexes in 3-space). Algebraic complexes are studied, as well as the nature of a complex in the neighborhood of its singular and special lines. The fundamental equations are derived.

Chapter VII is concerned with surfaces of a complex (*C*-surfaces). Elemental *C*-surfaces and elemental *C*-congruences are discussed, and a normal curvature for nondevelopable *C*-surfaces is defined. Definitions are given for two *C*-surfaces to be autoparallel, and for two nondevelopable *C*-surfaces to be extremal surfaces. It is shown that a necessary and sufficient condition that two nondevelopable *C*-surfaces be autoparallel is that they be extremals. A non-extremal surface has two curvatures which satisfy generalized Frenet equations and which determine the surface uniquely to within initial conditions.

Absolute pseudoparallelism, absolute autoparallelism, and absolute teleparallelism are discussed briefly.

Under the mapping onto the K-quadric a congruence of a complex is represented by a K-surface contained in the 3-dimensional K-manifold which is the image of the complex. A normal surface of a congruence is defined in Chapter VIII, and it is shown that this surface is projectively orthogonal to every surface through the same line as the normal surface. Generalized fundamental equations are derived and used in determining whether there exists a C-congruence with two particular preassigned tensors.

Chapter IX deals with surfaces of a congruence of a complex (called CC-surfaces). Elemental principal CC-surfaces, characterized by extreme values of the outer curvature, are studied. Spheroidal lines, which are lines on which the elemental principal CC-surfaces are not determined uniquely, are discussed. Inner and outer curvatures of a CC-surface are defined. The inner curvature determines the CC-surface in the congruence to within initial conditions. These curvatures are closely related to the first curvature of the surface considered as a surface of the complex. The concept of torsion of a CC-surface leads to an equation connecting the invariants of the CC-surface. Brief mention is made of asymptotic surfaces of C-congruences.

The last book is devoted to a study of the entire line space represented by the K-quadric. This space is shown to be conformal-euclidean. A section is devoted to metric line space and there it is shown that the curvature tensor can be made to vanish. A short section is devoted to the resulting analagmatic geometry.

There are naturally advantages and disadvantages to the exclusive use of the mapping of line-space onto the K-quadric. Questions concerning line manifolds can be answered fully but, as the author himself points out, questions of point geometry must be left unanswered.

The exposition is clear throughout. Although this book has been written as a textbook, it is highly specialized and will probably be used more as a reference than as a text. The author has placed problems in the text, not at the end of sections, but where they arise naturally in the theory. The proofs of some of the simpler theorems have been omitted and left as exercises.

There is no bibliography, and this is a serious defect. Only a few references to other works are given. There is too much "organization" in the writing. For example, there are Remark I, Agreement (1,1), Theorem (1, 1), Definition (1, 1), etc.

There are a few minor misprints, but on the whole the book has been very well printed and proofread—this latter not always being the case today. The book is a valuable addition to the literature on line geometry, and a good translation into English makes it available to many more readers.

ALICE T. SCHAFER

Drei Perlen der Zahlentheorie. By A. J. Chintschin. Trans. from the 2d (1948) Russian ed. by W. v. Klemm. Berlin, Akademie-Verlag, 1951. 61 pp. 6.50 DM.

Three pearls of number theory. By A. Y. Khinchin. Trans. from the 2d (1948) Russian ed. by F. Bagemihl, H. Komm, and W. Seidel. Rochester, Graylock, 1952. 64 pp. $2.00.

The author, one of the leading Russian mathematicians, attempts to present three important recent results in such a way that they can be understood without much knowledge of number theory. He tries to create admirers of number theory by showing that elementary number theory is not yet a finished field since highly interesting new results were obtained by ingenious methods during the last few years, and further progress can be expected.

The author has been extremely successful in writing an excellent book for trained mathematicians. However, it is stated in the German edition that the book can be read by students of the upper grades of high schools and amateurs of mathematics and in the American edition that it can be understood by beginning college students. It is the reviewer's opinion that this is impossible. No such reader could study it with success. Even if he could understand some pages, he would not recognize the beauty of the results and their proofs.

The simplest part of the book is certainly the first chapter. The reviewer has proved its results in his classes at the University of Berlin and at the University of North Carolina, and he knows from this experience that it is not easy to present these theorems even to students who had taken a course in number theory.

In the first chapter the author proves the following theorem of van der Waerden published under the title *Beweis einer Baudetschen Vermutung*, Nieuw Archief voor Wiskunde (2) vol. 15 (1927) pp. 212–216. Let k and l be arbitrary integers. There exists a constant $W=W(k, l)$ such that for any distribution of the numbers $1, 2, \cdots, W$ into k classes at least one of the classes contains an arithmetic progression of l terms.

Khinchin states the history of this theorem as follows. When he arrived in Goettingen in the summer of 1928, the topic of the day was this result of van der Waerden proved in Goettingen a few days before. One of the young mathematicians there had come upon this problem. Everyone believed it was a simple problem, but all attempts by the mathematicians of Goettingen and many foreign visitors there remained unsuccessful. After several weeks of strenuous efforts the problem was solved by van der Waerden who was in Goettingen at that time.

This history is not quite correct. While van der Waerden attributes the problem to Baudet, it is much older and was an unsolved problem for many years. It is due to I. Schur. In 1906, E. Jacobsthal had published his results on sequences of 3 and 4 quadratic residues. Schur conjectured then that there exist sequences of l quadratic residues for every l and all sufficiently large primes, and he conjectured for the proof of this result the proposition attributed to Baudet for $k=2$ in a little stronger form, namely that the difference of the sequence belongs to the same class as the sequence.

van der Waerden did not find his proof only a few days before Khinchin arrived in Goettingen in the summer of 1928; he had read it as a paper already in September 1927 at the meeting of the Deutsche Mathematiker-Vereinigung. The paper was published in 1927. The reviewer was present when von Neumann, just returned from the meeting of the D.M.V., came to see Schur to inform him that his conjecture was proved. Schur was highly excited, but on the other hand disappointed that only the weaker form of the conjecture was proved which did not give the theorem on the sequences of residues. But a few days later, the reviewer succeeded in proving Schur's conjecture for the quadratic residues, the corresponding theorems for the quadratic non-residues, the kth power residues, and later the theorem for the $k-1$ classes of kth power non-residues while Schur proved the stronger form of van der Waerden's theorem (Sitzungsber. Preussische Akademie d. Wiss. Phys.-Math. Kl. (1928) pp. 9–16 and (1931) pp. 329–341). For the proof of each of these results the theorem of van der Waerden is used. Although these results are the only known applications of the theorem of van der Waerden they are not mentioned in the book. One may wonder why the theorem of van der Waerden is called a theorem of number theory without these applications since actually it is a theorem of combinatorial analysis.

In the second chapter the theorem of Mann is proved. In his important investigations on Goldbach's theorem, L. Schnirelmann defined the sum of two sets $A=(a_i)$ and $B=(b_j)$ of integers as the set containing the elements of A, those of B, and all numbers of the form a_i+b_j. Let $A(n)$ be the number of elements $a_i \leqq n$. We define the Schnirelmann density $d(A)$ as the lower bound of the quotients $A(n)/n$. Schnirelmann proved $d(A+B)=1$ for $d(A)+d(B)\geqq 1$, and Landau conjectured in 1931 that

$$d(A + B) \geqq d(A) + d(B) \text{ if } d(A) + d(B) < 1.$$

This problem was studied in a number of papers, but for some years it was proved only for special cases, and other estimates were obtained (see the report of H. Rohrbach, Jber. Deutschen Math. Verein. vol. 48 (1938) pp. 199–236). These papers were so interesting since everything had to be obtained only from the definition of Schnirelmann density. In 1941, the reviewer gave a course on additive number theory whose only aim was to develop everything then known about the problem. At the end of the semester one of the students, H. B. Mann, succeeded in proving the conjecture. His proof is very ingenious, but difficult. One year later, a simpler proof was given by Artin and Scherk. The latter proof is given in the book.

The third chapter of the book is the most valuable. It gives an elementary proof of Waring's problem, which for hundreds of years was one of the most famous unsolved problems. The proofs of Hilbert and of Hardy-Littlewood are very difficult and use complicated analytic methods. In 1942, the young Russian mathematician Linnik published an elementary proof of this theorem. But this proof was not accessible to those who cannot read Russian. While Linnik's paper is only 6 pages long, Khinchin needs 28 pages for the presentation to facilitate the understanding. It is of great importance that this proof is now available in English and German.

ALFRED BRAUER

Existence theorems for ordinary differential equations. By F. J. Murray and K. S. Miller. New York University Press, 1954. 10+154 pp. $5.00.

The following quotation from the Introduction of this book indicates the scope and level of treatment, as well as the aims of the authors: "We assume a knowledge of basic real variable theory (and for certain specialized results only, of elementary functions of a complex variable) and establish the fundamental existence theorems for ordinary differential equations which are the culmination of the nineteenth-century development. We do not consider the elementary methods for solving certain special differential equations nor the more advanced specialized topics. By restricting ourselves in this fashion we hope to obtain a logically coherent discussion which has educational value for the student in a certain stage of his mathematical development."

A brief summary of the contents, chapter by chapter, is as follows: 1. Existence theorems of Peano type; 2. Implicit function theorem, with applications to differential equations not solved explicitly for the derivatives of highest order; 3. Uniqueness of solutions of differential equations under assumption of Lipschitz condition; 4. Existence theorem of Picard-Lindelöf type, with theorems on the continuity of solutions as functions of initial values, and as functions of parameters; 5. Differentiability properties of solutions as functions of initial values, and as functions of parameters, together with an existence theorem for differential equations in the complex plane; 6. Existence theorem for a system of linear differential equations of the first order and properties of solutions of such a system, including a discussion of the monodromic group in the complex variable case. The reader is led slowly through the proofs of the general theorems; indeed, in Chapters 1, 3 and 4 the considered theorems are established firstly for a single equation of the first order, and subsequently for a system of such equations.

The text contains a few illustrative examples; there are no additional exercises for the student, however, aside from occasional theorems that are stated with the suggestion that the reader supply the proofs. The book is almost devoid of references, with the entire list consisting of three in the Introduction, scattered references in the text to Ince's *Ordinary differential equations*, and in Chapter 6 five references on the Jordan canonical form for matrices.

To the reviewer it appears highly regrettable that the authors have not seen fit to introduce vector and matrix notation prior to the last chapter, where such is used in a limited fashion; certainly this innovation of Peano in his 1888 paper has contributed greatly to the reduction of cumbersome notation. In the present treatment one searches in vain for the niceties of detail afforded by judicious use of Taylor's formula with integral form of remainder, especially in the consideration of dependence of solutions on initial values, or for the treatment of differential systems involving parameters through the adjunction of suitable differential equations. In these aspects this work is inferior to the concise treatment presented in the Appendix of G. A. Bliss' *Lectures on the calculus of variations*.

The following errors were noted: (a) the statements in Section 8 of Chapter 5 on "higher derivatives with respect to x initial conditions" are manifestly false unless hypotheses are amplified to include for the involved functions the existence and continuity of certain partial derivatives with respect to x; (b) at various places in Chapters 3, 4, 5 there are statements to the effect that for a function $f(y_1, \cdots, y_n, x)$ the existence and boundedness of partial derivatives $\partial f/\partial y_j$, $(j=1, \cdots, n)$, in an open set $\mathfrak{A}$ of $(y_1, \cdots, y_n, x)$ space implies that in $\mathfrak{A}$ the function f satisfies a Lipschitz condition in $(y_1, \cdots, y_n)$, which is clearly not true without the added assumption of convexity of the intersections of $\mathfrak{A}$ with hyperplanes $x=$constant.

W. T. REID

Analog methods in computation and simulation. By W. W. Soroka. New York, McGraw-Hill, 1954. 14+390 pp. $7.50.

Modern technology must be based on a scientific analysis of the problems considered. At first glance this might seem to require a precise solution or at least a numerically valid solution to the mathematical formulation of the problem. However, this is not quite true. The basic information necessary for technical decisions may be available from the study of an analogous system under the control of the investigator. In these circumstances mathematics plays a somewhat different role. The basic problem is not the solution of the mathematical problem, but the establishment of the analogy, that is the similarity of the mathematical equations governing the two systems. The important question is the uniqueness of the solution rather than its construction.

The principle of analogy is well established in engineering. For many years, problems in power distribution, the vibrational response of elastic structures and their stress distribution, air vehicle stability, and a whole host of model studies have been based on analogy. Since the war, there has been a considerable development of electrical analog equipment which has considerable advantages in flexibility of set up, availability, and ease of operation over most other types. It is also true, however, that the many nonelectrical analogies have continued to advance. Each of these tends to have a field of optimum application where the results obtained by the specified method are the most appropriate available.

The present book is organized to survey the field relative to the various engineering applications. There is considerable introductory material relative to the realization of mathematical operations. The major analogies treated are those based on "lumped" electrical circuit theory including commercial electronic differential analyzers and the network analogies for elasticity and the theory of structures, those associated with the finite difference expressions for partial differential equations and membrane and conducting sheet analogies.

The book begins with chapters on mechanical and electrical computing elements which are followed by a description of machines for simultaneous linear algebraic equations and a chapter on nonlinear equation solvers, harmonic analyzers, and the conduction sheet analogy for the complex plane. The mechanical and electrical differential analyzers are each treated in a chapter. The concluding chapters develop the analogies between dynamical and electrical systems, the analogies on which the finite difference solutions of partial differential equations are based, and the above mentioned analogies for the membrane and conducting sheets.

As an engineering text book this appears to be excellent. The derivations are specific to the application and at the mathematical level associated with college elective courses on differential equations. This book would also be a good introduction to analogies for the mathematician interested in the myriad mathematical problems associated with this field.

FRANCIS J. MURRAY

The kinematics of vorticity. By C. Truesdell. Bloomington, Indiana University Press, 1954. 20+232 pp. $6.00.

This is a work packed with theorems of a general type concerning the vorticity of a fluid. The author uses freely the classical theory of vectors in three dimensions, the theory of dyadics and, to some extent, the tensor calculus. After various geometrical preliminaries, and the definitions of velocity, acceleration, expansion, deformation, etc., of a fluid motion, the vorticity is defined and its various interpretations are given. The vorticity field and the notions of vortex lines and tubes are next discussed, with their bearing on circulation. The measure of vorticity is given a chapter to itself and this is followed by one on vorticity averages. Bernoullian theorems are then investigated, by which are meant formulae for the squared speed of the fluid and the scalar potential of the flow. The two final chapters deal with the convection and diffusion of vorticity and with circulation-preserving motions.

I have attempted to estimate the number of theorems contained in the book and have concluded that there is probably an average of two per page, making a total of perhaps 400. Though they are expressed in the terminology of hydrodynamics, they are essentially theorems in pure mathematics, expressing relations between vectors and their integrals over volumes and surfaces. That the vectors are called velocity, acceleration, vorticity, etc., is not an essential feature of the treatment. One is brought up therefore against the question: What is the central problem of applied mathematics? To one school of thought, it is the establishment of a body of theorems and methods in pure mathematics that might conceivably be useful in dealing with physical situations. A hydrodynamist is an expert in vector analysis, who may never have watched water flowing; a relativist is an expert in differential geometry who may never have heard of the perihelion of Mercury. According to this school, Truesdell's book is an excellent monograph on hydrodynamics, compact, succinct and packed with information. But there is another school which holds that the central problem consists of sizing up the physical situation, of estimating what are the essential variables, and what the relations between them may be. In some twenty years' experience of teaching applied mathematics—not to speak of the attempts to carry out my own researches!—I have noticed that it is the translation of a physical situation into mathematical terms which is found difficult by students. The subsequent pure mathematics are a relatively minor obstacle. From this point of view Truesdell's book suffers from the defect that the reader is given no clue as to whether—if at all—any particular result or theorem is applicable to a physical situation nor, if it is, what the nature of the situation may be. The name of Horace Lamb often occurs in the text, as is to be expected, and it is interesting to compare an example of Truesdell's exposition with Lamb's. The former, in §72, considers "Lamb surfaces" which are simultaneously vortex and stream surfaces; that such surfaces exist in any circulation-preserving motion with steady vorticity is proved as a theorem in pure mathematics and there the matter is left. The reader is given no clue as to whether such motions are ever encountered in nature and, if they are, under what circumstances. In contrast, when Lamb proves the same theorem (H. Lamb, *Hydrodynamics*, 5th ed., Cambridge University Press, 1924, §165), he immediately produces two illustrations of the kinds of physical situation wherein it might be useful. To Lamb, pure mathematics is a tool, to Truesdell it would seem to be an end in itself. In defence of the latter's method it must be said that illustrations greatly extend the length of a book and that fourteen pages of references should surely be a sufficient guide to possible physical applications.

A curious feature of Truesdell's treatment of vorticity is his extreme insistence on the importance of three spatial dimensions (see, for example, the remarks on pp. 3, 59, 77). It is true that if vorticity is to be expressed as a *vector*, three dimensions are necessary; but the spinning or rotational characteristics of a fluid can hardly be dependent on the particular mathematical form used to express them. Indeed, if a 4-dimensional treatment is employed the vorticity is no longer a vector but is described by three of the components of a second-rank antisymmetrical tensor. Evidently Truesdell means that his particular formalism, in terms of 3-vectors, is valid only in three dimensions which is no doubt the case; but over-emphasis on this point of view leads to remarks such as that on p. 77, relative to Lagrange's acceleration formula. Whilst it is true that the particular formula (38.2) holds only in 3 dimensions, it is also true that a corresponding formula, involving the vorticity tensor, holds in 4 dimensions. Thus it does not appear to be correct to state without qualifications that "for the existence of Lagrange's formula it is requisite that the number of dimensions be three."

But when all has been said, one important fact emerges: this book is a valuable compendium of results that every expert in hydrodynamics, gas-dynamics or dynamical meteorology will want to keep by his side and refer to frequently.

G. C. McVittie

Introduction to integral geometry. By L. A. Santaló. Paris, Hermann, 1953. 127 pp. 1500 fr.

Integral geometry is the name given by Blaschke to a branch of geometry which originated with problems on geometrical probabilities and deals with relations between measures of geometrical figures. One should perhaps consider as its father the English geometer W. F. Crofton, while Poincaré left an important mark by introducing the kinematic measure (and by writing a book on geometrical probabilities).[1] In 1934 Blaschke and his coworkers started a series of papers on the subject. The author of this book made some of the most beautiful contributions to it in that period and, in the twenty years which followed, has consistently added new results to it. It is therefore gratifying that a book by the author now exists in the literature. The book is elementary in nature. Its first two parts presuppose only some knowledge of calculus and the last part some projective geometry and a little more maturity.

Part I, on the metric integral geometry of the plane, studies the densities of points, lines. pairs of points, etc. and culminates with Poincaré's kinematic density and Blaschke's kinematic formula. One of the most interesting applications is the author's proof of the isoperimetric inequality in the plane. In spite of the elementary nature of plane geometry, this is still the most fascinating chapter of the book. The presentation is informal and lucid. The author has certainly succeeded in leading up effortlessly to many beautiful results. One should, however, not be led to think that everything in this subject is so simple. The author has purposely neglected the set-theoretic difficulties. Otherwise even the proof of the simplest of Crofton's formulas would have taken pages, while a proof of the kinematic formula for fairly general domains would be quite complicated.

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[1] As a matter of historic interest mention should be made of a set of lecture notes by G. Herglotz, which introduced many of the early workers into the subject.

Part II generalizes the results of plane integral geometry to surfaces. The generalization is complete in the case of surfaces of constant curvature. In the general case a density for geodesics and the kinematic density are defined. To those familiar with modern notions in differential geometry it would be conceptually simpler to define the kinematic density as one in the circle bundle of unit tangent vectors and the density for geodesics as one in a leaved structure (structure feuilletée of Ehresmann). The kinematic density on a surface is probably an important notion. It may be of interest to mention that it was used by E. Hopf to prove his theorem that, if a closed surface has no conjugate points, its Euler characteristic is $\leqq 0$ (Proc. Nat. Acad. Sci. U. S. A. vol 34 (1948) pp. 47–51.

Part III is concerned with integral geometry in a general homogeneous space and constitutes one half of the book. It begins with a brief introduction to the local theory of Lie groups, following E. Cartan. Using this general theory, a necessary and sufficient condition is established for the existence of an invariant density when the homogenous space is acted on by a connected group. The general results are then applied to special cases: the Cayley plane, affine and projective geometry. Perhaps the most interesting result is the interpretation of Siegel's proof of the Minkowski-Hlawka theorem as a result of integral geometry. It seems to the reviewer that immediate further results are to be found in the case when the isotropy group of the homogeneous space is compact.

The question of studying the relations between measures in homogeneous spaces with the same group is a natural one, be it the Euclidean plane and the space of its lines or the affine plane and the space of its lattices in the sense of the geometry of numbers. The final word on these questions has not been said. We may hope that the publication of this book will contribute to further progress in this neglected but highly fascinating branch of geometry.

S. S. CHERN

A mathematician's Miscellany. By J. E. Littlewood. London, Methuen, 1953. 8+136 pp. 15s.

From the introduction: "A Miscellany is a collection without a natural ordering relation; I shall not attempt a spurious unity by imposing artificial ones. . . . The other quality is lightness, notwithstanding the highbrow pieces; my aim is entertainment and there will be no uplift. I must leave this to the judgment of my readers, but I shall have failed where they find anything cheap or trivial." There is nothing cheap or trivial in the book, because there is so much of Littlewood's mathematical personality in it. For the same reason, and despite the author's disclaimer, there *is* uplift in the book. It is indeed more than entertainment to learn Littlewood's views, to see his tastes, to feel "what makes him click." Even his entertaining game of debunking ("every mathematical proof is a debunking of some sort") becomes an experience when Littlewood discusses the discovery of Neptune.

The book is intended for the mathematical amateur. But the amateur whom the author envisages is of the rare type who frowns on a repetition of what can be found in Courant-Robbins. Littlewood's masterful style and lack of ostentation should have a wide appeal, but presumably the book will be enjoyed mostly by professional mathematicians who are used to concentration and who have the basic experience of mathematics as art.

The opening chapter is entitled "Mathematics with minimum 'raw material'." In rapid succession (sometimes in telegram style) the author reviews a series of puzzles, paradoxes, unexpected arguments, and proofs. They all illustrate the nature of mathematical reasoning, its power, or its appeal. "Official" mathematics is represented by a few items ranging from an isoperimetric inequality (whose claim to interest lies in the simplicity of the proof) to the highbrow convexity theorem of M. Riesz. Thorin's proof is given, and the reader is led to share Littlewood's excitement when an upper bound is taken "with respect to a variable that is not there!"

Next we come to a short digression on the Tripos, and then to "Cross-purposes, unconscious assumptions, howlers, etc." This chapter is, in part, hilarious, although it contains serious and noteworthy views on style and rigor. A satirical gem is a proof of Weierstrass' approximation theorem presented with all the horrors of bad mathematical manners.

There follow "The Zoo" (conformal mapping), "Ballistics," and "The Dilemma of Probability Theory." This theory, it is said, is genuine mathematics, but when the student of probability interprets its relation to experience then he is merely philosophizing. That is true, but it is not clear why this perfectly general dilemma should be connected with probability more than, say, ballistics. In fact, "in actual experience" (in particular before breakfast) multiplication tends to be noncommutative and should Littlewood defend number theory against this attack, he would be merely philosophizing. However, that he can do this in a provocative and worthwhile fashion is shown in the chapter "From Fermat's Last Theorem to the Abolition of Capital Punishment." In a few crisp lines the author takes us from ideals in algebra to functions and thence to a function whose domain is the historical time and whose range lies in the sample space of the Universe. Such a function is a sort of historical record (past or future) and we are confronted with the notion (or lack thereof) of determinism.

We come now to an extensive description of Littlewood's mathematical education and of conditions before 1907, and then to reprints of articles from the Mathematical Gazette. The latter include a few book reviews, a discourse on "Newton and the Attraction of a Sphere" and a causerie on "Large Numbers."

Of an altogether different nature are the final chapters "The Discovery of Neptune" and "The Adams-Airy Affair." In an elaborate and exceedingly interesting study the author shows how Neptune *could* have been discovered by most elementary means; he discusses the importance and bearing of accidental factors on the actual historical development and makes a few cogent remarks on the rôle of the principal actors. The reviewer feels that he has greatly profited from this study, but in view of his difficulties with details it is only fair to confess that he, for one, falls short of Littlewood's ideal of an amateur. A more exhaustive and detailed version of this noteworthy investigation would seem to be a worthwhile contribution to science.

The book concludes with purest mathematics: "A lion and a man in a closed arena have equal maximum speeds. What tactics should the lion employ to be sure of his meal?" The problem is old, the optimal strategy well-known and, what's more, "obvious". The joke is that the good old solution is false. A simple, but ingenious, construction shows that the man can save his life (provided, of course, the lion is a mathematical point). The proof is due to A. S. Besicovitch, who never published it. In fact, when congratulated on the idea, Besicovitch snorted indignantly: "It is only a joke." Yet, according to Littlewood, "A good mathematical joke is better, and better mathematics, than a dozen mediocre papers."

WILLIAM FELLER

Gruppentheorie. By A. G. Kurosch. With a supplement by B. H. Neumann. Berlin, Akademie-Verlag, 1953. 12+418 pp. 28.00 DM.

This is a German translation of the first Russian edition (1944) of Kurosch's *Teoriya Grupp.* Although it appeared after the second Russian edition (1952), it is well worth owning for itself, even for those who read Russian easily. The supplement by B. H. Neumann is particularly valuable.

In a very refreshing way Kurosch frees the study of groups from unnecessary assumptions of finiteness. Free groups, defining relations, and free products appear early and account for chapters 4 and 10. The study of Abelian groups is very extensive and occupies chapters 5, 8, and 9. This includes the theorem of Ulm which completely characterizes countable periodic Abelian groups. Chapter 7 covers solvable and nilpotent groups. This has been an area in which research has been active in recent years, the theory of special groups having been developed by the Russian school. In chapter 11 Kurosch gives most of what was known about subgroup lattices at the time he wrote.

The German edition adds to the 1944 edition in two ways. First the bibliography has been brought up to date and numerous footnotes relate the more recent work to the text. But most notably there is an extensive supplement by B. H. Neumann giving among other topics the theory of amalgamated products. Also three important examples due to Graham Higman are included. One of these is a finitely generated group isomorphic to a proper subgroup. A more complicated example of this was given earlier by Neumann himself. This shows the Hopf conjecture to be false.

Unfortunately, in eliminating finiteness Kurosch has thrown out the baby with the bath and there remain only faint traces of finite groups. He promises another volume on finite groups which will indeed be welcome.

MARSHALL HALL, JR.

Theoretical elasticity. By A. E. Green and W. Zerna. Oxford University Press, 1954. 14+442 pp. $8.00.

This book is mainly concerned with three areas in elasticity: nonlinear elasticity, complex variable methods in linear elasticity, and shell theories. Since even these are not treated comprehensively, it might more aptly be titled "Special Topics in Theoretical Elasticity." On the other hand, no other single work makes as serious an attempt to cover both linear and nonlinear elasticity.

Chapter I is devoted to purely mathematical results which are needed for later analyses. In Chapter II, the theory of stress and strain is discussed and the equations of the general theory are obtained. The authors restrict themselves to convected coordinates, i.e., to moving coordinate systems in which the coordinate surfaces are material surfaces. The reviewer and several other workers in elasticity prefer not to impose this restriction, but this is largely a matter of taste.

In Chapter III, the authors include those general solutions which were available in the literature at the time the book was written. In Chapter IV, the general theory of small deformations superposed on large is developed for isotropic materials and a number of solutions of the resulting equations are given. Aside from some results given in Chapter V concerning the form of the strain energy for anisotropic materials, this constitutes their treatment of nonlinear elasticity. So many results in nonlinear elasticity have since been published that their treatment already seems rather out of date.

Chapter V gives a presentation of the linear theory, and a number of solutions of these equations. On the whole, their work seems quite reliable, but their discussion of boundary value problems and uniqueness theorems on p. 164 is extremely slipshod. According to their remarks, prescribing the surface tractions determines a unique solution of the equations of motion, which is definitely untrue.

Chapter VI is devoted mainly to explaining the use of complex variable methods for plane strain of isotropic and anisotropic materials and Chapter VII extends these methods to two-dimensional problems in plate theory. In Chapters VIII and IX, these methods are applied to obtain numerous solutions. Included are interesting comparisons of solutions for isotropic and anisotropic materials.

The remainder of the book is devoted to shell theories obtainable from the linear theory by approximation. As the authors remark, "The methods used and the approximations which are made differ with almost every writer. . . . " In view of this, it would seem appropriate to make an attempt to give a rigorous justification of the approximations which they use, to give a critical appraisal of existing theories, to indicate how results obtained from different approximate theories compare or to otherwise clarify the situation. Instead, the authors follow the usual procedure of making more or less plausible approximations to obtain tractable equations which are, at least in some cases, different from others which have been proposed. Some solutions are given.

Those interested in the topics covered by this book should find this a useful reference, particularly for solutions of particular problems. It is not simply a rehash of the literature, a number of hitherto unpublished results having been included. The bibliography, while not complete, is sufficiently extensive to be quite useful. At present, it is the only single work which could reasonably serve as an introduction to both linear and nonlinear elasticity. It is perhaps too specialized to be well suited for this purpose, though a comprehensive treatment would fill several volumes.

J. L. ERICKSEN

Ricci-Calculus. An introduction to tensor analysis and its geometrical applications. By J. A. Schouten. 2d ed. Berlin, Springer, 1954. 20+516 pp. 55 DM; clothbound, 58.60 DM.

Since the publication in 1901 of the famous paper on absolute differential calculus by G. Ricci and T. Levi-Civita which established the foundation of the so-called Ricci-Calculus and especially since the publication in 1916 of the theory of general relativity by A. Einstein, the importance of Ricci-Calculus in its geometrical and physical applications has been universally recognized. The first edition of this book, published in German in 1923, covered all the researches made until then and played an important instructive role in this new branch of geometry. But since then tensor calculus was further developed to a great extent and many new notions were introduced, for example, normal coordinates, the symbolism of exterior differential forms, infinitesimal deformations, Lie derivatives, subprojective spaces and their generalizations in Hermitian geometry. To cover these new notions, J. A. Schouten and D. J. Struik published in 1935 and 1938 their *Einführung in die neueren Methoden der Differentialgeometrie* I and II. In this book they introduced the so-called kernel-index method which is a characteristic of Schouten's school.

Since 1935 Ricci-Calculus was again further developed. For example, the projective and conformal geometries have been studied in great detail from various points of view; Finsler and Cartan spaces, general spaces of paths and those of K-spreads were introduced, the motions in these spaces were studied by the use of Lie derivatives; and the ideas of harmonic spaces and of spaces of recurrent curvature were developed by British mathematicians. The book under review was written to cover these new developments in the Ricci-Calculus and the author tries to retain the instructive and the encyclopedic character which "Der Ricci-Kalkül" has. Thus, although the book is entitled "Ricci-Calculus, the second edition," it is an entirely new book.

It consists of eight chapters. The first chapter is devoted to tensor algebra. In the first section, an n-dimensional affine space E_n is defined and the coordinate transformations and point transformations are discussed. In this section the principle of the kernel-index method is explained. We use the same kernel letter to indicate a geometric object and different kinds of indices to indicate its components with respect to different kinds of coordinates. So, with coordinate transformations the kernel letters do not change but a new set of running and fixed indices is introduced. But with point transformations the kernel letters change and the running and fixed indices remain the same.

In the following sections, the author defines and discusses successively quantities in E_n, algebraic operations applied to quantities, subspaces in E_n, rank of a quantity with respect to one or more indices, symmetric tensors, multivectors, tensors of valence 2, introduction of a metric in an E_n, hybrid quantities, and abridged notations. In the last section, the author gives a survey of many different notations. At the end of Chapter I, we read: "Young authors especially, who cannot yet foresee all the consequences, should abstain as far as possible from introducing ad hoc abbreviations."

Chapter II is devoted to analytic preliminaries. The author starts by defining the arithmetic n-dimensional space $\mathfrak{A}_n$ and the geometrical n-dimensional space X_n. The development is similar to that of O. Veblen and J. H. C. Whitehead. Then he discusses geometric objects in X_n, the X_m in X_n, the (non-holonomic) X_n^m in X_n, operators Rot and Div, Pfaff's problem, the theorem of Stokes, anholonomic coordinates, the Lie derivatives and the Lagrange derivatives. In the last section, he compares his own symbolism with that of E. Cartan. This section should be useful for readers who are already familiar with notations other than those of the present book.

In Chapter III, the author comes to the discussion of linear connexions. After preliminary discussions in E_n, he defines the parallel displacements in X_n in a quite general way. Then there follow discussions on the torsion tensor, the curvature tensor, identities satisfied by the curvature tensor, integrability conditions of some tensor equations, geodesics and normal coordinates, Fermi coordinates and formulas in anholonomic coordinates. At the end of this chapter, the author again compares his formulas with those of E. Cartan. The last section is devoted to the linear connexions depending on a non-symmetric fundamental tensor, which A. Einstein used in 1945 to obtain a unified field theory.

Chapter IV is devoted to Lie groups and linear connexions. The classical theory of Lie groups is developed by an ingenious use of the Ricci-Calculus. This chapter deals with finite continuous groups, the parameter groups and the adjoint groups of a finite continuous group, finite continuous transformation groups, the geometry of group space, invariants of a transformation group in the X_n of the ξ^κ, invariants of a group in group space, and properties of integrable groups. Unfortunately the theory of simple and semi-simple groups, developed by E. Cartan and the author himself, is excluded.

In Chapter V, the author discusses imbedding and curvature. He starts with curves and hypersurfaces in a Riemannian space and in a space with a linear connexion. He next develops the theory of general (holonomic and anholonomic) subspaces making use of the D-symbolism due to van der Waerden and Bortolotti. The chapter ends with discussions on product spaces.

Chapter VI is devoted to projective and conformal transformations of connexions. The author first discusses projective transformations in a space with a symmetric connexion and those in a Riemannian space. He next deals with the behavior of affine conics under a projective transformation. In §4, the projective connexions of T. Y. Thomas are discussed but without using one supernumerary coordinate. In §§5 and 6, the author discusses the conformal transformations in a general Riemannian space and also in an Einstein space. Next comes the theory of conformal connexions of J. M. Thomas. The rest of the chapter is devoted to the subprojective connexions of Kagan, Adati's problem on subprojective spaces, subprojective transformations of a linear connexion, and concircular transformations in a Riemannian space.

In Chapter VII, variations and deformations are discussed. The author first explains the general method for dealing with the deformation problems and gets some of the classical results on motions and affine motions in a Riemannian space and in a space with a linear connexion, as well as some recent results by I. P. Egorov, G. Vranceanu, and the reviewer. He then comes to the discussions of deformations of subspaces. The latter part of the chapter is devoted to the discussions of holonomy groups and the symmetric space of E. Cartan. In the last section, the author applies the method of moving frames of E. Cartan to the study of holonomy groups and of symmetric spaces using his own symbolism. It will be very useful to see the relation between the symbolism of E. Cartan and that of the author.

The last chapter entitled "Miscellaneous examples" contains discussions of harmonic Riemannian spaces, connexions in complex spaces, and spaces of recurrent curvature.

The book has an extensive bibliography. It contains 1400 papers by 350 authors. The author says that in order to retain something of the encyclopedic character of the first edition, he selected the titles in such a way that the reader interested in some topic will always find at least a few titles which can lead him on to more references. The bibliography indicates where each paper is quoted in the text, which is very convenient for the reader.

The main topics in differential geometry which are not treated in this book are the geometries of Finsler and Cartan spaces and their generalizations, the geometries of paths and K-spreads and their generalizations, projective and conformal geometries with the use of supernumerary coordinates, the geometry under contact transformations, the geometry in almost complex spaces, the theory of general geometric objects, the differential geometry of fibre bundles, etc. The reviewer sincerely hopes that the author may soon have another opportunity to give a survey of these interesting topics.

This excellent book by an author who has since 1918 always been a pioneer of research in the fields of differential geometry will serve not only as "an introduction to tensor analysis and its geometrical applications" but also as an encyclopedia for the differential geometers of the front line and it will give all the information in the small necessary for the development of differential geometry in the large.

KENTARO YANO

Lectures on partial differential equations. By I. G. Petrovsky. Trans. by A. Shenitzer. New York, Interscience, 1954. 10+245 pp. $5.75.

This book is intended to be an introductory and self-contained text in the theory of partial differential equations. That it fulfills this purpose is a result not only of the excellent presentation of the author but also of the excellent translation of A. Shenitzer which preserves the easy, flowing style of the original.

The book is divided into four chapters: the classical trichotomy, hyperbolic, elliptic and parabolic equations, and an introductory chapter devoted to motivation and notation.

The first chapter begins with a derivation of the various classes of equations mentioned above from prototype physical problems. Beginning the study of existence and uniqueness, the author presents the Cauchy-Kowalewski theorem for the Cauchy problem with analytic initial conditions. This leads to the concept of characteristics, and to the question of existence and uniqueness of solutions with nonanalytic initial conditions. There is a very thorough discussion of this topic, in which the author brings the reader up to date on some of the latest developments in the field. It is characteristic of the book that the author continually emphasizes the latest advances as well as the classical results and singles out the outstanding open problems. The remainder of the chapter is devoted to the question of the reduction of equations to canonical form, thus preparing the way for the division of the three remaining chapters into the study of hyperbolic, elliptic and parabolic equations.

The second chapter is devoted to the study of hyperbolic equations, a subject particularly dear to the author's heart. It begins with an example due to Hadamard illustrating the discontinuous dependence on the initial values of the solution of the Cauchy problem for the Laplace equation. This motivates a discussion of the generalized solutions of Sobolev. Existence and uniqueness theorems for the wave equation and hyperbolic systems are then presented, followed by the derivation of the explicit formulas for the solution due to Kirchhoff, Poisson and d'Alembert. Using these, the stability of the solution as a functional of the initial values is demonstrated for the particular equations above. Again using the explicit formulas, the concept of diffusion and its dependence upon dimension is treated. Pulling over to a siding temporarily, the author discusses the Lorentz transformation, and the mathematics of the special theory of relativity. This part of the second chapter concludes with a survey of the basic results for the Cauchy problem for general hyperbolic systems and a discussion of the finite difference approach of Courant, Friedrichs and Lewy.

In the second part of the chapter we meet the wave equation with boundary as well as initial conditions. Uniqueness and continuous dependence upon the initial values are again demonstrated under simple conditions on the region. To illustrate the application of the method of Fourier, based upon separation of variables and the solution of eigenvalue problems, a one-dimensional problem is treated in detail. In order to keep the text self-contained, a large section is now devoted to a derivation of the properties of eigenfunctions and eigenvalues, including the minimax principle of Courant. Under simple conditions, the author establishes Parseval's identity and the convergence of the Fourier series obtained formally. To illustrate an alternate approach, it is shown how a Green's function may be used to convert the boundary value problem into that of solving a Fredholm integral equation. A multi-dimensional problem is now treated to demonstrate the points of similarity in the basic approach by means of Fourier's method, and some of the new difficulties that arise. Bessel functions occur in the particular example given. The chapter concludes with a short sketch of the variational techniques of Rayleigh, Ritz, and Galerkin.

The topic of elliptic equations occupies the third chapter. After a short proof, due to Privalov, of the minimum-maximum theorem for solutions of the Laplace equation, the author turns to the Dirichlet problem for the circle. Using the Poisson integral representation, the standard sequence of theorems concerning harmonic functions is derived. The existence of a solution for general regions is then proved using the Poincaré-Perron concept of superharmonic functions. In order, the author discusses the exterior boundary value problem, the Neumann problem, potential theory, and the application of potential theory to the solution of boundary value problems. Following these results, there is a brief discussion of the approximate solution of the Dirichlet problem by means of finite differences, following an approach due to Lusternik. As in the previous chapters, this chapter closes with a survey of some of the most important results for elliptic equations.

The last chapter is a very brief one, and sketches the application of some of the techniques developed in the previous chapters to the solution of some simple problems in the theory of heat conduction.

The only fault in the book is a small one—there is no index. The bibliography is given in footnotes, and this also is not completely desirable. Apart from these minor items, the book is highly to be recommended. It is printed very attractively, reads very smoothly, and all in all is to be regarded as an elegant introduction to an attractive field of mathematics.

RICHARD BELLMAN

Vorlesungen über Approximationstheorie. By N. I. Achieser. Berlin, Akademie-Verlag, 1953. 10+309 pp., 10 figures. 29.00 DM.

This splendid book (translated from the Russian edition of 1947) gives much more than the title promises. Besides a discussion of specific problems of approximation it provides also an introduction to many different parts of analysis, as can be seen from the following list of topics in Chapter I: Elements of functional analysis. Chapter II: Approximation in C (Chebyshev approximation). Chapter III: Fourier analysis: L^2-theory, Fejér's theorem, Watson transforms, conjugate functions. Chapter IV: Entire functions of exponential type bounded on the real axis, S. Bernstein's inequalities for the derivatives of these functions and generalizations. Chapter V: Problems of best approximation by trigonometric polynomials and by entire functions of exponential type. Chapter VI: Wiener's Tauberian theorem.

The presentation is very clear and interesting. The author has achieved a happy mixture of the abstract and the concrete. Definitions are immediately followed by examples and general theorems are strikingly applied to special cases. For example, the definition of the modulus of continuity is illustrated by a computation of the modulus of continuity of Weierstrass' nondifferentiable function. The judicious blending of modern and classical methods also appears in the proofs. In particular, the chapter on Wiener's theorem is probably the easiest account of the subject available in print, although the motivation for the notions used (normed ring, ideal) may be obscure to the beginner.

The "general problem of approximation" is formulated as follows by the author: $f(x)$ and $F(x; A)$ are two functions of a normed function-space; the second of these depends on a number of parameters A. It is required to choose the values of the parameters in such a way that $\|f-F\|$ becomes a minimum. The most important case is the linear case, $F(x; A)=F(x; a_1, a_2, \cdots, a_n)=\sum_{k=1}^{n} a_k f_k$ where $\{f_k\}$ $(k=1, 2, \cdots)$ is a given set of functions. The book contains concise proofs of the classical results of S. Bernstein, Chebyshev, Haar, D. Jackson and de la Vallée Poussin concerning the existence and uniqueness of the extremal functions for various choices of $\{f_k\}$ in the spaces L^p and C. The problem of the closeness of the approximation which can be achieved also receives careful attention. Chapters IV and V will be of particular interest to Western readers, since they contain an account of fairly recent Russian investigations. A large part of Chapter IV is concerned with generalizations of Bernstein's theorem: If $f(z)$ is an entire function of order 1, type σ, bounded by M on the real axis, then $|f'(x)| \leqq \sigma M$ (x real). The ideas underlying this work are the same as those used by R. C. Buck (see R. P. Boas, *Entire functions*). As an example of other work in this chapter I quote Krein's theorem: $\lambda(t)=\int_{-\infty}^{\infty} e^{ixt} d\omega(x)$ $(|t| \leqq \sigma)$, $\int_{-\infty}^{\infty} |d\omega| \leqq M$ is equivalent to $|\sum_{k=1}^{n} c_k \lambda(t_k)| \leqq M \sup_x |\sum_{k=1}^{n} c_k e^{it_k x}|$ for all integers n, all sequences of complex numbers $\{c_k\}$ and all choices of the t_k in $-\sigma \leqq t \leqq \sigma$. Chapter V is concerned with approximation by functions in $G=G(T)$, the class of all functions which are restrictions to the real axis of entire functions of order 1, type $\sigma<T$. For several important classes K of functions it is possible to determine explicitly the "best approximation" $\sup_{f \in K} \inf_{g \in G} \|f-g\|$, where the norm is either the C-norm or the L^p-norm $(p \geqq 1)$. Jackson's theorem, "If $f(x)$ has period 2π and $|f^{(r)}(x)| \leqq 1$, then there is a trigonometric polynomial $T(x)$ of order $<n$ such that $|f(x)-T(x)| \leqq A_r n^{-r}$," can be deduced from these results together with the best possible value of the constant A_r. There is an appendix which contains interesting special results, most of them concerning closure and best approxima-

tion by polynomials. An example is the following theorem. Let $\sigma(t)$ be a nondecreasing function, $\sigma'(t)$ the derivative of its absolutely continuous part. Let $L^p(d\sigma)$ ($p \geqq 1$) be the space of functions $f(t)$ ($0 \leqq t < 2\pi$) with $\|f\|^p = \int_0^{2\pi} |f(t)|^p d\sigma(t)$. The set $\{1, e^{it}, e^{2it}, \cdots \}$ is total in $L^p(d\sigma)$ if and only if $\int_0^{2\pi} |\log \sigma'(t)| dt = \infty$.

The book is handsomely printed, but the list of typographical errors at the end is far from complete. In particular, the reader should be encouraged to read p. 233 before p. 232.

W. H. J. FUCHS

BRIEF MENTION

Equazioni differenziali. By F. G. Tricomi. 2d ed. Torino, Einaudi, 1953. 353 pp. 4000 lire.

The revised edition of this book follows very closely the pattern of the first edition, which was reviewed in this Bulletin vol. 56 (1950) pp. 195–196. The most important cases of inclusion of new material are: (i) Chapter II has been augmented by an introduction to the subject of relaxation oscillations; (ii) Chapter IV has been revised considerably, to provide a more comprehensive treatment of the asymptotic character of solutions of differential equations of the form $y'' + Q(x)y = 0$.

Details of discussion have been altered in various instances, notably in Chapter IV in the treatment of the polynomials of Laguerre and Legendre. Material on the "method of Fubini" that formed an Appendix in the initial edition has been incorporated in Chapter IV; also, a number of new references have been added to the bibliography.

In this new edition the author has produced a commendable improvement of the highly interesting and valuable first edition.

W. T. REID

Linear operators. Spectral theory and some other applications. By R. G. Cooke. London, Macmillan, 1953. 12+454 pp. $10.00.

This book contains an introductory chapter, and then a chapter on quantum mechanics. These are followed by a long section (three chapters) devoted to various proofs of the spectral theorem. For a textbook treatment of this, the reviewer prefers the snappy handling in [1]. A sixth chapter is concerned with "projective convergence and limit in matrix spaces and rings." Chapter 7, the final chapter, is a self-contained exposition of the elements of the theory of Banach algebras. The theory of Banach algebras without unit element is included (i.e. the theory involving adjunction of a unit element, with ideals defined in the larger ring). Applications are given, e.g. Wiener's theorem: If t is a real variable, the necessary and sufficient condition that the reciprocal of an absolutely convergent trigonometric series $\sum_{-\infty}^{\infty} a_n e^{int}$ should be expressible as an absolutely convergent trigonometric series is that $\sum$ never be 0. Representation theory for positive linear functionals also appears, and the chapter is on the whole a good exposition, and a good outline of the subject for the nonspecialist.

BIBLIOGRAPHY

1. P. R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity*, New York, Chelsea, 1951.

JOEL BRENNER

Theory of functions of a complex variable. Vol. 2. By C. Carathéodory. Trans. by F. Steinhardt. New York, Chelsea, 1954. 220 pp.

The German edition was reviewed in this Bulletin vol. 57 [redacted] p. 190.

Contributions to the solution of systems of linear equations and the determination of eigenvalues. Ed. by O. Taussky. (National Bureau of Standards Applied Mathematics Series, vol. 39.) Washington, Government Printing Office, 1954. 4+139 pp. $2.00.

This volume contains 7 papers forming a sequel to vol. 29 of this series (this Bulletin vol. 60, p. 414).

Les fonctions de Bessel et leurs applications en physique. By G. Goudet. 2d ed. Paris, Masson, 1954. 4+90 pp. 600 fr.

The first edition appeared in 1943. This little book is a volume in the series Collection d'ouvrages de mathématiques à l'usage des physiciens.

Selected papers in statistics and probability. By A. Wald. Ed. for the Institute of Mathematical Statistics by T. W. Anderson and others. 10+702 pp. $8.00.

This volume collects almost all of Wald's work on statistics and probability except that which is covered by his books. It also contains a discussion of the papers and a bibliography.

VERSITY

NONLINEAR DIFFERENTIAL EQUATIONS

EDMUND PINNEY

1. Introduction.

A few nonlinear differential equations have known exact solutions, but many which are important in applications do not. Sometimes these equations may be linearized by an expansion process in which nonlinear terms are discarded. When nonlinear terms make vital contributions to the solution this cannot be done, but sometimes it is enough to retain a few "small" ones. Then a perturbation theory may be used to obtain the solution. A differential equation may sometimes be approximated by an equation with "small" nonlinearities in more than one way, giving rise to different solutions valid over different ranges of its parameters.

There are two types of small nonlinearity problems. In the first type the nonlinearities occur in the most highly differentiated terms. These are very important in several physical theories. Carrier refers to them as "boundary layer problems" [1; 2] in recognition of the application in which they had their first important development. They include many important nonlinear partial differential equations problems, as well as some ordinary nonlinear differential equations in which such phenomena as relaxation oscillations occur. Boundary layer problems are usually closely tied in with applications. Their theories have not yet received very general or exhaustive development, and much art and ingenuity has been called for in the work that has been done ([1]–[3]).

The second type of nonlinearity problem is that in which nonlinearities do not occur in the most highly differentiated terms. In this case the theory has been developed farther, and something more nearly resembling a general method of attack is possible. Actually several such methods have been developed. Each has its own special merits and limitations. I will discuss one such method. This particular method has the advantage of wide scope and practicality of application, but is limited to a class of differential equations which is associated with nonconservative physical systems.

This method offers nothing new in the case of ordinary nonlinear differential equations of the second order, but has a practical advantage in the case of systems of equations (or, what comes to the same

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thing, vector differential equations of higher than the second order), certain functional equations, such as difference-differential equations, and some partial differential equations. Strictly formal solutions may be obtained by a number of devices analogous to the methods of van der Pol, Poincaré, Kryloff and Bogoliuboff, etc., with fewer manipulations than required by the present method, which is rigorous. This is all right when the necessary existence theorems are well known and fairly easily usable as in the case of second order ordinary differential equations. However in the more complicated cases mentioned, such existence theorems are frequently unknown, and, if known, are usually confined to the case of periodic solutions (ruling out the possibility of two oscillations having incommensurate periods) and are usually very difficult to apply. Thus they typically require calculations equivalent to obtaining the characteristic exponents arising in the application of Floquet theory to systems of linear equations with periodic coefficients. This is a formidable problem even when the original equation is a second order ordinary differential equation, but in that case the theory (Hill theory) is already known.

The present theory gets around this difficulty by basing the proof of the existence of a solution upon the "trend" functions that one would have to calculate anyway in order to obtain an approximation to the solution. Of course this process does not always work. As a general rule it does work in the case of equations arising in nonconservative physical systems. In this method an approximate solution to the nonlinear equation is developed, based on the linear system in which nonlinear terms are neglected. Such a theory cannot in itself settle the question of unboundedness of the solution, for, as the dependent variable increases, the nonlinear terms must ultimately dominate, thus invalidating the base of approximation. Therefore a prediction that the solution tends to infinity merely means that the theory fails. Indeed, the theory may fail for a prediction of large finite values of the solution when certain conditions fail to be satisfied. This will be called *large solution failure.*

The method is applied to van der Pol's equation in §2, and carried through in detail. At the end of the section its principal features are recapitulated. About half of the calculation is devoted to proving the existence of the solution. In §3 the method is applied to the much more elaborate problem of systems of nonlinear equations. Results only are given. In §4 it is applied to a still more elaborate analysis of a partial differential equation occurring in a transmission line problem. For brevity much of the complete analysis is omitted, but enough is given to illustrate the ideas as well as some of the special problems and complexities that may be expected in nonlinear partial differential equations.

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2. **van der Pol's equation.** van der Pol's equation is

$$\frac{d^2y}{dt^2} + \epsilon(y^2 - 1)\frac{dy}{dt} + y = 0, \tag{2.1}$$

for ϵ small and positive. This may be written

$$y''(t) + y(t) = f(t), \tag{2.2}$$

where

$$f(t) = \epsilon[1 - y^2(t)]y'(t). \tag{2.3}$$

Equation (2.2) has a particular solution

$$\int_0^t \sin(t - \tau)f(\tau)d\tau,$$

so by (2.3),

$$\begin{aligned} y(t) = {} & y(0)\cos t + y'(0)\sin t \\ & + \epsilon\int_0^t \sin(t - \tau)[1 - y^2(\tau)]y'(\tau)d\tau. \end{aligned} \tag{2.4}$$

This may be written

$$y(t) = a_+(t)e^{it} + a_-(t)e^{-it}, \tag{2.5}$$

where

$$a_\pm(t) = \frac{1}{2}y(0) \pm \frac{1}{2i}y'(0) \pm \frac{\epsilon}{2i}\int_0^t e^{\mp i\tau}[1 - y^2(\tau)]y'(\tau)d\tau. \tag{2.6}$$

Differentiating,

$$a_\pm'(t) = \pm\frac{\epsilon}{2i}e^{\mp it}[1 - y^2(t)]y'(t). \tag{2.7}$$

Then by (2.5),

$$y'(t) = ia_+(t)e^{it} - ia_-(t)e^{-it}. \tag{2.8}$$

So far this development is exact, but it provides no method to compute the quantities $a_\pm(t)$. Our main problem is to obtain approximate expressions for these quantities. As noted in the introduction, we shall use an approximate procedure which is valid only if the solution is bounded.

Let us provisionally assume that for some quantity K, independent of t,

$$(2.9) \qquad |a_\pm(t)| < K$$

for all $t \geqq 0$. Then we will show below that $a_\pm(t)$ may be approximated by known functions $\rho_\pm(t)$, the error bound $E = E(\epsilon, K)$ being expressed in terms of ϵ and K:

$$(2.10) \qquad |a_\pm(t) - \rho_\pm(t)| < E(\epsilon, K).$$

Now let us not assume that (2.9) holds for all $t \geqq 0$, but that $|a_\pm(0)| < K$. It will appear that the $\rho_\pm(t)$ functions contain arbitrary constants which may be set so that $\rho_\pm(0) = a_\pm(0)$. Then the $\rho_\pm(t)$ are completely known and an upper bound K' may be calculated:

$$|\rho_\pm(t)| \leqq K'$$

for all $t \geqq 0$. Suppose K may be selected so that for ϵ in some neighborhood of $\epsilon = 0$,

$$(2.11) \qquad K' < K, \qquad E(\epsilon, K) < K - K'.$$

Then by (2.10) and the triangle inequality it follows that (2.9) holds for all $t \geqq 0$ for ϵ in this neighborhood. It appears then that the boundedness of the $a_\pm(t)$ is intimately associated with that of $\rho_\pm(t)$. So long as we are able to select a K greater than K' (which may depend on ϵ) for which (2.11) holds, the validity of (2.9) follows for all $t \geqq 0$. When this cannot be done, the theory suffers large solution failure.

It should be emphasized that K is not necessarily a least upper bound to the quantities $|a_\pm(t)|$. Indeed, the present theory will not enable us to calculate this bound. K may depend on ϵ. Without loss in generality we can, and will, for convenience, assume that K does not tend to 0 as $\epsilon \rightarrow 0$, i.e., $1/K = O(1)$. Without specifying K further we make the assumption (2.9), and proceed to calculate the function $E(\epsilon, K)$. With this and K', obtained from $\rho_\pm(t)$, we then seek a K greater than K' satisfying (2.11). Usually there are infinitely many possible values of K. The best is that which renders $E(\epsilon, K)$ smallest.

Now, assuming (2.9), by (2.5), (2.8),

$$|y(t)| < 2K, \qquad |y'(t)| < 2K.$$

Therefore by (2.7), assuming (as we shall hereafter) that $K \geqq 1$,

$$(2.12) \qquad |a'_\pm(t)| < 4\epsilon K^3.$$

By (2.5), (2.7), (2.8),

$$(2.13)\quad a'_\pm(t) = \frac{1}{2}\epsilon\left[(1 - a_+a_-)(a_\pm - a_\mp e^{\mp 2it}) - a_\pm^3 e^{\pm 2it} + a_\mp^3 e^{\mp 4it}\right].$$

Now define

$$(2.14)\quad \bar{a}_\pm(t) = a_\pm(t) \mp \frac{\epsilon}{4i}(1 - a_+a_-)a_\mp e^{\mp 2it} \pm \frac{\epsilon}{4i} a_\pm^3 e^{\pm 2it} \pm \frac{\epsilon}{8i} a_\mp^3 e^{\mp 4it}.$$

Therefore by (2.9),

$$(2.15)\quad \left| \bar{a}_\pm(t) - a_\pm(t) \right| < \frac{7}{8}\epsilon K^3.$$

Differentiating (2.14) and inserting (2.12) and (2.13),

$$\left| \bar{a}'_\pm(t) - \frac{1}{2}\epsilon(1 - a_+a_-)a_\pm \right| < \frac{17}{2}\epsilon^2 K^5.$$

Then by (2.15),

$$(2.16)\quad \bar{a}'_\pm(t) = \frac{1}{2}\epsilon(1 - \bar{a}_+\bar{a}_-)\bar{a}_\pm + R_\pm,$$

where

$$(2.17)\quad \left| R_\pm \right| < \frac{41}{4}\epsilon^2 K^5.$$

We are interested in the degree to which the functions $\bar{a}_\pm(t)$ may be approximated by the solutions $\rho_\pm(t)$ of the equations

$$(2.18)\quad \rho'_\pm(t) = \frac{1}{2}\epsilon(1 - \rho_+\rho_-)\rho_\pm,$$

which are the same as those in (2.16) except for the $R_\pm$ terms. Their solution is

$$(2.19)\quad \rho_\pm(t) = r(t)e^{\pm i\phi}, \qquad r(t) = (1 + Ce^{-\epsilon t})^{-1/2},$$

where C and ϕ are real constants.

The "trend functions" $\rho_\pm(t)$ tend to $e^{\pm i\phi}$ as $t \to \infty$. What about the quantities $\bar{a}_\pm(t)$? To answer this, construct an r, θ, t-cylindrical coordinate system, and plot the surfaces $r = r(t)$, which are surfaces of revolution about the t-axis. This will give a one-parameter family of surfaces filling the space. These surfaces will be called "trend surfaces." As $t \to \infty$, each trend surface approaches the cylinder $r = 1$ asymptotically. The space curves $r = r(t)$, $\theta = \phi$ will be called "trend lines." Now write

$$\bar{a}_{\pm}(t) = A(t)e^{\pm i\theta(t)} \tag{2.20}$$

where $A(t)$ and $\theta(t)$ are real. Then by (2.16),

$$A'(t) = \frac{1}{2}\epsilon(1 - A^2)A + R, \qquad \theta'(t) = I/A, \tag{2.21}$$

where

$$R \pm iI = R_{\pm}e^{\mp i\theta}. \tag{2.22}$$

The space curve $r = A(t)$, $\theta = \theta(t)$ will at every point cut a local trend line from which its direction angles will differ by R/A and I/A.

Specifically, write $\bar{a}_{\pm}(0) = A_0 e^{\pm i\phi_0}$, and let $\rho_{\pm}(t)$ denote the trend function whose trend line passes through $(A_0, \phi_0, 0)$, i.e., $\rho_{\pm}(0) = \bar{a}_{\pm}(0)$, and $r(0) = A_0$, $\phi = \phi_0$. How far away from $\rho_{\pm}(t)$ can $\bar{a}_{\pm}(t)$ get for $t > 0$? In other words, how far can $A(t)$ get from $r(t)$, and $\theta(t)$ from ϕ_0? First, suppose $A(t) > r(t)$. Then $A(t) - r(t)$ grows only if [redacted] $-r'(t) > 0$. By (2.19), (2.21), this implies

$$\frac{1}{2}\epsilon[(1 - A^2)A - (1 - r^2)r] + R > 0.$$

Therefore

$$(A - r)(A^2 + Ar + r^2 - 1) < 2R/\epsilon.$$

Therefore when $A_0 > 1/3$,

$$A - r < 2R/[\epsilon(3A_0^2 - 1)],$$

so by (2.17), (2.22),

$$A(t) - r(t) < (99/4)\epsilon K^5/(3A_0^2 - 1).$$

When $A_0 < 1/3$ the analysis is slightly more complicated because the trend lines themselves are at first diverging, but we again get $A(t) - r(t) = O(\epsilon K^5)$. The same order estimate is obtained when $A(t) < r(t)$. By (2.17), (2.21), (2.22), $\theta(t) = \phi + tO(\epsilon^2 K^5)$,

$$\bar{a}_{\pm}(t) = r(t) \exp(\pm i[\phi + tO(\epsilon^2 K^5)]) + O(\epsilon K^5).$$

From (2.15) we have $a_{\pm}(t) = \rho_{\pm}(t) \exp(\pm itO(\epsilon^2 K^5)) + O(\epsilon K^5)$. In view of (2.19), $|\rho_{\pm}(t)|$ has an upper bound $K' = \max(1, A_0)$. Since $A_0 = O(1)$ we can take $K' = O(1)$. Then (2.11) is satisfied if $K - K' > O(\epsilon K^5)$. This is clearly possible with $K = O(1)$, so that $a_{\pm}(t) = \rho_{\pm}(t) \cdot \exp(\pm itO(\epsilon^2)) + O(\epsilon)$. By (2.5), (2.8), (2.19),

$$\begin{aligned} y(t) &= 2r(t)\cos\left[\{1+O(\epsilon^2)\}t+\phi\right]+O(\epsilon), \\ y'(t) &= -2r(t)\sin\left[\{1+O(\epsilon^2)\}t+\phi\right]+O(\epsilon). \end{aligned} \tag{2.23}$$

The principal features of this method may now be pointed out. First the equation is written in the form (2.2) in which the principal linear part is on the left while the term on the right contains the nonlinear terms. This equation is "solved" in the same manner as the linear nonhomogeneous equation is solved, getting an integral equation (2.4). We write this in a form (2.5) resembling the solution to the linear homogeneous equation ($f=0$ in (2.2)) except that the quantities $a_\pm(t)$ are not now constant. However they have the property that when differentiated, giving (2.7), the integral sign is removed. Substituting in the expressions for y and y' in terms of $a_\pm$, the first order differential equations for $a_\pm(t)$ are given in (2.13). These equations are more complicated than the original equation (2.1), but they provide the basis for making an approximation. The intuitive reason to suspect this virtue in (2.13) is as follows. Since the equation has only a small nonlinearity when ϵ is small, its solution will be almost like a linear solution. Therefore the quantities $a_\pm(t)$, although not actually constant, should vary only slowly with t. Therefore on the right-hand side of (2.13) the terms involving $e^{\mp 2it}$, $e^{\pm 2it}$, and $e^{\mp 4it}$ oscillate in almost the same way as these quantities do themselves and so tend to cancel themselves out over a long t-interval in comparison with the other, nonoscillatory, terms. The introduction of the quantities $\bar{a}_\pm(t)$ and $\rho_\pm(t)$ leads to simplified equations in which the oscillatory terms are absent. The $\rho_\pm(t)$ quantities are approximations to the $a_\pm(t)$ quantities because of their exponential approach to their asymptotic limits. An approximate solution to the differential equation may then be written in terms of them. The accuracy of this approximation involves the analysis of the K and K' quantities and $E(\epsilon, K)$ as indicated in (2.9)–(2.11).

These same ideas may be applied to other and more complicated nonlinear situations. In fact, it is in the more complicated problems that this method has an advantage of practicality which it does not have in this problem.

3. Systems of nonlinear differential equations. The foregoing procedure may be applied to systems of equations. For brevity, results only will be given here. We shall consider a system

$$\frac{dy_i}{dt} = \sum_{\alpha=1}^{m} A_{i\alpha} y_\alpha + \epsilon f_i(y, t), \qquad i = 1, 2, \cdots, m, \tag{3.1}$$

the quantities in which are defined as follows:

(i) the $A_{i\alpha}$ are constants and ϵ is a small parameter;

(ii) $y=y(t)$ is the matrix $(y_1(t), y_2(t), \cdots, y_m(t))$;

(iii) $f_i(y, t)$ is a polynomial of degree n in the elements of y; the terms of degree n have coefficients that are all $O(1)$ in ϵ as $\epsilon \to 0$;

(iv) $f_i(y, t)$ depends on t explicitly only as a polynomial in circular functions of t;

(v) for $b_1, b_2, \cdots, b_m$ arbitrarily preassigned quantities,

$$y_i(0) = b_i, \qquad i = 1, 2, \cdots, m. \tag{3.2}$$

It is an elementary matter to express these equations and conditions more completely in matrix notation, omitting the subscripts and summation sign from (3.1), say. However many of our manipulations lie outside the limited class that can be done with this abbreviated notation, so the present more descriptive but slightly longer hybrid notation seems more efficient.

To discuss the system (3.1), let S denote the set of characteristic roots z_j (of multiplicity μ_j) of the characteristic equation

$$D(z) \equiv \left| A_{ij} - z\delta_{ij} \right| = 0, \tag{3.3}$$

and let $D_{ji}(z)$ be the cofactor of $A_{ij}-z\delta_{ij}$ in the determinant of (3.3). Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be an arbitrarily chosen set of constants not all of which are zero, and let $\nu_1, \nu_2, \cdots, \nu_m$ be another such set. Define the quantities

$$D_r(z) = \sum_{s=1}^{m} \lambda_s D_{rs}(z), \qquad \overline{D}_s(z) = \sum_{r=1}^{m} \nu_r D_{rs}(z),$$

$$\overline{D}(z) = \sum_{r=1}^{m} \nu_r D_r(z) = \sum_{s=1}^{m} \lambda_s \overline{D}_s(z) = \sum_{r=1}^{m} \sum_{s=1}^{m} \lambda_s \nu_r D_{rs}(z), \tag{3.4}$$

$$\overline{D}_{i\alpha}(z) = \sum_{r=1}^{m} \sum_{s=1}^{m} \lambda_s \nu_r D_{is;r\alpha}(z),$$

where $D_{is;r\alpha}(z)$ is the cofactor of $A_{\alpha r}-z\delta_{\alpha r}$ in the determinant $D_{is}(z)$. Define

$$G_i(z) = - D_i(z)/\overline{D}(z), \tag{3.5}$$

$$F(z, y, t) = \sum_{\alpha=1}^{m} \overline{D}_\alpha(z) f_\alpha(y, t). \tag{3.6}$$

The quantities $y_i(t)$ may be expressed in terms of certain quantities $a_{j,s}(t)$ in a manner analogous to (2.5):

$$y_i(t) = \sum_{S} e^{z_j t} \sum_{q=0}^{\mu_j - 1} \frac{\mu_j ! G_i^{(q)}(z_j)}{q! D^{(\mu_j)}(z_j)} a_{j,\mu_j-1-q}(t), \tag{3.7}$$

where the $a_{j,q}(t)$ quantities may be expressed in terms of integrals of $F(z, y, t)$ in a manner analogous to (2.6), and satisfy differential equations analogous to (2.7).

Again, as in §2, we are interested in bounded solutions and introduce a quantity K of the same type as that of §2, except that we have

$$\left| e^{z_j t} a_{jq}(t) \right| < K \quad \text{for} \quad z_j \text{ in } S, \qquad q = 0, 1, \cdots, \mu_j - 1, \tag{3.8}$$

instead of (2.9). As in (2.11) we require that

$$K > K', \qquad E(\epsilon, K) < K - K' \tag{3.9}$$

as $\epsilon \to 0$ to avoid large solution failure.

The quantities $y_i(t)$ in (3.7) are substituted into (3.6) and simplified. We write

$$F(z, y(t), t) = \sum_{S} e^{z_j t} g_j(z, a(t), t) + \sum\nolimits_{S'} e^{z'_j t} G_j(z, a(t)), \tag{3.10}$$

in which the following rules govern the indicated expansion:

(i) the terms included in the S summation must be such that

$$\frac{\partial}{\partial t} g_j(z, a, t) = e^{-z_j t} O(\epsilon K^{2n-1}); \tag{3.11}$$

(ii) the S' summation is over all terms in the expansion of the left-hand side of (3.10) not included in the S summation. However, we shall impose the further restriction that

$$1 / \left| z_j - z'_k \right| = O(1) \tag{3.12}$$

as $\epsilon \to 0$ for all z_j in S and all z'_k in S'.

Condition (i) simply defines the terms which are to be included in the S summation. However condition (3.12) in (ii) implies a new restriction on F and on the characteristic roots. A difficulty might seem to appear in the use of (3.11), because the quantity K is not yet known. In practice this is generally not significant because (3.11) usually implies a restriction upon certain parameters in the equations (3.1) which need not be made precise till later.

In place of equation (2.18), we now get "trend equations,"

$$\frac{d\rho_{jq}(t)}{dt} = \rho_{j,q-1}(t) + \epsilon g_j^{(q)}(z_j, \rho(t), t)/q! \tag{3.13}$$

for z_j in S, $q=0, 1, \cdots, \mu_j-1$, and for $\rho_{j,-1}(t)=0$. Here $g_j^{(q)}(z, a, t)$ denotes $(\partial^q/\partial z^q)g_j(z, a, t)$.

When the equations in (3.13) are solved, the "trend functions" $\rho_{jq}(t)$ can be analysed in a way similar to that used to analyse the trend functions in §2. $|\rho_{jq}(t)|$ may be plotted against t on a jq-trend modulus plane, and arg $[\rho_{jq}(t)]$ may be plotted against t on a "jq-trend argument cylinder," which is a cylindrical surface having t as axial distance and arg $[\rho_{jq}(t)]$ as azimuthal angle.

We shall designate as "principal trend functions" the set of trend functions $\rho_{jq}(t)$ which takes on some prescribed initial conditions. Since the solution of (3.13) involves constants of integration, this is only one set in a family. The behavior of others "near" this set is very important, as it was in §2.

THEOREM 1. *On each jq-trend modulus plane let* $\Delta|\rho_{jq}(t)|$ *denote the difference in ordinate between the principal jq-trend modulus and neighboring trend moduli; on each jq-trend argument cylinder let* Δ arg $[\rho_{jq}(t)]$ *denote the difference in asimuthal angle between the principal jq-trend argument and neighboring trend arguments. Suppose*

$$\Delta|\rho_{jq}(t)| = f_{jq}(t)e^{-(x_j+\kappa)t}, \qquad \Delta \arg [\rho_{jq}(t)] = g_{jq}(t)e^{-\kappa t}, \tag{3.14}$$

where $x_j = \mathrm{Re}\ (z_j)$, $\kappa>0$, *and*

$$f'_{jq}(t)/f_{jq}(t) \leqq 0, \qquad g'_{jq}(t)/g_{jq}(t) \leqq 0. \tag{3.15}$$

Then for $t \geqq 0$, $i=1, 2, \cdots, m$,

$$\begin{aligned} y_i(t) = {} & \sum_{S}{}_j\sum_{q=0}^{\mu_j-1} \frac{\mu_j! G_j^{(q)}(z_j)}{q! D^{(\mu_j)}(z_j)} e^{z_j t}\rho_{j,\, \mu_j-1-q}(t) \\ & + O(\epsilon K^n) + O(\epsilon^2 K^{2n-1}/\kappa). \end{aligned} \tag{3.16}$$

Unless $\kappa>0$ and the conditions in (3.15) are satisfied, the theory fails. When the conditions in (3.15) are not satisfied, sometimes a simple special discussion will insure the success of the theory, as in §2 in the case $A_0<1/3$.

Suppose a quantity K', independent of t, exists such that for z_j in S, $q=0, 1, \cdots, \mu_j-1$,

$$|e^{x_j t}\rho_{jq}(t)| \leqq K', \tag{3.17}$$

and that K', which may depend on ϵ, satisfies

$$\begin{aligned} & K' = o(\epsilon^{-1/(n-1)}), \qquad 1/K' = O(1), \\ & K' = o(\kappa^{1/(2n-1)}/\epsilon^{2/(2n-1)}), \end{aligned} \tag{3.18}$$

as $\epsilon \to 0$. Then (3.9) is satisfied for a value of K such that

$$K = O(K') \tag{3.19}$$

and the error estimate in (3.16), as well as the restriction implied in (3.11), may be made definite.

4. A partial differential equation application. A nonlinear transmission line. Our method may also be applied to partial differential equations problems which are not of the boundary layer type. The general features of the method are the same although there are some added complications of detail.

We shall consider resonance oscillations in a transmission line in which the series resistance, series inductance, and shunt capacitance are constant as usual, but in which the shunt conductance varies with transverse voltage. The mechanism by which this is accomplished is not our direct concern here, but there are such mechanisms, gaseous discharge tubes, for instance. The series resistance and shunt conductance will be assumed to be small.

We shall suppose the transmission line to have a finite length, and to be short circuited at the end. The case when the line is terminated by a nonzero impedance requires a slightly different treatment as will be noted below. The other end of the line will be driven by a harmonically oscillating generator of frequency α near one of the natural frequencies of the linearized system. Under these conditions the current flowing in the line and the voltage across the line may be expected to be rather large, i.e., a "resonant" condition prevails. If V is the transverse voltage across the line, x is the distance along the line, and t is time,

$$\begin{aligned} V &= V_0 \cos(\alpha t) &&\text{when} \quad x = 0, \\ V &= 0 &&\text{when} \quad x = l. \end{aligned} \tag{4.1}$$

In the range $0<x<l$, but not too near $x=0$ or $x=l$, V may be expected to be large compared to V_0.

Let L be the series inductance; R, the series resistance; C, the shunt capacitance; and G, the shunt conductance, all per unit length, and uniformly distributed. Let I be the current in the line. L, R, and C are constant, but G is a function of V. Then, by [4, p. 550],

$$\frac{\partial V}{\partial x} + L\frac{\partial I}{\partial t} + RI = 0, \quad \frac{\partial I}{\partial x} + C\frac{\partial V}{\partial t} + GV = 0. \tag{4.2}$$

It will be convenient to introduce dimensionless variables

$$X = \pi x/l, \qquad T = \pi ct/l \tag{4.3}$$

where

$$c = 1/(LC)^{1/2}. \tag{4.4}$$

Substituting these into (4.2) and eliminating I,

$$V_{XX} - V_{TT} = F(\epsilon V, \epsilon V_T) = u(X, T), \tag{4.5}$$

where ϵ is a conveniently chosen small quantity such that

$$F(\epsilon V, \epsilon V_T) = (\pi^2/l^2)\left\{\left[L\frac{d}{dV}(GV) + RC\right](\pi c/l)V_T + RGV\right\}. \tag{4.6}$$

F will be assumed to be a polynomial of degree N in its arguments.

Our problem is to solve equation (4.5) subject to the boundary conditions (4.1). Although the main ideas are the same as in §2, the manipulative machinery is substantially heavier, and for brevity much of it must be suppressed in this paper.

For m an integer define δ by

$$\alpha = m(1 + \delta)\pi c/l. \tag{4.7}$$

Then $|\delta| \ll 1$ implies resonance. The boundary conditions (4.1) may be written

$$\begin{aligned} V &= V_0 \cos m(1 + \delta)T \quad \text{when} \quad X = 0, \\ V &= 0 \quad\quad\quad\quad\quad\quad\quad \text{when} \quad X = \pi. \end{aligned} \tag{4.8}$$

Then a solution to (4.5), satisfying (4.8) is

$$\begin{aligned} V = V(X, T) &= (1 - X/\pi)\, V_0 \cos m(1 + \delta)T \\ &\quad - \frac{1}{2}\int_0^T \int_{X-T+\tau}^{X+T-\tau} U(\xi, \tau)\, d\xi d\tau, \end{aligned} \tag{4.9}$$

where

$$U(X, T) = u(X, T) - m^2(1 + \delta)^2(1 - X/\pi)V_0 \cos m(1 + \delta)T. \tag{4.10}$$

(4.9) is analogous to (2.4).

At this point it may be worth interpolating that had our transmission line been terminated by a nonzero impedance Z rather than short-circuited, instead of (4.9) we would have taken

$$\begin{aligned} V = V(X, T) &= V_0 e^{-XlR/(\pi Z)} \cos m(1 + \delta)(T - XcL/Z) \\ &\quad - \frac{1}{2}\int_0^T \int_{X-T+\tau}^{X+T-\tau} U(\xi, \tau)\, d\xi d\tau. \end{aligned} \tag{4.11}$$

The subsequent treatment of equation (4.11) would be closely analogous to that of equation (4.9).

To resume, (4.9) will satisfy the boundary conditions (4.8) provided the range of definition of the function $U(X, T)$ is extended beyond $0<X<\pi$ in such a way that it is odd in X and periodic of period 2π in X. It can be shown to be of bounded variation in X and therefore expandable in a Fourier sine series:

$$U(X, T) = i \sum_{n=-\infty}^{\infty} U_n(T)e^{inX}, \tag{4.12}$$

where

$$U_0(T)=0,$$

$$\begin{aligned} U_n(T) = & -(1/\pi)\int_0^\pi F(\epsilon V(\xi, T), \epsilon V_T(\xi, T)) \sin (n\xi)d\xi \\ & + (m^2/n\pi)(1+\delta)^2 V_0 \cos m(1+\delta)T \end{aligned} \tag{4.13}$$

for $n\neq 0$.

Define

$$a_n(s) = \int_0^s U_n(\tau)e^{-in\tau}d\tau, \tag{4.14}$$

so that

$$a_n'(s) = U_n(s)e^{-ins}, \tag{4.15}$$

and define

$$\begin{aligned} W(X, a(s)) = & \frac{1}{2}\int_0^s \left\{\int_0^{X-\tau} U(\xi, \tau)d\xi \right. \\ & \left. - \int_0^\pi U(\xi, \tau)(1-\xi/\pi)d\xi\right\} d\tau \\ = & \sum_{n=-\infty}^{\infty} (1/2n)a_n(s)e^{inX}. \end{aligned} \tag{4.16}$$

Then it may be shown that

$$\begin{aligned} V(X, T) = & (1-X/\pi)V_0 \cos m(1+\delta)T \\ & + W(T-X, a(T)) - W(T+X, a(T)). \end{aligned} \tag{4.17}$$

Returning to the problem in §2, equations (4.16) and (4.17) are analogous to (2.5). (4.14) is analogous to (2.6), and (4.15) is analogous to (2.7).

Again, as in §§2 and 3, we are interested in bounded solutions, and introduce a similar quantity K, except that in place of (2.9) we have, for $n \neq 0$,

$$|a_n(t)| < K/|n|. \tag{4.18}$$

As in (2.11), if large solution failure is to be avoided,

$$K > K', \qquad E(\epsilon, K) < K' - K. \tag{4.19}$$

In place of (2.18), we now get "trend equations,"

$$\rho_n'(T) = g_{n,n}(\rho(T), \delta T) \tag{4.20}$$

for $n \neq 0$, where

$$g_{n,k}(a(s), \sigma) = (1/2\pi) \int_{-\pi}^{\pi} U_n(a(s), \sigma, \tau) e^{-ik\tau} d\tau, \tag{4.21}$$

where

$$\begin{aligned} U_n(a(s), \sigma, T) = (m^2/n\pi)&(1 + \delta)^2 V_0 \cos m(T + \sigma) \\ &- (1/\pi) \int_0^{\pi} F(\epsilon[W(T - \xi, a(s)) - W(T + \xi, a(s))], \\ &\epsilon[W_1(T - \xi, a(s)) - W_1(T + \xi, a(s))]) \sin (n\xi) d\xi. \end{aligned} \tag{4.22}$$

The conditions that the "trend functions" $\rho_n(T)$ must satisfy in order to approximate the functions $a_n(T)$ are precisely those given in (3.14) and (3.15) with $x_j = 0$ and with the double jq subscripts replaced by n. If one solves (4.20) and verifies these conditions, then the solution is obtained by substituting

$$\begin{aligned} W(X, a(T)) = W(X, \rho(T)) &+ O(\delta/\kappa) + O(1) \\ &+ O(\epsilon^N K^N) + (\epsilon/\kappa)[O(1) + O(\epsilon^{2N-1} K^{2N-1})]. \end{aligned} \tag{4.23}$$

Now suppose a quantity K', independent of t, exists such that for $n \neq 0$,

$$|\rho_n(T)| \leqq K'/|n|, \tag{4.24}$$

and that K', which may depend on ϵ, satisfies

$$\begin{aligned} &K' = o(\epsilon^{-N/(N-1)}), \qquad &1/K' = O(1), \\ &\epsilon/\kappa = o(K'), \qquad &K' = o(\kappa^{1/[2(N-1)]} \epsilon^{-N/(N-1)}), \end{aligned} \tag{4.25}$$

as $\epsilon \to 0$. Then (4.19) is satisfied if

$$K = O(K') \tag{4.26}$$

and the error estimate in (4.23) may be made definite.

Since (4.20) represents an infinite system of differential equations, its solution may not be simple. One method of attack is to study a generating function

$$R(X, T) = \epsilon W(X - \delta T, \rho(T)) = \epsilon \sum_{n=-\infty}^{\infty} (1/2\pi)\rho_n(T)e^{in(X-\delta T)}. \tag{4.27}$$

This may be shown to satisfy the equation

$$\begin{aligned} &R_{XT}(X, T) + \delta R_{XX}(X, T) \\ &\qquad + (\epsilon/8\pi)\int_{-\pi}^{\pi}\{F(R(X, T) - R(\xi, T), R_X(X, T) - R_\xi(\xi, T)) \\ &\qquad\qquad - F(R(\xi, T) - R(X, T), R_\xi(\xi, T) - R_X(X, T))\}d\xi \\ &\qquad + \epsilon(m/2\pi)(1 + \delta)^2V_0 \sin (mX) = 0. \end{aligned} \tag{4.28}$$

Since F is a polynomial the integrand can be expanded. The integral is seen to be a polynomial in $R(X, T)$ and $R_X(X, T)$ whose coefficients are averages of polynomials of these two quantities taken over the range $-\pi < X < \pi$.

By (4.27), $R(X, T)$ must be periodic of period 2π in X. It is interesting to note that only odd terms in the expansion of F contribute to the integral in (4.28).

It may appear that we are replacing the partial differential equation (4.5) by a worse equation in (4.28). However (4.28) has some real advantages. In a problem of this type we are mainly interested in the "steady state" behavior of the system for t large. The physical nature of the problem strongly suggests that the quantities $\rho_n(T)e^{-in\delta T}$ approach constants asymptotically as $T \to \infty$, so that, by (4.27), the steady state behavior of the system is determined by $R(X, \infty) = R(X)$, say. By (4.28), $R(X)$ satisfies an ordinary differential equation of the second order whose coefficients contain mean values of polynomials in $R(X)$ and $R'(X)$. If the small nonlinear part of the shunt conductance is comparable in magnitude to the small linear part, then the equation for $R(X)$ is strongly nonlinear and its numerical computation is necessary. In this case the transmission line may be rather strongly excited, not only in the mode nearest the exciting frequency, but in all others as well. On the other hand, when the nonlinear part of the shunt conductance is small compared to the linear part, the equation for $R(X)$ is weakly nonlinear, and may be solved analytically by the usual perturbation methods. In this case, only the mode nearest the exciting frequency is excited strongly.

The validity of the solution corresponding to $R(X)$ can be established if conditions (3.14) and (3.15) are satisfied. To verify these we must know the asymptotic behavior of $\rho_n(T)$ as $T \to \infty$. This can be found from a perturbation analysis of the $R(X)$ solution. That is, we substitute

$$R(X, T) = R(X) + \eta(X, T) \tag{4.29}$$

into (4.28) and discard terms of higher powers than the first in η. This may turn out to be quite a difficult analysis comparable to, or worse than the analysis of the Hill equation.

Since our object is to illustrate a general method and to give no more than a glimpse of the special problems that are peculiar to special applications of this method, it does not seem appropriate to go further into the analysis of the function $R(X, T)$, since this is a matter of importance only to those who may be interested in this specific physical problem. This latter group may be satisfied with less than a complete analysis, for instance, with a determination of $R(X)$ only, the validity of the resulting solution being established by physical or experimental means.

REFERENCES

1. G. F. Carrier, *Boundary layer problems in applied mechanics*, "Advances in Applied Mechanics," vol. III, Academic Press, 1953.

2. ———, *Boundary layer problems in applied mathematics*, Communications on Pure and Applied Mathematics vol. 7 (1954) pp. 11–17.

3. M. J. Lighthill, *A technique for rendering approximate solutions to physical problems uniformly valid*, Philosophic Magazine vol. 7, XL (1949) p. 1179.

4. J. A. Stratton, *Electromagnetic theory*, McGraw-Hill, 1941.

UNIVERSITY OF CALIFORNIA, BERKELEY

ON THE CHARACTERS OF A SEMISIMPLE LIE GROUP

HARISH-CHANDRA

Let G be a connected semisimple Lie group and let Z denote its center. If π is a representation [2c] of G on a Hilbert space $\mathfrak{H}$ we consider the space V consisting of all finite linear combinations of elements of the form

$$\int f(x)\pi(x)\psi dx \qquad (f\in C_c^\infty(G),\ \psi\in\mathfrak{H}),$$

where dx is the Haar measure of G and $C_c^\infty(G)$ is the set of all (complex-valued) functions on G which are everywhere indefinitely differentiable and which vanish outside a compact set. V is called the Gårding subspace of $\mathfrak{H}$. Let R and C be the fields of real and complex numbers respectively and $\mathfrak{g}_0$ the Lie algebra of G. We complexify $\mathfrak{g}_0$ to $\mathfrak{g}$ and denote by $\mathfrak{B}$ the universal enveloping algebra of $\mathfrak{g}$ [2a]. Then there exists a (uniquely determined) representation π_V of $\mathfrak{B}$ on V such that $\pi_V(X)\psi=\lim_{t\to 0}(1/t)\{\pi(\exp tX)\psi-\psi\}$ $(X\in\mathfrak{g}_0,\ \psi\in V,\ t\in R)$. Let $\mathfrak{Z}$ denote the center of $\mathfrak{B}$. We say that π is quasi-simple if there exist homomorphisms η and χ of Z and $\mathfrak{Z}$ respectively into C such that $\pi(\zeta)\phi=\eta(\zeta)\phi$, $\pi_V(z)\psi=\chi(z)\psi$ for all $\zeta\in Z$, $z\in\mathfrak{Z}$, $\phi\in\mathfrak{H}$ and $\psi\in V$. η is then called the central character and χ the infinitesimal character of π. An irreducible unitary representation is automatically quasi-simple [5].

Let A be a bounded linear operator on $\mathfrak{H}$. We say that A is of the trace class or A has a trace if for every complete orthonormal set $(\psi_j)_{j\in J}$ in $\mathfrak{H}$ the series[1] $\sum_{j\in J}(\psi_j, A\psi_j)$ converges absolutely and its sum is independent of the choice of the complete orthonormal set.[2] We call this sum the trace of A and denote it by $\operatorname{Sp} A$. Now suppose π is quasi-simple and irreducible. Then it can be shown (see [2e]) that for any $f\in C_c^\infty(G)$ the operator $\int f(x)\pi(x)dx$ is of the trace class. If we denote its trace by $T_\pi(f)$ we get a linear function T_π on $C_c^\infty(G)$ which is actually a distribution (see [4; and 2e]). We call this distribution the character of π. Our object is to try to determine T_π.

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[1] As usual (ϕ, ψ) denotes the scalar product of the two elements ϕ and ψ in $\mathfrak{H}$.

[2] Actually it can be shown that this independence of the sum follows automatically from the absolute convergence of the series for every orthonormal base.

Let $x \rightarrow \mathrm{Ad}\,(x)$ $(x \in G)$ denote the adjoint representation of G. If λ is an indeterminate and I is the identity mapping of $\mathfrak{g}_0$ we consider the characteristic polynomial $P_x(\lambda) = \det\,(\lambda I - \mathrm{Ad}\,(x))$ of $\mathrm{Ad}\,(x)$. Let l be the highest integer such that $(\lambda-1)^l$ divides $P_x(\lambda)$ for every $x \in G$. We expand $P_x(\lambda)$ in powers of $(\lambda-1)$ and denote by $D(x)$ the coefficient of $(\lambda-1)^l$ in this expansion. Then it is clear that D is an analytic function on G which, in view of our definition of l, cannot be identically zero. The integer l is called the rank of G. Let S denote the set of all $x \in G$ for which $D(x)=0$. Then S is a closed, nowhere dense subset of G and its complement G' is open and everywhere dense in G. An element $x \in G$ is called singular or regular according as $x \in S$ or $x \in G'$. Since $\mathrm{Ad}\,(zx) = \mathrm{Ad}\,(x)$ $(z \in Z)$ it is obvious that $ZS = S$ and $ZG' = G'$. Also since $P_{yxy^{-1}}(\lambda) = P_x(\lambda)$ $(x, y \in G)$, it follows that $D(yxy^{-1}) = D(x)$ and therefore $ySy^{-1} = S$, $yG'y^{-1} = G'$ $(y \in G)$.

When speaking of a real differentiable (or analytic) manifold M let us agree to include the case when M is not connected but the various connected components of M, which are all manifolds in the usual sense, have the same dimension. Under this definition every open subset U of M is again a manifold. Let $C_c^\infty(U)$ denote the set of all complex-valued functions on M which are indefinitely differentiable and which vanish outside some compact subset of U. In particular suppose U is an open subset of G and F is a (complex-valued) function on U. We say that F is locally summable if it is summable on every compact subset of U with respect to the Haar measure of G. T being a distribution on G we say that $T=F$ on U if F is locally summable and

$$T(f) = \int f(x)F(x)dx$$

for all $f \in C_c^\infty(U)$. Our main result may now be stated as follows.

THEOREM 1. *Let π be an irreducible quasi-simple representation of G on $\mathfrak{H}$ and let T_π denote its character. Then there exists an analytic function F_π on G' such that $T_\pi = F_\pi$ on G'.*

Although in general G' is not connected, there always exist a finite number of connected components $G_1, \cdots, G_r$ of G' such that $ZG_i \cap ZG_j = \emptyset$ if $i \neq j$ and $G' = \bigcup_{i=1}^r ZG_i$. Moreover if η_π is the central character of π it is easy to show that $F_\pi(zx) = \eta_\pi(z) F_\pi(x)$ $(z \in Z, x \in G')$. Hence the knowledge of F_π on $G_1 \cup G_2 \cup \cdots \cup G_r$ is sufficient to determine it completely. On the other hand it is possible to give examples in which F_π vanishes everywhere on one of the components G_i without being zero identically on G'. However in case G is either a compact or a complex group, G' is connected and then F_π is completely determined by its restriction on any nonempty open subset of G'.

For any $x \in G$ and $f \in C_c^\infty(G)$ we define the function f^x by $f^x(y) = f(x^{-1}yx)$ $(y \in G)$. Then $f^x \in C_c^\infty(G)$ and $T_\pi(f^x) = T_\pi(f)$ (see [2e]). From this it follows that $F_\pi(xyx^{-1}) = F_\pi(y)$ $(x \in G, y \in G')$. Now let A be a maximal abelian subgroup of G which is not contained in S. (We call such a group a Cartan subgroup of G.) Obviously A is closed in G. Let $A' = A \cap G'$ and $V = \bigcup_{x \in G} xA'x^{-1}$. Then V is an open subset of G' and it is clear from the above remarks that F_π is determined completely on V as soon as we know it on A'. Let $\mathfrak{h}_0$ denote the subalgebra of $\mathfrak{g}_0$ corresponding to A. Then the complexification $\mathfrak{h}$ of $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}$. Let W be the Weyl group (see [2b]) of $\mathfrak{g}$ with respect to $\mathfrak{h}$ so that W is a finite group of nonsingular linear transformations of $\mathfrak{h}$. If Λ is a linear function on $\mathfrak{h}$ and $s \in W$ we define the linear function $s\Lambda$ by the rule $s\Lambda(H) = \Lambda(s^{-1}H)$ $(H \in \mathfrak{h})$. Let $(H_1, \cdots, H_l)$ be a base for $\mathfrak{h}$ over C. A function P on $\mathfrak{h}$ is called a polynomial function if there exists a polynomial $p(x_1, \cdots, x_l)$ in l variables $(x_1, \cdots, x_l)$ with coefficients in C such that $P(H) = p(a_1, \cdots, a_l)$ if $H = a_1H_1 + \cdots + a_lH_l (a_i \in C)$. The degree of p is also called the degree of P. Clearly these definitions are independent of the choice of the base $(H_1, \cdots, H_l)$.

THEOREM 2. *Let π and F_π be as in Theorem* 1. *Then there exists a linear function Λ on $\mathfrak{h}$ with the following property. For any $a \in A'$ we can choose polynomial functions p_s $(s \in W)$ on $\mathfrak{h}$ such that*

$$F_\pi(a \exp H) = |D(a \exp H)|^{-1/2} \sum_{s \in W} p_s(H) e^{s\Lambda(H)}$$

for all H lying sufficiently near zero in $\mathfrak{h}_0$. Λ is unique up to an operation of W and if N is the number of elements σ in W such that $\Lambda = \sigma\Lambda$, the degree of every p_s is necessarily smaller than N.

In particular if $s\Lambda \neq \Lambda$ except when s is the unit element of W, p_s must all be constants. (It is hardly necessary to point out the resemblance of the above formula to the one given by Weyl [6] for the irreducible characters of a compact semisimple Lie group. It should also be compared with the results of Gelfand and Naimark [7] on the unitary characters of the complex classical groups (see also [2f, p. 511])). Although A' is not necessarily connected, it is possible to select a finite set $B_1, \cdots, B_r$ of its connected components such that $A' = \bigcup_{i=1}^r ZB_i$. Therefore in order to determine F_π on A', it is sufficient to know η_π and the restrictions of F_π on some nonempty open subsets of $B_1, \cdots, B_r$. Hence if Λ is known, Theorem 2 gives us a formula for F_π on A' in terms of a finite number of undetermined constants. On the other hand we shall see presently that Λ is completely determined (up to an operation of W) by the infinitesimal character χ_π of π.

Two Cartan subgroups A_1 and A_2 are said to be conjugate if $A_2 = xA_1x^{-1}$ for some $x \in G$. It is always possible to choose a finite number of distinct Cartan subgroups $A_1, \cdots, A_k$ such that every Cartan subgroup is conjugate to exactly one of these. Then if $A_i' = A_i \cap G'$ and $V_i = \bigcup_{x \in G} xA_i'x^{-1}$, G' is the disjoint union of $V_1, \cdots, V_k$. This shows that if η_π and χ_π are known, F_π is completely determined in terms of a finite number of constants.[3]

Now we come to a brief outline of the proof. Let M be a differentiable manifold, Q a linear mapping of $C_c^\infty(M)$ into itself and x_0 a point in M. We say that Q is a differential operator at x_0 if there exists a coordinate system $(t_1, \cdots, t_m)$ valid on an open neighborhood U of x_0 and indefinitely differentiable functions $g_{i_1 i_2 \cdots i_p}$ on U $(1 \leq i_1, \cdots, i_p \leq m,\ 0 \leq p \leq q)$ such that if $f \in C_c^\infty(U)$, Qf is zero outside U and

$$Qf = \sum_{0 \leq p \leq q} \sum_{1 \leq i_1, \cdots, i_p \leq m} g_{i_1 i_2 \cdots i_p} \frac{\partial^p}{\partial t_{i_1} \cdots \partial t_{i_p}} f$$

on U. If Q is a differential operator at every point in M we say simply that it is a differential operator (on M). T being a distribution on M and Q a differential operator we define a distribution $Q'T$ as follows:

$$(Q'T)(f) = T(Qf) \qquad (f \in C_c^\infty(M)).$$

In particular if g is an indefinitely differentiable function on M it defines a differential operator $Q: f \to gf$ $(f \in C_c^\infty(M))$. In this case we write gT to denote $Q'T$ so that $(gT)(f) = T(gf)$. It is clear that the product of two differential operators is again a differential operator and therefore the differential operators form an algebra.

Coming back to G, we note that every $X \in \mathfrak{g}_0$ may be regarded as a differential operator on G as follows:

$$(Xf)(x) = \left\{ \frac{d}{dt} f(x \exp tX) \right\}_{t=0} \qquad (f \in C_c^\infty(G),\ x \in G,\ t \in R).$$

Thus it is easy to see that $\mathfrak{B}$ may be identified in a natural way with a subalgebra of the algebra of differential operators on G. Then for any $b \in \mathfrak{B}$ we have a linear transformation b' of the space of distributions on G. Since $(b_1b_2)'=b_2b_1$ $(b_1, b_2\in\mathfrak{B})$ the mapping $b\to b'$ is an anti-representation of $\mathfrak{B}$. Let ϕ denote the anti-automorphism of $\mathfrak{B}$ over C which is uniquely determined by the condition that $\phi(X)=-X$ $(X\in\mathfrak{g})$. Then $b\to(\phi(b))'$ $(b\in\mathfrak{B})$ is a representation of $\mathfrak{B}$. T being any distribution on G we now define $bT=(\phi(b))'T$ $(b\in\mathfrak{B})$. Then $(bT)(f)=T(\phi(b)f)$ $(f\in C_c^\infty(G))$. If χ_π is the infinitesimal character of π and $z\in\mathfrak{Z}$, $f\in C_c^\infty(G)$, it is easy to see that

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[3] Actually it is possible to improve Theorems 1 and 2 and show that T_π coincides with an analytic function on an open subset of G which, in general, is larger than G' and therefore has fewer connected components.

$$\int(\phi(z)f)(x)\pi(x)\psi dx=\pi_0(z)\left(\int f(x)\pi(x)\psi dx\right)$$
$$=\chi_\pi(z)\int f(x)\pi(x)\psi dx$$

for all $\psi\in\mathfrak{H}$. (Here π_0 is the representation of $\mathfrak{B}$ on the Gårding subspace of $\mathfrak{H}$.) From this it follows that $zT_\pi=\chi_\pi(z)T_\pi$ for all $z\in\mathfrak{Z}$. Hence T_π is an eigen-distribution for each differential operator in $\mathfrak{Z}$.

On the other hand let A be a Cartan subgroup of G. Put $A'=A\cap G'$ and $V=\bigcup_{x\in G} xA'x^{-1}$ as before. We regard A' and V as open submanifolds of A and G respectively. Let G^* be the factor space G/A consisting of cosets of the form xA $(x\in G)$. If $h\in A$ and $x^*\in G^*$ we define $h^{x^*}=xhx^{-1}$ where x is any element in the coset x^*. Let dh and dx^* respectively denote the Haar measure on A and the invariant measure on G^*. Then we have the following lemma.

LEMMA 1. *There exists a distribution τ_π on A' with the following property. If $f\in C_c^\infty(V)$, $T_\pi(f)=\tau_\pi(g)$ where g is the function in $C_c^\infty(A')$ given by*

$$g(h)=|D(h)|\int_{G^*}f(h^{x^*})dx^* \qquad (h\in A').$$

Let $\mathfrak{h}_0$ be the Lie algebra of A. Any element $H\in\mathfrak{h}_0$ may be regarded as a differential operator on A so that if $g\in C_c^\infty(A)$,

$$(Hg)(h)=\left\{\frac{d}{dt}g(h\exp tH)\right\}_{t=0} \qquad (h\in A, t\in R).$$

Let $\mathfrak{h}$ be the subspace of $\mathfrak{g}$ spanned by $\mathfrak{h}_0$ over C and $\mathfrak{U}$ the subalgebra of $\mathfrak{B}$ generated by $(1, \mathfrak{h})$. Then $\mathfrak{U}$ may be identified in a natural way with a subalgebra of the algebra of differential operators on A. For any distribution τ on A' and $u\in\mathfrak{U}$ we define a distribution $u\tau$ on A' as follows:

$$(u\tau)(g)=\tau(\phi(u)g) \qquad (g\in C_c^\infty(A')).$$

Here ϕ is the automorphism of $\mathfrak{U}$ given by $\phi(H)=-H$ $(H\in\mathfrak{h})$.

For every root α of $\mathfrak{g}$ (with respect to $\mathfrak{h}$), choose an element $X_\alpha \neq 0$ in $\mathfrak{g}$ such that $[H, X_\alpha] = \alpha(H)X_\alpha$ for all $H \in \mathfrak{h}$. We introduce some lexicographic order (see [2b]) in the set of all roots and denote by P the set of positive roots under this order. Put $\mathfrak{n} = \sum_{\alpha \in P} CX_\alpha$. Then for every $z \in \mathfrak{Z}$ there exists a unique element $\gamma'(z) \in \mathfrak{U}$ such that $z - \gamma'(z) \in \mathfrak{B}\mathfrak{n}$ (see [2b, p. 72]). If $2\rho = \sum_{\alpha \in P} \alpha$ there exists a unique automorphism λ of $\mathfrak{U}$ such that $\lambda(1) = 1$ and $\lambda(H) = H - \rho(H)$ $(H \in \mathfrak{h})$. We put $\gamma(z) = \lambda(\gamma'(z))$ $(z \in \mathfrak{Z})$. Let W be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$. It is clear that every $s \in W$ defines an automorphism $u \rightarrow u^s$ of $\mathfrak{U}$ such that $1^s = 1$ and $H^s = sH$ $(H \in \mathfrak{h})$. An element $u \in \mathfrak{U}$ is called an invariant if $u = u^s$ for all $s \in W$. Let J be the subalgebra of $\mathfrak{U}$ consisting of all invariants. Then (see [2b, Lemma 38]) the mapping $z \rightarrow \gamma(z)$ $(z \in \mathfrak{Z})$ defines an isomorphism of $\mathfrak{Z}$ onto J. Now let Λ be a linear function on $\mathfrak{h}$. We can extend it uniquely to a homomorphism of $\mathfrak{U}$ into C which takes the value 1 at 1. We agree to denote this extension also by Λ. Then as shown in [2b, Theorem 5] we can choose Λ in such a way that $\chi_\pi(z) = \Lambda(\gamma(z))$ for all $z \in \mathfrak{Z}$. Λ is determined up to an operation of W by this condition.

Let $\Delta(h) = |D(h)|^{1/2}$ $(h \in A')$. Then Δ is an analytic function on A' and therefore $\sigma_\pi = \Delta\tau_\pi$ is a well-defined distribution on A'. Now if we transform the differential equation $zT_\pi = \chi_\pi(z)T_\pi$ $(z \in \mathfrak{Z})$ for T_π into a differential equation for σ_π we get the following result which is one of the main steps of our argument.

LEMMA 2. *τ_π being as in Lemma 1 put $\sigma_\pi = \Delta\tau_\pi$. Then*

$$\gamma(z)\sigma_\pi = \chi_\pi(z)\sigma_\pi$$

for every $z \in \mathfrak{Z}$.

Now choose a linear function Λ on $\mathfrak{h}$ such that $\chi_\pi(z) = \Lambda(\gamma(z))$ for $z \in \mathfrak{Z}$. Let ζ be an indeterminate and u an element in $\mathfrak{U}$. Since $\mathfrak{U}$ is abelian we can consider the polynomial

$$\prod_{s \in W} (\zeta - u^s).$$

It is clear that every coefficient of this polynomial lies in J and therefore if w is the order of W, there exist uniquely determined elements $z_1(u), \cdots, z_w(u)$ in $\mathfrak{Z}$ such that

$$\prod_{s \in W}(\zeta - u^s) = \zeta^w + \gamma(z_1(u))\zeta^{w-1} + \gamma(z_2(u))\zeta^{w-2} + \cdots + \gamma(z_w(u)).$$

On replacing ζ by u we immediately get the identity

$$u^w + u^{w-1}\gamma(z_1(u)) + u^{w-2}\gamma(z_2(u)) + \cdots + \gamma(z_w(u)) = 0$$

in $\mathfrak{U}$ if we recall that $\mathfrak{U}$ is abelian. Now apply the left side to σ_π and use Lemma 2. Then

$$u^{w}\sigma_{\pi} + \chi_{\pi}(z_1(u))u^{w-1}\sigma_{\pi} + \cdots + \chi_{\pi}(z_w(u))\sigma_{\pi} = 0.$$

But $\chi_{\pi}(z_j(u)) = \Lambda(\gamma(z_j(u)))$, $1 \leqq j \leqq w$, and since Λ is a homomorphism of $\mathfrak{U}$ into C it is obvious that

$$\prod_{s\in W} (\zeta - \Lambda(u^s)) = \zeta^{w} + \Lambda(\gamma(z_1(u)))\zeta^{w-1} + \cdots + \Lambda(\gamma(z_w(u))).$$

Therefore the above differential equation for σ_{π} may be written in the form

$$\prod_{s\in W} (u - \Lambda(u^s))\sigma_{\pi} = 0.$$

However if $H_1, \cdots, H_l$ is a base for $\mathfrak{h}_0$ over R and $\square = H_1^2 + \cdots + H_l^2$ it is obvious that the differential equation

$$\prod_{s\in W} (\square - \Lambda(\square^s))\sigma_{\pi} = 0$$

is of the *elliptic* type (see Gårding [1]). Hence it follows from the work of Schwartz [4, p. 137] and John [3a, 3b] that σ_{π} must be an analytic function on A'. Now if we take into account Theorem 5 of [2b] and the fact that $\prod_{s\in W} (u - \Lambda(u^s))\sigma_{\pi} = 0$ for every $u \in \mathfrak{U}$, we get Theorem 2 without difficulty.

π being any quasi-simple irreducible representation we denote by T_{π}, η_{π}, and χ_{π} respectively the character, the central character, and the infinitesimal character of π. Also we denote by F_{π} the analytic function on G' such that $T_{\pi} = F_{\pi}$ on G'. Let T be a distribution on G. Since D is an analytic function on G the product $D^m T$ ($m \geqq 0$) is a well-defined distribution. It is then possible to prove the following result.

LEMMA 3. *There exists an integer* $m \geqq 0$ *with the following property. Suppose* $\pi_1, \cdots, \pi_k$ *is a finite set of quasi-simple irreducible representations and* $c_1F_{\pi_1} + \cdots + c_kF_{\pi_k} = 0$ ($c_i \in C$). *Then if* $T = c_1T_{\pi_1} + \cdots + c_kT_{\pi_k}$, $D^m T = 0$.

From this lemma one can deduce the following theorem.

THEOREM 3. *Let* π_0 *be a quasi-simple irreducible representation of* G *such that* F_{π_0} *is not identically zero. Then, apart from infinitesimal equivalence* [2d, p. 230], *there exist only a finite number of quasi-simple irreducible representations* π *such that* $F_{\pi} = F_{\pi_0}$.

Let $\eta \neq 0$ and $\chi \neq 0$ respectively be given homomorphisms of Z and $\mathfrak{Z}$ into C. Let ω denote the set of all quasi-simple irreducible representations π of G such that $\eta_\pi = \eta$ and $\chi_\pi = \chi$. As we have seen, it follows from Theorems 1 and 2 that the functions F_π ($\pi \in \omega$) span a finite-dimensional vector space over C. Hence if ω is not empty, we can choose a finite set of elements $\pi_1, \cdots, \pi_k$ in ω such that $F_{\pi_1}, \cdots, F_{\pi_k}$ form a base for this vector space. Then if π is any representation in ω, $F_\pi = c_1F_{\pi_1} + \cdots + c_kF_{\pi_k}$ ($c_j \in C$). If one could conclude from this equation that $T_\pi = c_1T_{\pi_1} + \cdots + c_kT_{\pi_k}$ it would follow (see [2e, Theorem 6]) that π is infinitesimally equivalent to some π_j ($1 \leq j \leq k$) and therefore, apart from infinitesimal equivalence, ω has only a finite number of representations. Therefore it is important to consider the following question.

Let $\pi_1, \cdots, \pi_k$ *be a finite set of quasi-simple irreducible representations of G such that* $c_1F_{\pi_1} + \cdots + c_kF_{\pi_k} = 0$ ($c_i \in C$). *Then is it always true that* $c_1T_{\pi_1} + \cdots + c_kT_{\pi_k} = 0$?

I believe the answer is yes but do not know how to prove it.

REFERENCES

1. L. Gårding, Math. Scand. vol. 1 (1953) pp. 55–72.
2a. Harish-Chandra, Ann. of Math. vol. 50 (1949) pp. 900–915.
2b. ———, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 28–96.
2c. ———, Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951) pp. 366–369.
2d. ———, Trans. Amer. Math. Soc. vol. 75 (1953) pp. 185–243.
2e. ———, Trans. Amer. Math. Soc. vol. 76 (1954) pp. 234–253.
2f. ———, Trans. Amer. Math. Soc. vol. 76 (1954) pp. 485–528.
3a. F. John, Comm. Pure Appl. Math. vol. 3 (1950) pp. 273–304.
3b. ———, *Proceedings of the symposium on spectral theory and differential problems*, Stillwater, Okla., 1951, pp. 113–175.
4. L. Schwartz, *Théorie des distributions* I, Paris, Hermann, 1950.
5. I. E. Segal, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 13–15.
6. H. Weyl, Math. Zeit. vol. 24 (1925) pp. 328–395.
7. I. M. Gelfand and M. A. Naimark, Trudi Mat. Inst. Steklova vol. 36 (1950)

COLUMBIA UNIVERSITY

TOPOLOGY OF LIE GROUPS AND CHARACTERISTIC CLASSES[1]

ARMAND BOREL

1. Introduction. The notion of continuous group, later called Lie group, introduced by S. Lie in the nineteenth century, has classically a local character. Although global Lie groups were also sometimes considered, it is only after 1920 that this concept was clearly formulated. We recall that a Lie group in the large is first a manifold, i.e., a topological Hausdorff space admitting a covering by open sets, each of which is homeomorphic to euclidian n-space; second it is a group; third it is a topological group, i.e., the product $x \cdot y$ of x and y and the inverse x^{-1} are continuous functions of their arguments; and finally it is required that there exist coordinates in a neighbourhood V of the identity element e such that if x, y, and $x \cdot y$ are in V, the coordinates of $x \cdot y$ are analytic functions in the coordinates of x and y. Gleason, Montgomery, and Zippin recently proved that the last condition follows from the others, thus solving Hilbert's fifth problem, with which we shall not be concerned here.

As soon as the concept was defined with precision, there arose the problem of studying topological properties of such group-manifolds. Indeed, in the first paper which systematically considers global Lie groups, H. Weyl's famous paper on linear representations [81], a key result which states that the fundamental group of a compact semisimple Lie group is finite is topological in nature. The question was next considered by E. Cartan in several papers, and later on by many mathematicians; as a matter of fact, it was often generalized in order to include also the study of "homogeneous spaces," i.e., manifolds which admit a transitive Lie group of homeomorphisms. A very complete survey of the work done in this field up to 1951 has been published in this Bulletin by H. Samelson [67]. Although of course some overlap is unavoidable, the present report is meant as a sequel and will therefore concentrate mainly on developments which occurred during these very last years. It will be devoted for the greater

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[1] This report also surveys material expounded at the Summer Mathematical Institute on Lie groups and Lie algebras, Colby College, 1953. The author expresses his hearty thanks to Dr. W. G. Lister, who prepared mimeographed notes of these lectures, which were helpful in the preparation of this paper.

part to homological properties of compact Lie groups, their classifying spaces and coset spaces. To complete the picture we also discuss at the end homotopy groups, noncompact Lie groups and their coset spaces and homogeneous complex spaces.

2. **Outline of methods.** There have been several lines of approach to the study of homological properties of compact Lie groups and coset spaces which, in first approximation, may be divided into three groups; methods of differential geometry, of algebraic topology, and the use of Morse theory.

The first direction was initiated by E. Cartan [16] who showed that the study of exterior differential forms on a coset space G/U invariant under the operations of G allows one to compute the Betti numbers of G/U using de Rham's theorems (which were conjectured by Cartan for that purpose, and proved soon afterwards by de Rham). This method was applied to the classical groups by R. Brauer and to certain symmetric spaces by C. Ehresmann, and Iwamoto. Implicit in Cartan's construction was the notion of cohomology ring of a Lie algebra modulo a subalgebra which was explicitly formulated by Chevalley-Eilenberg [25] and which became Koszul's principal tool [41]. Under the influence of A. Weil this point of view was broadened and led to a theory of differentiable principal bundles which gave a framework to a modern exposition of E. Cartan's theory of connections as well as new tools to study homology of coset spaces. The work of H. Cartan, C. Chevalley, J. L. Koszul and A. Weil in this direction has been summarized in [20; 24; 42]. Some of the main points are an algebraic transgression theorem in the Weil algebra (whose topological analog will be discussed in §9), a theorem of Chevalley connecting the Betti numbers of a Lie group and the invariants of its Weyl group (see §9), a result of Cartan on homogeneous spaces (see §13), and a theory, due to Koszul, of a certain type of differential algebras. Besides its topological applications, the latter allows us to give a homological formulation of Hilbert's theory of syzygies; it was later on generalized by H. Cartan-S. Eilenberg (see their forthcoming book, *Homological Algebra*, Princeton Series, no. 19). This method makes full use of differential forms, and as far as topology is concerned, gives results only about real cohomology. Already discussed in [67], it will not be dealt with here anymore, since most of the results directly relevant to our subject have been obtained later on by topological methods, to which we now turn our attention. Roughly speaking, these and the results to which they lead may also be divided into three groups.

The first group (see §§6, 7) consists of general properties, "general" in the sense that they derive solely from the existence of a nontrivial product, say with unit, sometimes assumed to be associative. The interest of this approach was displayed first by H. Hopf [38] who showed that the real cohomology algebra of a compact connected manifold endowed with such a product is a Grassmann algebra. Subsequent work along these lines has been done by H. Samelson [66], J. Leray [45], and the author [2].

The second group uses fiber bundle theory and in particular Leray's spectral sequence. Its starting point is the existence of universal bundles (see §8 for definition) and is at the source of a great part of the recent progress in this field. It may be viewed as a study of relations between on one hand cohomological and group-theoretical properties of a compact Lie group G, and on the other hand universal properties of characteristic classes of bundles with structural group G. Thus it connects topology of Lie groups with a problem of fiber bundle theory, and the simultaneous consideration of these two questions has allowed us to gain new information on both. The main results are described in §§8, 9 and the most important examples are given in §10; more or less direct applications to coset spaces and Lie groups are discussed in §§11 to 14.

The results obtained so far from these two points of view do not give a systematic and effective procedure to compute the cohomology over a field of characteristic not zero of a given coset space. In order to determine it (whenever possible), one has to use "special devices" taking advantage of some particular property of the space under consideration, and these make up the third group mentioned above. The main ones are cellular decompositions (see §15) and the study of special fiberings combined with the general results on universal bundles; §11 indicates the present state of knowledge concerning the cohomology of the compact simple Lie groups.

The use of Morse theory in these questions was recently initiated by R. Bott [13] who derives topological properties of certain coset spaces and of the space of loops on a Lie group by the study of geodesics on compact Lie groups. Their properties are obtained from the theory of singular elements, due to H. Weyl and E. Cartan, which is thus connected with topology in an entirely new way. The results announced in [13; 14], will be mentioned in §§11, 12, 15, 18.

3. **Algebra.** It will be convenient to recall in §§3, 4, 5 some known facts and definitions and to fix notations.

p will denote either a prime number or zero, Z_p (p prime) the field of integers mod p, Z_0 the field of rational numbers, R the real numbers, K an arbitrary field.

A module M over a ring A is *graded* if it is the direct sum of submodules to be denoted here by M^i (sometimes M_i), the index running through the positive integers; the degree d^0x of an element $x\neq 0$ is the smallest integer i such that $x\in\sum_{j\leq i} M^j$, and the elements of M^i are the homogeneous elements of degree i.

A *graded algebra* over a ring A is a graded module where $H^i\cdot H^j \subset H^{i+j}$; the unit, if any, is then contained in H^0. The algebra, or the product, is said to be *anticommutative* if

$$a\cdot b = (-1)^{i\cdot j} b\cdot a, \qquad (a\in H^i,\ b\in H^j),$$

when $A=Z_2$, this is just commutativity; if A is a field of characteristic $\neq 2$, then every homogeneous element of odd degree has square zero.

The exterior algebra of a vector space P over a field K (or of a free abelian group) is denoted $\bigwedge(P)$ or $\bigwedge(x_1, \cdots, x_s)$ where the x_i's form a base of P; the latter will always be graded, and the x_i be homogeneous; $\bigwedge(P)$ is then a graded algebra under the exterior product. When K has characteristic $\neq 2$, the exterior product is anticommutative if and only if the x_i have odd degrees.

The elements (x_i) of an algebra H form a *simple system* of generators if H is the weak direct sum of the A-modules generated by the unit, if any, and by the elements

$$x_{i_1}\cdot x_{i_2}\cdot \cdots \cdot x_{i_k} \qquad (i_1 < i_2 < \cdots < i_k;\ 1\leq k\leq s).$$

We write in this case $H=\Delta(x_1, \cdots, x_s)$. The additive basis is the same as for the exterior algebra $\bigwedge(x_1, \cdots, x_s)$, but the former differs in that no requirement is made about $x_ix_j+x_jx_i$.

As usual $A[x_1, \cdots, x_s]$ is the ring of polynomials in the indeterminates $x_1, \cdots, x_s$ with coefficients in the commutative ring A. We denote by $S(x_1, \cdots, x_s)$ the ring of symmetric polynomials in the x_i's and by $\sigma_i(x_1, \cdots, x_s)$ the ith elementary symmetric function.

If H_1 and H_2 are two graded algebras over a ring A, $H_1\otimes H_2$ is their (skew-)-tensor product over A: as a module, it is the usual tensor product, but the product is defined by

$$(a\otimes b)\cdot(c\otimes d) = (-1)^{ij}(a\cdot c\otimes b\cdot d), \qquad (b\in H_2^i,\ c\in H_1^j);$$

it has three gradings, but the one we shall consider usually is the total degree defined by

$$(H_1\otimes H_2)^i = \sum_{s+t=i} H_1^s\otimes H_2^t.$$

If H_1 and H_2 are anticommutative, so is $H_1 \otimes H_2$ with respect to the total degree.

4. Homology and cohomology groups. We shall not specify the homology and cohomology groups of a space X. Actually, since almost all the spaces considered, at least up to §16, are compact differentiable manifolds and hence finite polyhedra, most of the time it would suffice to use simplicial homology and cohomology groups. $H^i(X, A)$ (resp. $H_i(X, A)$) is the ith cohomology (resp. homology) group of X with coefficients in the abelian group A, and $H^*(X, A)$ (resp. $H_*(X, A)$) is the direct sum of the $H^i(X, A)$ (resp. $H_i(X, A)$). We shall in general consider cohomology so that, when A is a ring, we have a product in $H^*(X, A)$, the so-called cup-product; it is associative, distributive, adds the degrees, has a unit, spanning $H^0(X, A)$ when X is connected, and is anticommutative when A is commutative.

The homology and cohomology groups depend on the coefficients, the most complete information being obtained when A is the ring of integers Z. In this case we have (X being a finite polyhedron):

$$H^i(X, Z) = F^i + T^i$$

with F^i free abelian and finitely generated, T^i finite; the T^i define the *torsion of the space* X. In particular we say that X has p-torsion (p prime) if one of the T^i has order divisible by p; to simplify certain statements we also allow p to be zero and agree that a space is always without 0-torsion. The study of integral cohomology groups is often difficult and it is convenient to first investigate the cohomology over various fields of coefficients choosing one for each characteristic. Roughly speaking, we may say that the knowledge of $H^*(X, Z_0)$ gives information on the F^i, but says nothing about torsion; $H^*(X, Z_0)$ and $H^*(X, Z_p)$ give rather precise, though not complete, information on the p-primary components of the torsion groups T^i.

In case of a field of coefficients the $H^i(X, K)$ and $H_i(X, K)$ are vector spaces, dual to each other. The dimension of $H^i(X, Z_0)$ (resp. $H^i(X, Z_p)$) is the ith Betti number (resp. the ith Betti number mod p) of X. Finally we denote by $P_p(X, t)$ the *Poincaré polynomial*

$$P_p(X, t) = \sum_i \dim H^i(X, Z_p) \cdot t^i.$$

As is well known the Poincaré polynomial of a cartesian product $X \times Y$ is the product $P(X, t) \cdot P(Y, t)$, and more generally

$$H^*(X \times Y, K) = H^*(X, K) \otimes H^*(Y, K)$$

(and similarly for homology).

A continuous map $f:X\to Y$ induces a (degree preserving) homomorphism of $H^*(Y, A)$ (resp. $H_*(X, A)$) into $H^*(X, A)$ (resp. $H_*(Y, A)$) to be denoted by f^* (resp. f_*).

5. **Compact Lie groups.** We denote by G/U the space of left cosets of a Lie group G modulo a closed subgroup U, endowed with the usual quotient topology. It is an analytic manifold on which G acts transitively and analytically by means of left translations. Conversely any manifold on which a Lie group acts transitively is homeomorphic to a coset space G/U by a homeomorphism which commutes with the operations of G.

A Lie group is compact, or connected, if its underlying manifold is compact or connected. Two Lie groups are locally isomorphic if there exists a homeomorphism between two neighborhoods of the identities compatible with the product.

Any abelian compact connected n-dimensional Lie group is isomorphic to a torus T^n, i.e., to the direct product of n copies of the multiplicative group of complex numbers with norm 1. As is well known, the maximal tori of a compact Lie group G are conjugate to each other by inner automorphisms; their common dimension is the *rank* of G; a maximal torus will be in general denoted by T, since the omission of the rank will not bring confusion. A maximal torus has finite index in its normalizer N_T and the quotient N_T/T is a finite group, the Weyl group of G, to be denoted by $W(G)$; it has an obvious representation as automorphism group of T, which is faithful when G is connected.

A compact connected Lie group is simple if it has no proper closed invariant subgroups of strictly positive dimension, it is semi-simple if its center is finite. Compact connected Lie groups are locally isomorphic to (and in fact finitely covered by) direct products of tori and simple non-abelian groups, so that their classification reduces to that of simple groups. The different classes of locally isomorphic compact connected simple groups are usually denoted by the symbols A_r $(r\geqq 1)$, B_r $(r\geqq 2)$, C_r $(r\geqq 3)$, D_r $(r\geqq 4)$ (the classical structures), and G_2, F_4, E_6, E_7, E_8 (the exceptional structures); the corresponding groups have dimension $r(r+2)$, $r(2r+1)$, $r(2r+1)$, $r(2r-1)$, 14, 52, 78, 133, 248 respectively; the subscript denotes the rank. Each of these symbols may represent several groups, as will be discussed more thoroughly in §11. For the moment, we simply recall that the classical structures are represented by well known linear groups, namely:

A_r by the group $SU(r+1)$ of $r+1\times r+1$ complex unitary matrices of determinant $+1$.

B_r (resp. D_r) by the group $SO(2r+1)$ (resp. $SO(2r)$) or real orthogonal $2r+1\times 2r+1$ (resp. $2r\times 2r$) matrices of determinant $+1$, or by the spinor group *Spin* $(2r+1)$ (resp. *Spin* $(2r)$).

C_r by the group $Sp(r)$ of $r\times r$ quaternionic unitary matrices. They are of course all defined for $r\geqq 1$, but the restriction on the indices was made above in order to get each structure exactly once. We recall that

$$(5.1)\qquad A_1 = B_1 = C_1,\quad B_2 = C_2,\quad A_3 = D_3,\quad D_2 = A_1\times A_1.$$

The full group of real orthogonal (resp. complex unitary) matrices will be denoted by $O(n)$ (resp. $U(n)$).

6. **Hopf's and Samelson's theorems.** An H-space is a space endowed with a binary continuous law of composition with unit (in fact a weaker condition is usually postulated, but this is of no importance here). Hopf's theorem recalled in §2 admits the following generalization [2]: Let X be a connected finite polyhedron which is an H-space. Then $H^*(X, Z_p)$ has a minimal system of homogeneous generators (x_i), $(1\leqq i\leqq m)$, such that the monomials

$$x_1^{r_1}\cdot x_2^{r_2}\cdot\cdots\cdot x_m^{r_m}\qquad (0\leqq r_i < s_i,\ x_i^{s_i-1}\neq 0,\ x_i^{s_i}=0)$$

form an additive basis; moreover, for $p=0$, d^0x_i is odd and $s_i=2$; for $p=2$, s_i is a power of 2; for $p\neq 0, 2$, $s_i=2$ when d^0x_i is odd and s_i is a power of p when d^0x_i is even. A suitable modification of that statement applies also to infinite dimensional H-spaces, but will not be considered here. A consequence of this theorem is that if X has no p-torsion (resp. no torsion) its cohomology ring over Z_p (resp. Z) is the exterior algebra of a subspace (resp. free abelian group) generated by elements of odd degrees. As in Hopf's case, this follows from a purely algebraic result. Let us say that an algebra H over a field K is a Hopf algebra if it is graded by subspaces H^i $(i\geqq 0)$, has a unit generating H^0, is associative, distributive, anticommutative, and if there is a homomorphism h^* of H into $H\otimes H$ such that for x homogeneous and $\neq 0$ we have

$$h^*(x) = x\otimes 1 + 1\otimes x + \sum_1^k u_i\otimes v_i\quad (0 < d^0u_i < d^0x = d^0u_i + d^0v_i).$$

If X is an H-space, the product $h: X\times X\to X$ induces a map

$$h^*\colon H^*(X, Z_p)\to H^*(X\times X, Z_p)\cong H^*(X, Z_p)\otimes H^*(X, Z_p)$$

which satisfies the previous condition and the above theorem follows from the fact that every finite dimensional Hopf algebra over a perfect field has a system of generators (x_i) with the above mentioned properties. Standard facts about binomial coefficients show that the unit and x_i generate a subalgebra which is a Hopf algebra under the map h_i^* defined by

$$h_i^*(x_i) = x_i \otimes 1 + 1 \otimes x_i;$$

hence we may say that a Hopf algebra over a perfect field is isomorphic to the tensor product of Hopf algebras H_i which are generated by one element (besides the unit). This does not mean however that the isomorphism carries h^* over to the tensor product of the h_i^*. The existence of such an isomorphism is essentially equivalent to the existence of a system of generators which are "primitive" elements, i.e., elements for which

$$h^*(x) = x \otimes 1 + 1 \otimes x,$$

and examples show that this need not be the case. However, it is true in one important particular case: when H is an exterior algebra generated by elements of odd degrees and when h^* is "associative," i.e., when the two homomorphisms $h^* \otimes 1$ and $1 \otimes h^*$ of $H \otimes H$ into $(H \otimes H) \otimes H$ and $H \otimes (H \otimes H)$ become identical under the standard isomorphism of the two triple products. This is Samelson's theorem or more precisely the cohomological version of it (see also §7) proved in characteristic zero by Samelson [66], and later on by J. Leray [45], whose argument is valid for arbitrary characteristic, and even for the exterior algebra of a free abelian group. The associativity condition is of course satisfied in the case of H-spaces with an associative product, in particular for Lie groups, for which Samelson's theorem can also be obtained via transgression theorems [2, §20].

Even for cohomology algebras of Lie groups, the properties of h^*, which in fact are those of the Pontrjagin product discussed below, are not completely known; they seem to be closely connected with the universal spectral sequence (defined in §9), witness e.g. the fact that a universally transgressive element is primitive [2, §20]. This, combined with the results of §9, shows that, very often, $H^*(G, Z_p)$ is generated by primitive elements, but not much is known about h^* when this fails to be true, as e.g. in the case of $H^*(\text{Spin}\ (n), Z_2)$ $(n \geq 10)$ (see [5]).

We already pointed out that if the H-space X (always supposed to be a finite polyhedron) has no torsion, then $H^*(X, Z)$ is the exterior algebra of a free abelian group graded by odd degrees, and spanned by primitive elements when the product is associative. More generally the same conclusion applies to the quotient $H^*(X)$ of $H^*(X, Z)$ by its torsion ideal. In fact, the Künneth rule shows first that h^* induces a homomorphism of $H^*(X)$ into $H^*(X) \otimes H^*(X)$ satisfying Hopf's condition (6.1), then the generalized Hopf's theorem, applied to $H^*(X) \otimes Z_p$ for all p, gives the first statement and the second one follows from Leray's proof of Samelson's theorem (for compact Lie groups these facts are also proved in [29]). It follows that for $p \neq 2$, $H^*(X, Z_p)$ contains an exterior subalgebra having the same Poincaré polynomial as $H^*(X, Z_0)$; it would have some interest to know more about this embedding.

In the general case, nothing is known about the cohomology ring of an H-space over the integers or over a ring (say the integers modulo a power of a prime) which is not a field. A chief difficulty is of course that $H^*(X \times X, A)$ is not the tensor product of $H^*(X, A)$ with itself, but an extension of it by the Tor-product of $H^*(X, A)$ with itself.

7. **The Pontrjagin product.** Again let X be a connected H-space, with associative product h. The latter defines a map h_* of $H_*(X \times X, K)$, which we may identify with $H_*(X, K) \otimes H_*(X, K)$, into $H_*(X, K)$, and $h_*(a \otimes b)$ is called the Pontrjagin product of a and b: thus $H_*(X, K)$ becomes a ring under a product, first considered by Pontrjagin [62], which is associative, distributive, adds the degrees, has a unit spanning $H_0(X, K)$, but, unlike the cup-product, is not necessarily anticommutative, and that last fact makes its study more difficult. It may be remarked that the ring $H_*(X, K)$ together with the map d_* induced by the diagonal map $d: x \rightarrow (x, x)$ of X into $X \times X$ satisfies all conditions imposed on a Hopf algebra except for anticommutativity, but there is no structure theorem analogous to the generalized Hopf theorem of §6; in fact Bott-Samelson (Comment. Math. Helv. vol. 27 (1954) pp. 320–337) have given examples of infinite dimensional H-spaces whose homology rings are free associative algebras. Even for Lie groups the Pontrjagin product is not always anticommutative, as the following example shows [5]: $H_*(\text{Spin }(10), Z_2)$ has a simple system of generators x_3, x_5, x_6, x_7, x_9, x_{15} $(d^0 x_i = i)$, with the following relations:

$$x_i \cdot x_i = 0 \text{ (all } i), \qquad x_i x_j = x_j \cdot x_i \qquad (i < j, (i, j) \neq (6, 9)),$$

$$x_6 \cdot x_9 = x_9 \cdot x_6 + x_{15}.$$

This is so far the only case in which the homology ring of a Lie group over a field is completely known and is not an anticommutative exterior algebra; its structure is still reasonably simple and it does not seem too unlikely that homology rings of *finite-dimensional* H-spaces have more common properties than have been found so far.

The homomorphisms h^* and h_* are transposes of each other under the standard duality between homology and cohomology over a field; hence h^* and the Pontrjagin product determine each other. It is also easily seen that if $H^*(X, K)$ has a simple system of primitive generators (x_i) $(1 \leq i \leq m)$, then

$$H_*(X, K) = \bigwedge (y_1, \cdots, y_m) \qquad (d^0 y_i = d^0 x_i,\ i = 1, \cdots, m),$$

and conversely. By Samelson's theorem, the assumption is fulfilled in characteristic zero (and in fact the conclusion is Samelson's formulation of this result), and also in characteristic p when $H^*(X, Z_p)$ is an exterior algebra generated by elements of odd degrees.

The Pontrjagin product admits a useful generalization. Let M be a space on which an H-space X operates, i.e. we have a continuous map $g: M \times X \rightarrow X$ such that

$$(7.1) \qquad \begin{aligned} g(m, e) &= m \qquad (e \text{ the identity element}), \\ g(m, x \cdot y) &= g(g(m, x), y). \end{aligned}$$

There is induced a map g_* of $H_*(M \times X, K)$, which is identified with $H_*(M, K) \otimes H_*(X, K)$, into $H_*(M, K)$, and the image of $a \otimes b$ $(a \in H_*(M, K),\ b \in H_*(X, K))$ will also be called the Pontrjagin product of a and b. By duality, one also gets a pairing of $H^*(M, K)$ and $H^*(X, K)$ to $H^*(M, K)$. If X operates on another space N and if there is a continuous map $f: M \rightarrow N$ commuting with the operations of X, then f^* commutes with the Pontrjagin product in the obvious way. More generally, one may define a pairing of the spectral sequence (E_r) of f and of $H_*(X, K)$ to (E_r). When X is a Lie group, this was considered by J. Leray [46; 47] and later on, for fiber maps, by T. Kudo [44] and the author [2; 5]. An analogous situation, involving spaces of loops, is the subject of Bott-Samelson's paper already mentioned. This pairing allows one e.g. to prove the following generalization of some results of [2; 47; 66], (see [5, §3]): Let G be a compact connected Lie group operating on a space M and let f be a map of G into M commuting with the operations of G on M and onto itself by left translations. Assume $H^*(G, Z_p)$ to be an exterior algebra generated by elements of odd degrees. Then the image of f^* is generated by primitive elements, and if $p \neq 2$, $H^*(M, Z_p) = A \otimes B$ where f^* is 1-1 on A and annihilates the elements of strictly positive degrees in B.

8. **Characteristic classes, principal and universal bundles.** For more details about the facts of fiber bundle theory discussed in the sequel, we refer once and for all to [19; 48; 73].

The notion of characteristic class arose in 1935 in the work of Stiefel and Whitney about vector fields on manifolds. It is well known that on a compact manifold M it is not always possible to construct a field of nonzero tangent vectors, depending continuously on their origin. However, this can be done up to a finite number of points and to each of these points there corresponds a certain integer, the "index of singularity" of the vector field at the point. By a well known result of H. Hopf the sum of these indices is equal to the Euler-Poincaré characteristic of M (in our notations to $P_0(M, -1)$); in particular it is independent of the vector field. Stiefel and Whitney studied a generalization of the problem: the existence of k tangent vector fields, linearly independent at every point. To construct them, one starts from a triangulation of M, defines the vector fields arbitrarily at the vertices, then tries to extend the definition continuously to the edges, 2-dimensional faces, and so on. This is always possible up to the $(n-k)$-dimensional simplices of the triangulation, but not necessarily to the $(n-k+1)$-dimensional skeleton. Thus we are led to consider the "indices of singularity" attached to the $(n-k+1)$-dimensional simplices; to their sum there corresponds an $(n-k+1)$-dimensional cohomology class w_{n-k+1} which is called the $(n-k+1)$st Stiefel-Whitney class of M. The important point is that it depends only on M, and not on the construction of the vector fields. We shall omit discussion of the natural coefficient systems with respect to which w_i is taken, and for simplicity consider w_i to be an element of $H^i(M, Z_2)$. Let us just mention that when M is orientable and oriented, w_n is in reality an integral class, equal by Hopf's theorem to the fundamental class multiplied by the Euler-Poincaré characteristic.

In 1942 Pontrjagin attached to a compact oriented manifold M integral cohomology classes by a seemingly quite different procedure. He started from an embedding of M into some euclidian space E^N (N large), and by assigning to each point $P \in M$ the n-dimensional subspace parallel to the tangent space to M at P, he defined a map f of M into the Grassmann manifold $G^0_{N,n}$ of oriented n-dimensional subspaces of E^N. By a careful study of its cohomology, he singled out certain elements $p_i \in H^{4i}(G^0_{N,n}, Z)$ $(1 \leq 4i \leq n)$ which appeared to be basic and considered their images $f^*(p_i)$. These turned out to be independent of the embedding and therefore to be intrinsically attached to the given differentiable manifold. Subsequently, it was also shown that the Stiefel-Whitney classes can be defined in a similar fashion and, conversely, that the Pontrjagin classes are connected with certain problems on vector fields. Later on, S. S. Chern defined for a compact complex analytic manifold M of complex dimension n, classes $C_i \in H^{2i}(M, Z)$ $(0 \leq i \leq n)$ which may be characterised either as "obstructions" to the construction of complex vector fields or as images of certain elements in the cohomology of the complex Grassmann manifold $H_{N,n}$ of n-dimensional subspaces of the complex affine space C^N (N large), relative to a suitable map of M into $H_{N,n}$.

A first generalization of these ideas depends on the notion of *principal bundle*. A space E is a principal bundle with structural group G (or as we shall say, a G-bundle), if E is a transformation space for G and if no transformation other than the identity has a fixed point (we assume G to be compact, otherwise the definition is slightly more complicated). Then the orbits of G are all homeomorphic to G and are called the fibers. The space of orbits B and the map of E onto B associating with each point in E its orbit are called the base space and the projection. Examples: (1) Let E be a topological group containing G as a closed subgroup, and let G operate on E by right multiplication. The fibers are then the left cosets xG and B is the space of left cosets, denoted by E/G. (2) $B = M_n$ is an n-dimensional Riemannian manifold, E the set of all orthonormal frames on M_n (i.e. a point of E is an orthonormal basis of the tangent space at some point of M_n), with a suitable topology. The orthogonal group $O(n)$ operates on E in a natural way and E becomes an $O(n)$-bundle: the fibers are the frames with a given origin and the projection assigns to each frame its origin. Similarly, if M_n is orientable, the space of orthonormal frames corresponding to a given orientation is an $SO(n)$-bundle, and if M is a complex analytic hermitian manifold, the space of unitary orthonormal frames on M is a $U(n)$-bundle.

It next became apparent that the above characteristic classes may be associated with bundles of frames and moreover that the Stiefel-Whitney, Pontrjagin, and Chern classes may be defined for any $O(n)$, $SO(n)$, and $U(n)$-bundle respectively.

The final step in the generalization arose from the observation that the Grassmann manifolds were not necessary as "reference" spaces but could be replaced by what we now call classifying spaces. This generalization is valid for any compact Lie group G.

A principal bundle with structural group G is said to be ***n*-universal** (for G) if its homology (or homotopy) groups vanish up to n (except for H_0 of course). Such spaces exist for arbitrary compact Lie groups and any n, including $n = \infty$; we shall hereafter omit mentioning n. The base space B_G of the universal bundle E_G for G is called a *classify-*

ing space for G; its importance lies in the classification theorem which, roughly, states that the essentially different G-bundle structures with a given base B are in 1-1 correspondence with the homotopy classes of maps $f:B\to B_G$. Therefore, to any G-bundle over B there is attached a homomorphism f^* of $H^*(B_G, A)$ into $H^*(B, A)$, the *characteristic map* of the fibering, the image of which is called *the characteristic ring* of the fibering. The characteristic classes of Stiefel-Whitney, Pontrjagin, and Chern appear then as images of particular elements in $H^*(B_{O(n)}, Z_2)$, $H^*(B_{SO(n)}, Z)$, and $H^*(B_{U(n)}, Z)$ respectively. Thus $H^*(B_G, A)$ may be viewed as the ring of "universal" characteristic classes for G-bundles; its properties are universal in the sense that, by the classification theorem, they are valid in any G-bundle, and are therefore quite important. More generally there is the problem of investigating the relations between the cohomology rings of B_G and of G, in other words of the base and the fiber in the universal bundle E_G; this is precisely the problem alluded to in §2; it will be discussed in §9. In the study of this problem, an important role is played by a map $\rho(U, G)$ defined as follows. Let U be a closed subgroup of G. Then E_G is clearly a principal bundle with structural group U, with vanishing homotopy groups; hence it is also a universal bundle for U. The natural projection of E_G/U onto E_G/G may then be considered as a map of B_U into B_G, to be denoted by $\rho(U, G)$.

Without entering into details, let us just mention in passing that $\rho(U, G)$ is a fiber map of a fibering of B_U, with typical fiber G/U, base B_G, and that $\rho(U, G)$ also occurs in the problem of restricting the structural group of a fiber bundle: the structural group of a bundle with base B defined by a map $f:B\to B_G$ can be restricted to U if and only if there exists a map $g:B\to B_U$ such that $f=\rho(U, G)\cdot g$.

9. **Results on universal bundles. Invariants of the Weyl group.** To express these results, we shall use the notion of transgression in a fiber bundle, and first recall briefly one of its possible definitions (see [2, §5] for more details). Let E be a bundle with fiber F, base B, projection map π (e.g. a principal bundle, the only case in which this definition will be used below). Let π' (resp. i') be the map of cochains induced by π (resp. the inclusion of a fiber in E). An element $x\in H^s(F, A)$ is *transgressive* in E if there exists a cochain c on E such that $i'(c)$ is a cocycle of x and that its coboundary is of the form $\pi'(b)$, where b is some cochain on B, necessarily a cocycle. Its cohomology class $y\in H^{s+1}(B, A)$ is determined by x only modulo a certain subgroup L^{s+1} and the transgression is the map of the transgressive elements of $H^s(F, A)$ into $H^{s+1}(B, A)/L^{s+1}$ derived from

$x \to y$. In spite of the fact that y is not uniquely determined by x in general, we shall write $y = \tau(x)$ whenever y is obtained from x by this procedure.

Let now G be a compact connected Lie group. Then $x \in H^s(G, A)$ is ***universally transgressive*** if it is transgressive in E_G. The classification theorem shows that it is then transgressive in all G-bundles. The notion of transgression is interesting for at least two reasons: First, as we shall see, it is intimately related with the characteristic classes, and second a knowledge of the transgressive elements is quite useful in the computation of spectral sequences; this more technical point will not be illustrated here any further and we refer to [2; 5] for examples.

The study of the homological properties of the universal bundle, i.e., essentially of its spectral sequence, the ***universal spectral sequence*** for G, appears thus to be a basic problem. So far, only partial results are known, the main one being:

(A) *If $H^*(G, Z_p)$ is the exterior algebra of a subspace graded by odd degrees, then $H^*(G, Z_p) = \wedge(x_1, \cdots, x_m)$, with x_i universally transgressive, of odd degree, and $H^*(B_G, Z_p) = Z_p[y_1, \cdots, y_m]$ with $y_i = \tau(x_i)$ $(1 \leq i \leq m)$. Conversely if $H^*(B_G, Z_p) = Z_p[y_1, \cdots, y_m]$ with the y_i's of even degrees, then $H^*(G, Z_p) = \wedge(x_1, \cdots, x_m)$ with the x_i's universally transgressive and $y_i = \tau(x_i)$ $(1 \leq i \leq m)$.*

A similar result is valid over integers, when G has no torsion (see [2, Théorème 19.1; 5, Théorème 6.1]). The assumption of (A) is fulfilled when G has no p-torsion, and in particular in characteristic zero. An analogous, but weaker, result is:

(B) *If $H^*(G, Z_2)$ has a simple system (x_i) of universally transgressive generators, then $H^*(B_G, Z_2) = Z_2[y_1, \cdots, y_m]$, where $y_i = \tau(x_i)$ $(1 \leq i \leq m)$ and conversely.*

(See [2, Proposition 19.1; 5, Théorème 6.1]). In (B) we assume the existence of transgressive generators, whereas it is a part of the conclusion in (A); this assumption is often fulfilled, even when there is 2-torsion, e.g. in the case of $SO(n)$ $(n \geq 3)$, G_2, F_4, Spin (n) $(7 \leq n \leq 9)$. However it is not true for $H^*(\text{Spin}\ (n), Z_2)$ when $n \geq 10$ (see [5]).

Nothing is known up to now about $H^*(B_G, Z_p)$ outside the cases covered by (A) and (B); it seems that one has to expect quite different properties and that their study will require more knowledge about the behavior of reduced powers in the spectral sequence.

Let now T be a maximal torus of G. Then

$$H^*(T, Z) = \wedge(u_1, \cdots, u_r) \qquad (d^0 u_i = 1,\ r = \text{dimension of } T),$$

and by (A), or also by more direct considerations,

$$H^*(B_T, Z) = Z[v_1, \cdots, v_r] \qquad (v_i = \tau(u_i), (1 \leqq i \leqq r)).$$

The Weyl group $W(G)$ of G operates on T, hence on its cohomology, and also on $H^*(B_T, Z)$, as is easily seen; we denote by I_G the ring of polynomials in the latter invariant under $W(G)$; it is a direct summand, and $I_G \otimes Z_p$ is thus canonically embedded in $H^*(B_T, Z) \otimes Z_p$, which is just $H^*(B_T, Z_p)$, since B_T has no torsion; it is of course contained in the ring of invariants of $W(G)$ operating on $H^*(B_T, Z_p)$, and is equal to it when $p=0$, but may be different from it otherwise. If we consider real cohomology, then we may identify $H^*(B_T, R)$ with the ring of polynomials over the Lie algebra of T, the operations of $W(G)$ being the obvious ones, but we have of course to give the "dimension" 2 to the coordinates v_i when we consider them as elements of $H^*(B_T, R)$. The invariants of the Weyl group and the cohomology of B_G are connected by the following theorem:

(C) *Let T be a maximal torus of a compact connected Lie group G. Assume $H^*(G, Z_p)$ to be the exterior algebra of an s-dimensional subspace graded by odd degrees. Then $s = \dim T$ and $\rho^*(T, G)$ maps $H^*(B_G, Z_p)$ isomorphically onto $I_G \otimes Z_p$.*

A similar result is valid over integers when G has no torsion. This theorem is established in [2, §§26, 27], under the apparently slightly stronger (though in fact equivalent) assumption that G has no p-torsion, but the proof is the same. Also, [2] assumes that G/T has no p-torsion (resp. no torsion), a fact which has since been proved to be always true (see §15). Theorems (A) and (C) have assumptions always fulfilled in characteristic zero and show therefore that the ring of invariants of the Weyl group has r algebraically independent generators of (cohomological) dimensions $2m_1, \cdots, 2m_r$, where $H^*(G, R) = \bigwedge(x_1, \cdots, x_r)$ $(d^0 x_i = 2m_i - 1,\ i = 1, \cdots, r)$, a result first obtained by C. Chevalley [24] ($r = \operatorname{rank} G$).

Let U be a closed connected subgroup of G, and $S \subset T$ be maximal tori of U and G. When (A) can be applied to U and G, the study of the homological position of U in G, i.e., of the map $H^*(G, K) \rightarrow H^*(U, K)$ induced by the inclusion, is essentially equivalent to that of $\rho^*(U, G)$. If moreover (C) can be used, then $\rho^*(U, G)$ is determined by the behavior of the invariants of $W(G)$ under the map $\rho^*(S, T)$, which in turn is explicitly described by means of the homomorphism $H^1(T, K) \rightarrow H^1(S, K)$ defined by the inclusion $S \subset T$ (see [2, §§28 and 31] for some applications). If $G = U(n)$, $Sp(n)$, $SO(n)$, then the knowledge of $\rho^*(S, T)$ boils down to that of the weights (in E. Cartan's sense) of the linear group U. Thus we are led to connections between weights and homological position of subgroups of the classical linear groups. For real cohomology, and starting from a point of view more akin to [20; 24], they have been studied by E. B. Dynkin [27; 28; 30; 31a].

An analogous, though different, example of relations between classifying spaces and group theoretical properties is offered in cohomology mod 2 by $O(n)$. Here we take instead of a maximal torus the subgroup $Q(n)$ of diagonal matrices which is isomorphic to $(Z_2)^n$; by a standard result

$$H^*(B_{Q(n)}, Z_2) = Z_2[v_1, \cdots, v_n] \qquad (d^0 v_i = 1;\ i = 1, \cdots, n).$$

The quotient by $Q(n)$ of the normalizer of $Q(n)$ in $O(n)$ is readily seen to operate on $H^*(B_{Q(n)}, Z_2)$ as the full group of permutations of the v_i's. Then, in analogy with (C), it may be shown that $\rho^*(Q(n), O(n))$ maps $H^*(B_{O(n)}, Z_2)$ isomorphically onto the ring of symmetric functions in the v_i's, i.e., onto the ring of invariants of the "Weyl group" defined by using $Q(n)$ instead of T. Moreover, $\rho^*(Q(n), O(n))$ maps the ith universal Stiefel-Whitney class w_i (introduced in §8) onto the ith elementary symmetric function in the v_i's [3, Théorème 6.1]. A similar result is valid for the special orthogonal group.

Here again, not much is known outside these two cases. The latter one suggests the substitution of maximal abelian subgroups of type $(p, \cdots, p)$ for maximal tori when dealing mod p with a group having p-torsion, but examples show that one cannot expect relations as simple as for $O(n)$ in general, though on the other hand these subgroups seem definitely to be related to the p-torsion of the group (see §12).

10. **Examples.** We describe here briefly some relations between the results of §9 and the characteristic classes introduced in §8, using some facts on the cohomology of the classical groups (see §11 for references), and on their structure (see e.g. [74]).

$U(n)$ has rank n and no torsion. The Weyl group operating on $H^*(B_T, Z) = Z[v_1, \cdots, v_n]$ is the group of permutations of the v_i's, and $\rho^*(T, U(n))$ maps $H^*(B_{U(n)}, Z)$ isomorphically onto the ring $S(v_1, \cdots, v_n)$ of symmetric functions. The universal ith Chern class is mapped on the ith elementary symmetric function. Thus $H^*(B_{U(n)}, Z)$ is the ring of polynomials in the Chern classes, and moreover $H^*(U(n), Z) = \bigwedge(x_1, \cdots, x_n)$, where $C_i = \tau(x_i)$ and therefore $d^0 x_i = 2i - 1$.

The unitary symplectic group $Sp(n)$ has rank n and no torsion. $W(Sp(n))$ is the group of permutations of the v_i combined with arbitrary changes of signs; hence $I_{Sp(n)}=S(v_1^2, \cdots, v_n^2)$: the ring $H^*(B_{Sp(n)}, Z)$ is generated by elements of dimensions $4i$ $(1\leqq i\leqq n)$, and $H^*(Sp(n), Z)$ by elements of dimensions $4i-1$ $(1\leqq i\leqq n)$.

$SO(2n+1)$ (resp. $SO(2n)$), has rank n and its Weyl group consists of the permutations of the v_i's together with an arbitrary (resp. even) number of changes of signs. Hence $I_{SO(2n+1)}=S(v_1^2, \cdots, v_n^2)$ and $I_{SO(2n)}$ is generated by $S(v_1^2, \cdots, v_n^2)$ and the product $v_1 \cdots v_n$. For $p\neq 2$, $SO(m)$ has no p-torsion $(m=1, 2, 3, \cdots)$ and we may apply (A) and (C). Also, it turns out that $\rho^*(T, SO(m))$ maps the ith universal Pontrjagin class, reduced mod p, onto $\sigma_i(v_1^2, \cdots, v_n^2)$. In fact, this is also true over the integers, though in that case the kernel of $\rho^*(T, SO(m))$ is not zero. In the case $m=2n$, the product $v_1 \cdots v_n$ plays a special role, too; it is the image under $\rho^*(T, SO(2n))$ of the so called Euler-Poincaré class W_n. The latter is the unique element of $H^{2n}(B_{SO(2n)}, Z)$ having the following property: if M is a $2n$-dimensional orientable compact oriented manifold, and if $f: M\rightarrow B_{SO(2n)}$ is the map corresponding to the bundle of orthonormal frames (see §8), then $f^*(W_n)$ is equal to the generator of $H^{2n}(M, Z)$ singled out by the orientation, multiplied by the Euler-Poincaré characteristic of M.

Analogously, it may be shown that for $m=2n$, $2n+1$, and for $p\neq 2$, $\rho^*(T, O(m))$ maps $H^*(B_{O(m)}, Z_p)$ isomorphically onto $S(v_1^2, \cdots, v_n^2)$. One can also introduce a universal Pontrjagin class $P_i\in H^{4i}(B_{O(m)}, Z)$, (i.e., for $O(m)$ bundles and not as before for $SO(m)$ bundles); it is mapped onto $\sigma_i(v_1^2, \cdots, v_n^2)$ by $\rho^*(T, O(m))$ (also over integers).

The cohomology mod 2 and the relations with Stiefel-Whitney classes have already been mentioned in §9. A closer investigation shows that in $H^*(B_{O(n)}, Z)$ and $H^*(B_{SO(n)}, Z)$ all torsion elements have order 2. This implies that an element of the integral cohomology ring is completely determined by its reductions mod 2 and over the rationals. The ith Pontrjagin class may then be characterised as the element which is mapped onto the ith elementary symmetric function in the v_i's in $H^*(B_T, Z_0)$ and onto the square of the $2i$th elementary symmetric function in $H^*(B_{O(n)}, Z_2)$.

Some of the above results for the orthogonal groups are not in the literature, but follow without difficulty from known facts. A complete discussion may be found in mimeographed notes of lectures held by the author at the University of Chicago, Fall 1954.

The interpretation of characteristic classes as elementary functions gives immediate interpretations or proofs of their so-called Whitney duality properties. It also leads to a method of computation of their Steenrod reduced powers and, via transgression, of reduced powers in the classical Lie groups. This in turn has applications to problems of differential geometry like existence of vector fields or of almost complex structures on spheres (see [12]).

11. Homology and cohomology of compact simple Lie groups. The different classes of locally isomorphic compact simple Lie groups have been listed in §5. Each of them contains a simply connected representative, unique up to global isomorphism, and the other groups are quotients of the former by subgroups of its center.

For A_r, B_r, C_r the simply connected groups are

$$SU(r+1), \quad \mathrm{Spin}(2r+1), \quad Sp(r)$$

whose centers are cyclic of orders $r+1$, 2, 2. The quotients by the full centers are the groups $PU(r+1)$, $SO(2r+1)$, $PSp(r)$ of projective transformations induced in the complex r-dimensional, real $2r$-dimensional, quaternionic $(r-1)$-dimensional projective space respectively.

The simply connected group of structure D_r is Spin $(2r)$, its center is of order 4, cyclic if r is odd, noncyclic otherwise; the quotient of order 4 is the projective orthogonal group $PSO(2r)$. For r odd, there is one quotient of order 2, which is $SO(2r)$; for r even there are besides $SO(2r)$ the two "semi-spinor" groups. The latter are homeomorphic to each other for all r, and in particular for $r=4$, are isomorphic to $SO(8)$ by the triality principle; however, it is not known to the author whether the semi-spinor group is homeomorphic to the corresponding orthogonal group in general.

Finally the simply connected representatives of the structures G_2, F_4, E_6, E_7, E_8 have cyclic centers of orders 1, 1, 3, 2, 1. (For all this, see e.g. E. Cartan, Annali di Matematica vol. 4 (1927) pp. 209–256).

Up to now, the following information on the homological properties of these groups has been obtained.

(11.1) *The real cohomology for all groups* (see e.g. [2], [24], [66], and [67] for other references). Two locally isomorphic groups have the same real cohomology (see e.g. [62], it also follows from standard theorems on coverings, or from statement (C) in §9); hence it is a property of a class of locally isomorphic groups. For E_6, E_7, E_8 the degrees of the primitive generators are (3, 9, 11, 15, 17, 23), (3, 11, 15, 19, 23, 27, 35), (3, 15, 23, 27, 35, 39, 47, 59). For the other structures, see below.

(11.2) *The integral cohomology ring of $SU(n)$, $Sp(n)$, G_2, F_4.* The groups $SU(n)$ and $Sp(n)$ have no torsion [36; 62; 2] and

$$H^*(SU(n), Z) = \bigwedge(x_3, x_5, \cdots, x_{2n-1}),$$
$$H^*(Sp(n), Z) = \bigwedge(x_3, x_7, \cdots, x_{4n-1}), \qquad (d^0x_i = i).$$

G_2 has 2-torsion and $H^*(G_2, Z)$ has 2 generators h_3, h_{11} of degrees 3, 11 such that $h_3^4 = h_{11}^2 = 0$ and that $H^*(G_2, Z)$ is the weak direct sum of the infinite cyclic groups generated by 1, h_3, h_{11}, $h_3 \cdot h_{11}$ and of the groups of order 2 generated by h_3^2, h_3^3 (see [5, §17]).

F_4 has torsion coefficients of order 2, 3, 6, and $H^*(F_4, Z)$ is isomorphic to the product

$$U \otimes H^*(G_2, Z) \otimes \bigwedge(x) \qquad (d^0x = 15),$$

where U is a graded algebra with unit defined by

$$U^0 = U^{23} = Z, \quad U^8 = U^8 \cdot U^8 = U^{16} = Z_3, \quad U^i = 0 \text{ otherwise}$$

(see [5, §23]).

(11.3) Spin (n) $(n \leqq 6)$ *has no torsion*; $SO(n)$ $(n \geqq 3)$ *and* Spin (n) $(n \geqq 7)$ *have 2-torsion and all their torsion coefficients are equal to* 2 (see [2; 35; 56; 62] for $SO(n)$, and [5] for Spin (n)). Hence their integral cohomology rings are theoretically determined by their cohomology rings over Z_0 and Z_2; however the explicit formulas seem to be quite cumbersome and have not been written down.

Added in proof. Other proofs of the results about Spin(n) mentioned in (11.3), (11.4) have been recently given by S. Araki (Memoirs of the Faculty of Science. Kyusyu Imperial University, Series A, vol. 9 (1955) pp. 1–35). He uses a cellular decomposition which is a covering of the J. H. C. Whitehead cellular decomposition of $SO(n)$, the basic tool of Miller [56].

(11.4) *The cohomology* mod p (all p) *for* G_2, F_4 *and all the groups of the classical structures* (with the possible exception of the semispinor groups of type D_{2r} $(r \neq 2)$ in cohomology mod 2), with partial or complete information on Steenrod's reduced powers. In the case of G_2, F_4, $SU(n)$, $Sp(n)$ the cohomology ring can be read off (11.2). For the reduced powers see [5; 12]; for the Sq^i in $SO(n)$, see [3; 56]. Moreover, $H^*(G_2, Z_2)$ and $H^*(F_4, Z_2)$ have simple systems of universally transgressive generators of degrees (3, 5, 6) and (3, 5, 6, 15, 23) respectively.

Let s be the greatest power of 2 dividing a given integer n. Then

$$H^*(PSp(n), Z_2)$$
$$= Z_2[a]/(a^{4s}) \otimes \bigwedge(x_3, x_7, \cdots, x_{4s-1}, x_{4s+3}, \cdots, x_{4n-1}),$$

$(d^0a = 1,\ d^0x_i = i,\ 1 \leqq i \leqq 4n)$. There are analogous results for $PSO(2r)$

and for the quotients of $SU(n)$, which we do not recall here (see [5, §10]).

$H^*(SO(n), Z_2)$ has a simple system of universally transgressive elements (h_i) $(d^0h_i = i,\ i = 1, \cdots, n-1)$, with the relations

$$Sq^i h_j = \binom{j}{i} h_{i+j} \qquad (i \leq j,\ i + j \leq n - 1),\ Sq^i h_j = 0 \text{ otherwise.}$$

In particular, $h_i \cdot h_i = h_{2i}$ if $2i \leq n-1$, and $h_i \cdot h_i = 0$ otherwise. $H^*(PSO(2n), Z_2)$ is derived from this in the same way as $H^*(PSp(n), Z_2)$ from $H^*(Sp(n), Z_2)$.

Finally, let $s(n)$ be the integer such that $2^{s(n)-1} < n \leq 2^{s(n)}$. Then $H^*(\text{Spin}\ (n), Z_2)$ has a simple system of generators whose degrees form a sequence obtained from $3, 4, \cdots, n-1$ by erasing all powers of 2 and adding $2^{s(n)} - 1$ (one generator for each degree). The Sq^i are also known for the greater part [5, §13].

(11.5) *The groups of the structure E_6 (resp. E_7, E_8) have no p-torsion for $p \geq 7$ (resp. $p \geq 11$) (see [7]) and the group of structure E_8 has 2-torsion* [11, *see* §12].

(11.6) *Let G be compact, connected, simply connected, simple and non-abelian. Then $H_3(G, Z) = Z$.*

This is deduced by Hurewicz's isomorphism theorem from $\pi_1(G) = \pi_2(G) = 0$, $\pi_3(G) = Z$ (see §18).

(11.7) We have already pointed out that the homology ring (the multiplication being the Pontrjagin product) mod p is an exterior algebra when there is no p-torsion, in particular for $p = 0$. It follows also from §7 and (11.4) that $H_*(SO(n), Z_2)$, $H_*(G_2, Z_2)$, $H_*(F_4, Z_2)$ are exterior algebras generated by elements of degrees $(1, 2, \cdots, n-1)$, $(3, 5, 6)$, $(3, 5, 6, 15, 23)$ respectively, (see [5], and also [56] for $SO(n)$).

Finally, we point out that many results on the classical linear groups are particular cases of theorems on Stiefel manifolds (see the references given above).

12. **Remarks on cohomology of Lie groups and on Weyl groups.** Many of the results on the torsion of Lie groups listed in §11 are established by use of special properties of the individual groups. However even if the three last exceptional groups could be handled in the same way, this would not constitute a fully satisfactory solution of the problem, the ultimate goal being to arrive at a systematic procedure for computing cohomology from the group theoretical, or infinitesimal, properties. For real cohomology this is attained by Chevalley's theorem (see §9) which relates Betti numbers and in-

variants of the Weyl group. Another method was already contained in E. Cartan's work [16], but it led to computations too cumbersome to be applied to the last 4 exceptional groups. No such result is yet available in cohomology mod p for $p\neq 0$, though there is evidence of relations between torsion and group-theoretical properties. An example [7] is the theorem stating that if the prime p does not divide the order of the Weyl group of a compact connected Lie group G, then G has no p-torsion. For E_6, E_7, E_8 the orders of the Weyl groups are $2^7\cdot 3^4\cdot 5$, $9!\cdot 8$, $10!\cdot 3\cdot 2^6$ respectively, whence the first part of (11.5). Also all results known so far agree with the following *conjecture*: If p is greater than the coefficients of the highest root, expressed as linear combination of fundamental roots, then the simply connected group G has no p-torsion. For the simply connected representatives of the structures E_6, E_7, E_8 this would imply no p-torsion for $p\geqq 5$, $p\geqq 5$, $p\geqq 7$ respectively.

In this connection, let us also mention that if a compact connected Lie group of rank r contains an abelian subgroup isomorphic to $(Z_p)^k$ with $k>r$, then it has p-torsion [11]; in application E_8 has 2-torsion. Also, if $H^*(G, Z_2)$ has a simple system of s universally transgressive generators, then it does not contain a subgroup isomorphic to $(Z_2)^t$ with $t>s$; the upper bound of t is equal to s for $G=U(n)$, $SU(n)$, $Sp(n)$, $SO(n)$ (any n), G_2, F_4. In other cases, the connection between abelian subgroups of type $(p, \cdots, p)$ and p-torsion is not as simple. Anyway, in all examples known to the author, it seems to be true that G has p-torsion if and only if it has an abelian subgroup of type $(p, \cdots, p)$ not contained in a maximal torus.

The connections between Weyl groups and homology suggest some problems about the former. Viewed as a transformation group of the universal covering R^r of a maximal torus T of a compact connected Lie group G of rank r, the Weyl group is a finite linear group, generated by reflections in hyperplanes, which is moreover "crystallographic," i.e. it leaves a lattice of rank r invariant or, equivalently, is represented by matrices with integral coefficients in a suitable basis. Conversely, every crystallographic group generated by reflections is a Weyl group, as was checked by E. Stiefel [74] and later on given an a priori proof by C. Chevalley (C. R. Acad. Sci. Paris vol. 227 (1948) pp. 1136–1138) and Harish-Chandra (Trans. Amer. Math. Soc. vol. 70 (1951) pp. 28–96, Theorem 1). We recall that two locally isomorphic, not isomorphic, groups have isomorphic Weyl groups but different lattices [74]. We have seen (§9) that the ring of invariant polynomials with real coefficients of a Weyl group $W(G)$ has r algebraically independent generators ($r=$dimension of the vector space on which $W(G)$ operates, i.e., the rank of G). This can in fact be proved directly for all finite groups generated by reflections (Chevalley), crystallographic or not. In the crystallographic case, it does not seem to be known whether a similar result is valid for invariants with integral coefficients in the symmetric algebra over Z of an invariant lattice: also, the invariants of a finite group generated by reflections of a vector space over a field of characteristic p have not yet been considered except to prove that Chevalley's theorem holds true for a group obtained by "reduction mod p" from a Weyl group W, given by integral matrices, when p is prime to the order of the group [7].

Added in proof. For Chevalley's theorem, see a forthcoming note of his in the American Journal of Mathematics.

An interesting question about Weyl groups also arises in connection with a result of Bott [13]. Let G be a compact, connected and simply connected Lie group and Ω_G the space of closed paths on G, with a fixed origin. Then Bott has proved that Ω_G has no torsion, and moreover he gives an explicit method to get the Betti numbers of Ω_G out of the "diagram" of G (i.e., essentially the planes of reflections of the elements of $W(G)$ and the invariant lattice). On the other hand, these Betti numbers can be read off from the degrees of the invariants of $W(G)$ with real coefficients (using Chevalley's theorem and spectral sequences arguments). Thus a connection between these degrees and the diagram which so far has been proved only with the help of rather difficult topological methods. For more details and other problems about these finite groups, we refer to G. C. Shephard [72].

13. **Cohomology of homogeneous spaces.** The most comprehensive result on the real cohomology of a homogeneous space is due to H. Cartan [20] (see [2, §26] for a topological proof). It says that $H^*(G/U, R)$ (G compact connected, U closed connected) is isomorphic to the cohomology ring of the tensor product $H^*(B_U, R) \otimes H^*(G, R)$ relative to a certain coboundary operator completely (and explicitly) determined by the transgression in E_G and by a homomorphism which turns out to be equivalent to $\rho^*(U, G)$. Many particular results obtained previously by Samelson, Leray, Koszul can be derived from it, either directly or using Koszul's theory of homology of S-modules [42]. Particular cases of H. Cartan's theorem are to be found in [65].

Cartan's theorem shows how to compute $H^*(G/U, R)$, knowing $H^*(G, R)$, $H^*(U, R)$, and $\rho^*(U, G)$, i.e. in the last analysis out of group theoretical information, namely the invariants of the Weyl groups of G and U and the position of a maximal torus of U in a maximal torus of G. This reduction of cohomology theory to group theory has not been yet achieved in characteristic $p \neq 0$ (except however for characteristic classes of the tangential structure, see §14); the best tools presently available are spectral sequences; one of them [2, Théorème 22.1] is so to say a weak analog of H. Cartan's algebra mentioned above and it seems possible that H. Cartan's theorem is also valid mod p when G and U have no p-torsion or, more generally, when their cohomology rings mod p are generated by universally transgressive elements.

The proof of (C) in §9 also gives information on $H^*(G/U, Z_p)$ when G and U have the same rank and no p-torsion [2, §§29, 30]. In that case G/U has no p-torsion, $H^*(G/U, Z_p)$ is isomorphic to the quotient of $I_U \otimes Z_p$ by the ideal generated by the elements of strictly positive degrees in $I_G \otimes Z_p$, and its Poincaré polynomial is

$$P_p(G/U, t) = (1 - t^{2m_1}) \cdots (1 - t^{2m_r}) \cdot (1 - t^{2n_1})^{-1} \cdots (1 - t^{2n_r})^{-1},$$

$(2m_1 - 1, \cdots, 2m_r - 1)$ (resp. $2n_1 - 1, \cdots, 2n_r - 1$) being the degrees of the primitive elements in $H^*(G, Z_p)$ (resp. $H^*(U, Z_0)$). This is valid in particular for $p = 0$ and the last equality is then the Hirsch formula, conjectured by G. Hirsch, proved by H. Cartan-J. L. Koszul [20; 42] and by J. Leray [49] (see also [2, §26]). Also, if only U has no p-torsion then G/U has no p-torsion and (13.1) is again valid although the multiplicative structure may be rather different. Similar statements are valid for integral cohomology when the assumption "no p-torsion" is replaced by "no torsion." These results allow us to describe rather completely the cohomology of certain classical coset spaces, like complex Grassmann manifolds or complex quadrics [2, §31].

In §9 it was pointed out that results similar to (C) hold for the cohomology mod 2 of the orthogonal groups and their classifying spaces if maximal tori are replaced by maximal abelian subgroups of type $(2, \cdots, 2)$. This analogy can be pushed further and leads in certain cases to results paralleling those above for $H^*(G/U, Z_2)$ when G and U have the same "2-rank," i.e., have a maximal abelian subgroup of type $(2, \cdots, 2)$ in common. One gets a "Hirsch formula" mod 2, in which degrees of primitive elements are replaced by degrees of the members of simple systems of universally transgressive generators [3].

At the other extreme so to say is the case where U is a circle. Koszul had shown that $P_0(G/T^1, t)$ (G compact, semi-simple) is obtained from $P_0(G, t)$ simply by writing $(1+t^2)$ for $(1+t^3)$; this follows also from H. Cartan's theorem, which moreover gives the multiplicative structure of $H^*(G/T^1, R)$. The cohomology mod p of that quotient can also be investigated when $H^*(G, Z_p)$ has a simple system of universally transgressive generators; here $P_p(G/U, t)$ is $P_p(G, t)$ with one of the factors $(1+t^{2k+1})$ replaced by $(1+t+\cdots+t^{2k})$, but k may now be $\neq 1$ and its value depends in fact on the position of subgroup in the group. This case has not yet been treated in the literature but it is quite analogous to, only slightly simpler than, the study of the quotient G/Z_p, which is done in [5, §10]; the common feature of these two problems is that $H^*(B_U, Z_p)$ is or is "almost" a ring of polynomials with one generator.

14. **Characteristic classes of homogeneous spaces.** A coset space G/U is a differentiable manifold; hence we may consider the Pontrjagin or Stiefel-Whitney classes of its tangential structure, or also its Chern classes when it carries an almost complex or a complex analytic structure invariant under G. It turns out that they can be expressed by means of group theoretical invariants, namely, by roots [9]. We content ourselves with some brief indications, assuming familiarity of the reader with the theory of roots of complex or compact semisimple Lie algebras (see e.g. [26; 74; 81]). The roots are usually defined as linear forms on the Lie algebra of a maximal torus T but they can in a natural way be identified with elements of $H^1(T, Z)$, and, via transgression, with elements of $H^2(G/T, Z)$. Now to any system of positive roots (a_i), $(i=1, \cdots, m$, $\dim G=\dim T+2m)$, there is associated a complex analytic structure on G/T (about which more will be said in §15), invariant under G. Its ith Chern class is then the ith elementary symmetric function in the a_i's. If G and its closed connected subgroup U have the same rank and if G/U carries an invariant complex analytic structure, then the map f^* induced by the natural projection of G/T onto G/U maps the ith Chern class on the ith elementary symmetric function in the roots of G which are not roots of U. Similarily, the Pontrjagin classes are symmetric functions in squares of roots (and vanish for G/T). When G and U have different ranks, one must take the roots of G with respect to a maximal torus of U. The Stiefel-Whitney classes (mod 2) may be also connected with symmetric functions not exactly in the roots but in what may be called the "2-roots," i.e., essentially the characters of a maximal abelian subgroup of type $(2, 2, \cdots, 2)$ in the adjoint representation. These facts are interesting for G/T, where they allow us to establish relations between topological properties, theorem of Riemann-Roch, and formulas of representation theory [9].

15. G/T **and related spaces. Cellular decompositions.** We say that

a compact manifold has a cellular decomposition if there is a partition of M into a finite number of embedded submanifolds M_i, the "open cells," each homeomorphic to some affine space, and having a set-theoretical boundary made up of open cells of strictly smaller dimensions. The closures of the open cells then form a cellular decomposition of M in the sense of combinatorial topology. In the same way a compact complex analytic manifold has a complex analytic cellular decomposition if the M_i are embedded complex analytic submanifolds, bi-holomorphically homeomorphic to complex affine spaces. In his Thesis [33] Ehresmann constructed complex analytic cellular decompositions for certain algebraic homogeneous spaces (Grassmann manifolds, complex quadrics, $SO(2n)/U(n)$, $Sp(n)/U(n)$, and some others), using Schubert systems. This can be done in fact for all algebraic varieties which admit a transitive compact semi-simple group of complex analytic homeomorphisms, by a group theoretical method [6]. The manifolds in question are the quotients G/U, where G is compact semi-simple and U is the centralizer of a toral subgroup of G. The complexification G^c of G also operates on G/U, and the latter may be identified with a quotient of G^c by a complex subgroup V containing a maximal connected solvable subgroup L of G^c (and, conversely, every quotient G^c/V with $V \supset L$ is algebraic); the group L is generated by a Cartan subalgebra and the root vectors corresponding to a system of positive roots. The open cells are then just the orbits of L acting in the obvious way on G^c/V; they are also birationally and biregularly equivalent to complex affine spaces, and it follows that these coset spaces are rational varieties; this last fact was first proved by M. Goto [37] and, as a matter of fact, precisely by construction of the open cell of highest dimension. Among these spaces we find G/T, where T is as usual a maximal torus of G, which may also be written G^c/L. In this case the cells are in 1-1 correspondence with the elements of the Weyl group $W(G)$ of G, which implies that the Euler-Poincaré characteristic $\chi(G/T)$ of G/T is equal to the order of the Weyl group. This gives a new proof of a result of A. Weil (C. R. Acad. Sci. Paris vol. 200 (1935) pp. 518–520) which was later proved again by Hopf-Samelson [39]: more precisely [39] gives $\chi(G/U)$ for any coset space G/U with G compact, but this follows by easy fiber bundle arguments once $\chi(G/T)$ is known. (*Added in proof.* The previous cellular decomposition of G/T is also obtained in a paper by C. Chevalley on simple groups, to appear in The Tohoku Mathematical Journal.)

Since they carry a complex analytic cellular decomposition, the coset spaces considered above have no torsion, a fact first proved by

R. Bott [13], who moreover gives an expression of the Poincaré polynomial of the space computable out of the diagram of singular elements of G. Both procedures are group theoretical but have the disadvantage of not yielding any information on the cup-product in $H^*(G/U, Z)$. For G/T another approach has been devised by Bott and Samelson [14]. They first construct a space, say Y_G, of the same dimension as G/T, which is a multiple fibering with 2-dimensional spheres as fibers, and whose cohomology ring is determined by the Cartan integers of G. Then they prove the existence of a map f of degree 1 of Y_G onto G/T, and show that the 1-1 image of f^* is a direct summand completely characterized by the expressions of the roots of G as linear combinations of fundamental roots. It may be hoped that this will for instance allow one to prove *a priori* a fact *checked* by the author [7] for all simple groups G: If p is strictly greater than the coefficients of the dominant root, expressed as sum of fundamental roots, then $H^*(G/T, Z_p)$ is generated by 1 and by $H^2(G/T, Z_p)$, and, consequently, B_G has no p-torsion, when G is simply connected.

Ehresmann [34] and Nordon [61] have studied cellular decompositions of some real algebraic manifolds, notably the real "flag" manifolds $O(n)/O(n_1)\times \cdots \times O(n_k)$ $(n_1+\cdots+n_k=n)$. A new difficulty arises because the cells may have all possible dimensions and are not always cycles, so that incidence numbers must be computed. It was shown that these cells are cycles mod 2 and that all torsion coefficients of the integral homology groups are equal to 2. These cells may also be defined as orbits, in essentially the same way as in the complex case, and it might have some interest to know whether this method extends to a wider class of real algebraic coset spaces and if so, whether these spaces have the same torsion properties as the flag manifolds.

16. **Homogeneous complex spaces.** In connection with the manifolds discussed in §15, we now digress slightly to report on recent investigations about homogeneous complex spaces.

A coset space G/U is homogeneous complex (resp. homogeneous kählerian) if it carries a complex analytic structure (resp. and a kählerian metric) invariant under G. The quotients G/U (G compact, U centralizer of a torus) considered in §15 are exactly the compact homogeneous kählerian spaces with finite fundamental group and are in fact all simply connected. We have pointed out that they are rational algebraic varieties; they have been studied from a different point of view by J. Tits [77; 78].

More generally, Wang [79] has determined all compact simply connected homogeneous complex spaces. These are precisely the even dimensional quotients G/U where G is compact and U is (locally) the product of a torus T^s by the semi-simple part of the centralizer of some torus (containing T^s of course, but possibly bigger). Among them we find the even dimensional compact groups, a particular case also discussed by H. Samelson [68]. In the nonalgebraic case, not much is known about compact homogeneous complex spaces with infinite fundamental group: if it is algebraic, then it is globally a direct product of a rational coset space (see above) by an algebraic complex torus (not yet published by the author); if it is homogeneous kählerian, then it is locally the product of a torus by a simply connected kählerian homogeneous space [52]; if the compact complex manifold is "complex parallisable", it is the quotient of a complex Lie group by a discrete subgroup [80].

In the noncompact case, some results have been obtained about homogeneous kählerian or homogeneous symplectic (i.e. carrying an invariant nondegenerate closed exterior 2-form) coset spaces. For instance [6], if G is semi-simple, with maximal compact subgroup K, then G/U is homogeneous kählerian if and only if U is compact, centralizer of a torus, and G/K is hermitian symmetric. (For hermitian symmetric spaces, see [10].) Related results are obtained by Koszul [43] who investigates necessary conditions for a homogeneous space to carry a nondegenerate 2-form derived from an invariant volume element, in the same way as Bergmann's metric is derived from the kernel function. Under that assumption, he moreover proves e.g. that if G is effective, then its center must be discrete.

A bounded domain in C^n carries a kählerian metric (the Bergmann metric) invariant under all complex analytic homeomorphisms; hence it is homogeneous kählerian whenever it is homogeneous complex. E. Cartan determined all those which are symmetric ([17], see also [10]). They are products of irreducible ones, the latter ones being exactly the quotients G/K (G simple noncompact, K maximal compact subgroup) where K has a nondiscrete center. To be more accurate, E. Cartan had checked case by case (except for possibly two cases) that all these spaces are equivalent to bounded domains; recently this has been proved *a priori*, by a general argument, for all of them by Harish-Chandra (yet unpublished). E. Cartan posed the question as to whether there are bounded homogeneous domains which are not symmetric. Only partial answers have been given so far: the bounded domain G/U is symmetric if the linear isotropy group is irreducible [50] (also contained in [43]) or if G is semi-simple [6; 43].

17. Noncompact Lie groups and their coset spaces. A fundamental theorem due to E. Cartan-Malcev-Iwasawa (see [67] for references) states that a connected Lie group G is homeomorphic to the direct product of a maximal compact subgroup K by a euclidian space. More precisely, there exists a subspace F homeomorphic to R^m such that each $g \in G$ may be written in one and only one way as product $k \cdot f$, with $k \in K$, $f \in F$, depending continuously on g. We write $G = K \cdot F$ or $G = K \cdot R^m$. Moreover, every compact subgroup of G is conjugate by an inner automorphism to a subgroup of K.

The circumstances are more complicated for a coset space G/U, even when U is connected. Let us write

$$G = K \cdot R^s, \quad U = L \cdot R^t$$

$$(K, \text{ resp. } L, \text{ compact maximal in } G, \text{ resp. } U, K \supset L).$$

Then $s \geqq t$ and G/U has the same homological properties as $K/L \times R^{s-t}$ [4], and in fact K/L is a deformation retract of G/H [58]: this implies in particular that G/U has at most 2 ends in Freudenthal's sense [4]. The space G/U is homeomorphic to the product $K/L \times R^{s-t}$ when $s-t=1$ [4] or also when G is solvable (Chevalley, see [57]; G/U is then a product of circles and straight lines). It is natural to ask whether the above decompositions of G and U may be chosen in a "coherent" way, i.e., with $R^t \subset R^s$. This apparently difficult question has not yet been fully answered. Mostow [58] has shown, under a rather often fulfilled condition, which might conceivably be superfluous, that there exist subspaces E, F of G, homeomorphic to euclidian spaces, generated, by means of the exponential, by subspaces of the Lie algebra of G invariant under the inner automorphisms of L, such that

$$G = K \cdot F \cdot E, \quad U = L \cdot F.$$

It follows that G/U has a fibering with base K/L and typical fiber $E = R^{s-t}$. Mostow's condition is that U be "self-adjoint modulo the radical $R(G)$ of G," i.e., $UR(G)/R(G)$ must be invariant under one characteristic involutive automorphism of the semi-simple group $G/R(G)$. This condition is always fulfilled when U is semi-simple [59]. It has also been shown in [58] that if G/U is acyclic (i.e., as is easily seen, if and only if $K = L$) then G/U is homeomorphic to a euclidian space (without any other condition on U). Some acyclic homogeneous spaces are discussed in [15].

The quotients G/U, with G connected noncompact, U nonconnected, have been investigated so far only when G is nilpotent [54; 55; 60], or solvable [57]. In the former case G/U is the product of a euclidian space by a compact homogeneous space of a nilpotent group; in the latter case there exists a subgroup U^* of finite index of U, containing the commutator group of U, such that G/U^* is the product of a euclidian space by a compact coset space of a solvable group. The example of the Möbius strip shows that one may have to go to a covering to get a product decomposition. Since a connected and simply connected solvable group G is homeomorphic to euclidian space (see e.g. [18]), it is immediately seen that G/U is a classifying space, in the sense of §8, for its fundamental group U/U_0, (U_0 connected component of the identity in U). It follows then from standard facts of fiber bundle theory that if G/U and G'/U' (G, G' simply connected solvable) have isomorphic fundamental groups, then they have the same homotopy type. If they are moreover compact, then they are actually homeomorphic [57]. When G, G' are nilpotent simply connected, and U, U' are discrete, this homeomorphism is induced by an isomorphism of G onto G' carrying U onto U' [54], but this need not be so in the solvable case. Malcev has proved that a nilpotent Lie group G has a discrete subgroup U with compact quotient G/U if and only if its Lie algebra has rational constants of structure, relative to a suitable basis; this is not generally true for solvable groups, in fact, the affine group of the real line has no discrete subgroup with a compact quotient, as may be seen without difficulty.

A compact coset space G/U with infinite fundamental group is not in general the quotient of a compact Lie group, and little is known about its homological properties, except for nilpotent G. In that case G/U is orientable and its Euler-Poincaré characteristic is ≥ 0 [60]; the second property is also shared by the quotients of compact groups [39] and it is not known whether this is true for all compact homogeneous spaces. For nilpotent G again, $H^*(G/U, R)$ is isomorphic to the cohomology ring of the Lie algebra of G modulo the Lie algebra of U, in the sense of Chevalley-Eilenberg; this had been proved by Matsushima [55] in dimensions 1, 2 and was generalized by K. Nomizu [60] for all dimensions. Here again, easy examples show that it may be wrong for solvable G. Even in the nilpotent case, $H^*(G/U, R)$ is not always isomorphic to the cohomology ring of invariant differentiable forms [60], in contrast with E. Cartan's result [16] about compact groups G.

A discrete subgroup is the fundamental group of the quotient of a connected nilpotent (resp. solvable) Lie group if and only if it is a torsion free finitely generated nilpotent group [54], (resp. the extension of a torsion free finitely generated nilpotent group by a finitely generated free abelian group (H. C. Wang, *Discrete subgroups of solvable groups* I, to appear in the Annals of Mathematics)).

18. **Homotopy groups of Lie groups.** The nth homotopy group $\pi_n(X)$ or $\pi_n(X, P)$ of a space X is the set of all homotopy classes of continuous maps of the n-dimensional sphere S_n into X, sending a fixed point of S_n onto a fixed point $P \in X$, endowed with a suitable law of composition (see e.g. [73]), which is commutative for $n \geqq 2$. The group $\pi_1(X)$ is the fundamental group of X, it is also abelian when X is a topological group (or an H-space). $\pi_n(X \times Y)$ is the direct sum of $\pi_n(X)$ and $\pi_n(Y)$; also a space and a covering space have isomorphic nth homotopy group for $n \geqq 2$. Hence, taking into account the theorem recalled at the beginning of §17, it is enough, when dealing with Lie groups, to consider compact semi-simple (or even simple) Lie groups.

In the introduction, we mentioned H. Weyl's theorem to the effect that $\pi_1(G)$ is finite for G compact connected, semi-simple ([81]; see also [69] and [67] for references to other proofs). Extending Weyl's argument further, E. Cartan [18] showed that $\pi_2(G)=0$. It follows also directly from the fact that the space of loops on G has no torsion (for simply connected G) [13]; it can also be obtained by an easy homotopy sequence argument from the absence of torsion on G/T, discussed in §15. Bott's result also implies that $\pi_3(G)=Z$, for G compact connected, simple, non-abelian.

By use of Morse theory, Bott and Samelson have obtained the following result on $\pi_4(G)$, for G compact connected, simply connected, simple. Let T be a maximal torus, R^r its universal covering, Γ the inverse image of the identity of T in R^r, and let a be the dominant root with respect to some lexicographic order of the roots of G. Then $\pi_4(G)=Z_2$ (resp. $\pi_4(G)=0$) if the hyperplane $a=1$ has a point (resp. no point) of Γ (yet unpublished).

It was pointed out several times that the cohomology ring over the reals of a compact connected Lie group G is an exterior algebra generated by r elements ($r=$rank G) of odd degrees, say (m_i) $(1 \leqq i \leqq r)$. In other words, $H^*(G, R)$ is isomorphic to the real cohomology ring of a product of spheres of dimensions $m_1, \cdots, m_r$, which will be denoted by X_G. This of course leads to the problem of knowing to what extent G and X_G have similar properties. L. S. Pontrjagin [63] showed that $SU(n)$ $(n \geqq 3)$, which has the same integral cohomology ring as $X_{SU(n)}$, is not the product of S_3 by some space, hence is not homeomorphic to $X_{SU(n)}$. His proof used homotopy, but the theorem can also be obtained in the framework of homology, by showing that there is a nonzero Steenrod square Sq^2 in $H^*(SU(n), Z_2)$. In a similar way, Steenrod's reduced powers allow one to show that $Sp(n)$ $(n \geqq 2)$ is not homeomorphic to $X_{Sp(n)}$ [12]. The groups G_2, F_4, E_8, having torsion, are not homeomorphic to the corresponding products of spheres, and it is rather likely to be also true for E_6, E_7 so that finally it seems highly probable that no simple group of rank >1 is homeomorphic to a product of spheres.

However, G and X_G have many common properties, discovered by J-P. Serre [70]. For instance, there exists a map of G into X_G which induces an isomorphism of $\pi_n(G) \otimes Z_0$ onto $\pi_n(X_G) \otimes Z_0$ (G is compact semi-simple, connected). Known results on spheres show then that the rank of $\pi_n(G)$ is equal to the number of m_i (as defined above) equal to n $(n \geqq 1)$; in particular, $\pi_n(G)$ is finite if $n \neq m_i$ $(1 \leqq i \leqq r)$, e.g., if it is even. Let G be compact connected, simply connected semi-simple, of dimension m and rank r. Then if $p \geqq m/r - 1$, there is a map of X_G into G which induces an isomorphism of the p-primary component of $\pi_n(X_G)$ onto the p-primary component of $\pi_n(G)$ $(n \geqq 2)$ ([70, §5], we have also included the exceptional groups in the statement by making use of [7]); moreover, if G is a classical group, such a map exists only if $p \geqq m/r - 1$ [70]; however, the latter inequality is not necessary for the exceptional groups, because it follows from Toda's result $\pi_{10}(G_2) = Z_8$ (see below) that there is such a map for $G = G_2$, and $p = 3$ (see [70, last remark]).

The foregoing gives all known homotopy-properties having a general character. Other results have been obtained for the classical groups, G_2, F_4 by extensive use of homotopy theory. We summarize the main ones.

First, we remind the reader of the isomorphisms (5.1) and we recall the isomorphisms

$$\begin{array}{rcll}
\pi_i(SO(m)) & = & \pi_i(SO(n)) & (n, m \geqq i + 2) \\
\pi_i(SU(m)) & = & \pi_i(SU(n)) & (n, m \geqq (i + 1)/2), \\
\pi_i(Sp(m)) & = & \pi_i(Sp(n)) & n, m \geqq (i - 1)/4), \\
\pi_i(G_2) & = & \pi_i(\text{Spin }(7)) & (i \leqq 5), \\
\pi_i(F_4) & = & \pi_i\ (\text{Spin }(9)) & (i \leqq 6), \\
\pi_i\ (\text{Spin }(9)) & = & \pi_i\ (\text{Spin }(7)) & (i \leqq 13),
\end{array}$$

which follow from the homotopy sequence of the fiberings:

$$\begin{array}{cc}
SO(n)/SO(n - 1) = S_{n-1}, & SU(n)/SU(n - 1) = S_{2n-1}, \\
Sp(n)/Sp(n - 1) = S_{4n-1}, & \text{Spin }(7)/G_2 = S_7, \\
F_4/\text{Spin }(9) = W, & \text{Spin }(9)/\text{Spin }(7) = S_{15}
\end{array}$$

where W is the projective plane over the Cayley numbers (see [1] for the last three fiberings).

The following table lists the groups $\pi_i(G)$ for $i \leqq 13$, G a classical group of G_2, F_4 as given to the author by H. Toda. We write m, for Z_m and Z respectively. Some of these results had been established previously by several authors: For $SU(2) = S_3$, see notably [71; 73]; H. Toda, Jour. Inst. Polyt. Osaka City Univ. vol. 3 (1952) pp. 43–82, C. R. Acad. Sci. Paris vol. 240 (1955) pp. 42–44, 147–149; J-P. Serre, C. R. Acad. Sci. Paris vol. 234 (1952) pp. 1243–1245. For π_4, π_5, π_6 of the classical groups, see [73; 32; 12] respectively. The groups $\pi_7(SO(n))$, $\pi_8(SO(n))$ have been determined by Serre [71], Sugawara [75; 76], Paechter (not yet published), the two last named authors giving also the value of $\pi_9(SO(n))$. In [76] we also find partial information on $\pi_{10}(SO(n))$, $\pi_{11}(SO(n))$ and [12] gives also some results on p-primary components of higher homotopy groups of the classical groups.

	4	5	6	7	8	9	10	11	12	13
SU (2)	2	2	12	2	2	3	15	2	2+2	2+4+2
SU (3)	0	∞	6	0	12	3	30	4	60	6
SU (4)	0	∞	0	∞	24	2	120+2	4	60	4
SU (5)	0	∞	0	∞	0	∞	120	0	360	4
SU (6)	0	∞	0	∞	0	∞	3	∞	720	0
SU (7)	0	∞	0	∞	0	∞	3	∞	0	∞
Sp (2)	2	2	0	∞	0	0	120	2	2+2	4+2
Sp (3)	2	2	0	∞	0	0	3	∞	2	2
SO (8)	0	0	0	∞+∞	2+2+2	2+2+2	24+24	∞+2	0	2+2
SO (9)	0	0	0	∞	2+2	2+2	24	∞+2	0	2
SO (10)	0	0	0	∞	2	∞+2	12	∞	12	2
SO (11)	0	0	0	∞	2	2	6	∞	2	2+2
SO (12)	0	0	0	∞	2	2	3	∞+∞	2+2	2+2
SO (13)	0	0	0	∞	2	2	3	∞	2	2
SO (14)	0	0	0	∞	2	2	3	∞	0	∞
SO (15)	0	0	0	∞	2	2	3	∞	0	0
G_2	0	0	3	0	2	6	3	∞+2	0	3
F_4	0	0	0	0	2	2	0	∞+2	0	0

In this connection, let us mention a question raised by J-P. Serre: As far as homology or infinitesimal theory are concerned, the groups $SO(2n+1)$ and $Sp(n)$ "differ only with respect to the prime 2." Is it then true that $\pi_i(SO(2n+1))$ and $\pi_i(Sp(n))$ have isomorphic p-primary components for all odd primes? Using the above results, Serre has proved that it is indeed the case for $n = 3$. For $n = 1$, 2 it is of course implied by (5.1).

Very little is known about homotopy groups of homogeneous spaces, and up to now, only those of the Stiefel manifolds have been subject to rather systematic investigations, see J. H. C. Whitehead, Proc. London Math. Soc. vol. 48 (1944) pp. 243–291, vol. 49 (1947) pp. 479–481; [32]; [44] and a forthcoming paper of G. F. Paechter.

BIBLIOGRAPHY

As regards the papers published up to 1951, we refer to [67] for a complete list and mention here mainly those which are connected with the text.

1. A. Borel, *Le plan projectif des octaves et les sphères comme espaces homogènes*, C. R. Acad. Sci. Paris vol. 230 (1950) pp. 1378–1380.

2. ———, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. vol. 57 (1953) pp. 115–207.

3. ———, *La cohomologie mod 2 de certains espaces homogènes*, Comment. Math. Helv. vol. 27 (1953) pp. 165–197.

4. ———, *Les bouts des espaces homogènes de groupes de Lie*, Ann. of Math. vol. 58 (1953) pp. 443–457.

5. ———, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. J. Math. vol. 76 (1954) pp. 273–342.

6. ———, *Kählerian coset spaces of semi-simple Lie groups*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 1147–1151.

7. ———, *Sur la torsion des groupes de Lie*, J. Math. Pures Appl., to appear.

8. A. Borel and C. Chevalley, *The Betti numbers of the exceptional groups*, Memoirs of the American Mathematical Society, no. 14, 1955, pp. 1–9.

9. A. Borel and F. Hirzebruch, *On characteristic classes of homogeneous spaces*, to appear.

10. A. Borel and A. Lichnerowicz, *Espaces riemanniens et hermitiens symétriques*, C. R. Acad. Sci. Paris vol. 234 (1952) pp. 2332–2334.

11. A. Borel and J-P. Serre, *Sur certains sous-groupes des groupes de Lie compacts*, Comment. Math. Helv. vol. 27 (1953) pp. 128–139.

12. ———, *Groupes de Lie et puissances réduites de Steenrod*, Amer. J. Math. vol. 73 (1953) pp. 409–448.

13. R. Bott, *On torsion in Lie groups*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 586–588.

14. R. Bott and H. Samelson, *On the cohomology ring of G/T*, Ibid. vol. 41 (1955).

15. L. Calabi, *I gruppi semisemplici di Lie che operano sullo spazio euclideo ad n dimensioni*, Rend. Mat. e Applicazioni Ser. V vol. 11 (1952) pp. 1–5.

16. E. Cartan, *Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces*, Annales de la Société Polonaise de Mathématique vol. 8 (1929) pp. 181–225; *Oeuvres complètes*, Part I, vol. 2, Paris, Gauthier-Villars, 1952, pp. 1081–1125.

17. ———, *Sur les domaines bornés homogènes de l'espace de n variables complexes*, Abh. Math. Sem. Hamburgischen Univ. vol. 11 (1935) pp. 116–162; *Oeuvres complètes*, Part. I, vol. 2, Paris, Gauthier-Villars, 1952, pp. 1259–1305.

18. ———, *La topologie des espaces représentatifs des groupes de Lie*, Actualités Scientifiques et Industrielles, no. 358, Paris, Hermann, 1936, *Oeuvres complètes*, Part I, vol. 2, Paris, Gauthier-Villars, 1952, pp. 1307–1330.

19. H. Cartan, *Séminaire de Topologie de l'E.N.S.* II, Paris, 1949–1950, (Notes polycopiées).

20. ———, a. *Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie*, Colloque de Topologie (espaces fibrés), Bruxelles, 1950, Liège et Paris, 1951, pp. 15–27; b. *La transgression dans un groupe de Lie et dans un espace fibré principal*, ibid. pp. 57–71.

21. S. S. Chern, *Characteristic classes of hermitian manifolds*, Ann. of Math. vol. 47 (1946) pp. 85–121.

22. ———, *Topics in differential geometry*, Institute for Advanced Study, Princeton, 1951 (mimeographed notes).

23. ———, *On the characteristic classes of complex sphere bundles and algebraic varieties*, Amer. J. Math. vol. 75 (1953) pp. 565–597.

24. C. Chevalley, *The Betti numbers of the exceptional Lie groups*, Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, Providence, American Mathematical Society, 1952, vol. 2, pp. 21–24.

25. C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 85–124.

26. E. Dynkin, *The structure of semi-simple Lie algebras*, Uspehi Matematičeskih Nauk (N.S.) vol. 2 (1947) pp. 59–127, translated in Amer. Math. Soc. Translation no. 17, 1950.

27. ———, *Topological invariants of linear representations of the unitary group*, C. R. Acad. Sci. URSS (N.S.) vol. 85 (1952) pp. 697–699.

28. ———, *A connection between homologies of a compact Lie group and its subgroups*, ibid. vol. 87 (1952) pp. 333–336.

29. ———, *Construction of primitive cycles in compact Lie groups*, ibid. vol. 91 (1953) pp. 201–204.

30. ———, *Homological characterisations of homomorphisms of compact Lie groups*, ibid. pp. 1007–1009.

31. ———, *Homologies of compact Lie groups*, Uspehi Matematičeskih Nauk (N.S.) vol. 5 (1953) pp. 73–120.

31a. ———, *Homological characterisations of homomorphisms of compact Lie groups*, Mat. Sbornik N.S. vol. 35 (1954) pp. 129–173.

32. B. Eckmann, *Espaces fibrés et homotopie*, Colloque de Topologie Algébrique (espaces fibrés), Bruxelles, 1950, Liège et Paris, 1951, pp. 83–89.

33. C. Ehresmann, *Sur la topologie de certains espaces homogènes*, Ann. of Math. vol. 35 (1934) pp. 396–443.

34. ———, *Sur la topologie de certaines variétés algébriques réelles*, J. Math. Pures Appl. vol. 16 (1937) pp. 69–100.

35. ———, *Sur la variété des génératrices planes d'une quadrique réelle et sur la topologie du groupe orthogonal à n variables*, C. R. Acad. Sci. Paris vol. 208 (1939) pp. 321–323.

36. ———, *Sur la topologie des groupes simples clos*, ibid. pp. 1263–1265.

37. M. Goto, *On algebraic homogeneous spaces*, Amer. J. Math. vol. 76 (1954) pp. 811–818.

38. H. Hopf, *Ueber die Topologie der Gruppen-Mannigfaltigkeiten und ihrer Verallgemeinerungen*, Ann. of Math. vol. 42 (1941) pp. 22–52.

39. H. Hopf and H. Samelson, *Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen*, Comment Math. Helv. vol. 13 (1940–1941) pp. 240–251.

40. I. M. James, *A note on factor spaces*, J. London Math. Soc. vol. 28 (1953) pp. 278–285.

41. J. L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France vol. 78 (1950) pp. 65–127.

42. ———, *Sur un type d'algèbres différentielles en rapport avec la transgression*, Colloque de Topologie Algébrique (espaces fibrés), Bruxelles, 1950, Liège et Paris, 1951, pp. 73–81.

43. ———, *Sur la forme hermitienne canonique des espaces homogènes complexes*, to appear.

44. T. Kudo, *Homological structure of fibre bundles*, Jour. Inst. of Polytechnics, Osaka City University vol. 2 (1952) pp. 101–140.

45. J. Leray, *Sur la forme des espaces topologiques et sur les points fixes des représentations*, J. Math. Pures Appl. vol. 54 (1945) pp. 95–167.

46. ———, *Espaces où opère un groupe de Lie compact connexe*, C. R. Acad. Sci. Paris vol. 228 (1949) pp. 1545–1547.

47. ———, *Applications continues commutant avec les éléments d'un groupe de Lie*, ibid. pp. 1784–1786.

48. ———, *L'homologie d'un espace fibré dont la fibre est connexe*, J. Math. Pures Appl. vol. 29 (1950) pp. 169–213.

49. ———, *Sur l'homologie des groupes de Lie, des espaces homogènes et des espaces fibrés principaux*, Colloque de Topologie Algébrique, Bruxelles, 1950, Liège et Paris, 1951, pp. 101–115.

50. A. Lichnerowicz, *Variétés pseudo-kählériennes à courbure de Ricci non nulle; application aux domaines bornés de C^n*, C. R. Acad. Sci. Paris vol. 235 (1952) pp. 12–14.

51. ———, *Sur les espaces homogènes kählériens*, ibid. vol. 237 (1953) pp. 695–697.

52. ———, *Espaces homogènes kählériens*, Colloque de Géométrie différentielle, Strasbourg, 1953, Publ. du C.N.R.S. Paris, 1953, pp. 171–184.

53. ———, *Un théorème sur les espaces homogènes complexes*, Archiv der Mathematik vol. 5 (1954) pp. 207–215.

54. A. Malcev, *On a class of homogeneous spaces*, Izvestiya Akademii Nauk SSSR Ser. Math. vol. 13 (1949) pp. 9–32, translated in Amer. Math. Soc. Translation no. 42, 1951.

55. Y. Matsushima, *On the discrete subgroups and homogeneous spaces of nilpotent Lie groups*, Nagoya Math. Jour. vol. 2 (1951) pp. 95–110.

56. C. E. Miller, *The topology of rotation groups*, Ann. of Math. vol. 57 (1953) pp. 95–110.

57. G. D. Mostow, *Factor spaces of solvable groups*, Ann. of Math. vol. 60 (1954) pp. 1–27.

58. ———, *On covariant fiberings of Klein spaces*, Amer. J. Math. vol. 77 (1955),

59. ———, *Decomposition theorems for semi-simple Lie groups*, Memoirs of the American Mathematical Society, no. 14, 1955.

60. K. Nomizu, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, Ann. of Math. vol. 59 (1954) pp. 531–538.

61. J. Nordon, *Les éléments d'homologie des quadriques et des hyperquadriques*, Bull. Soc. Math. France vol. 74 (1946) pp. 11–129.

62. L. S. Pontrjagin, *Homologies in compact Lie groups*, Math. Sbornik (N.S.) vol. 6 (1939) pp. 389–422.

63. ———, *Ueber die topologische Struktur der Lie'schen Gruppen*, Comment Math. Helv. vol. 13 (1940–1941) pp. 227–238.

64. ———, *Characteristic cycles on differentiable manifolds*, Mat. Sbornik (N.S.) vol. 21 (1947) pp. 233–284.

65. I. Z. Rosenknop, *Homology groups of homogeneous spaces*, C. R. Acad. Sci. URSS (N.S.) vol. 85 (1952) pp. 1219–1221.

66. H. Samelson, *Beiträge zur Topologie der Gruppen-Mannigfaltigkeiten*, Ann. of Math. vol. 42 (1941) pp. 1091–1137.

67. ———, *Topology of Lie groups*, Bull. Amer. Math. Soc. vol. 58 (1952) pp. 2–37.

68. ———, *A class of complex analytic manifolds*, Portugaliae Math. vol. 12 (1953) pp. 129–132.

69. I. Satake, *On a theorem of E. Cartan*, J. Math. Soc. Japan vol. 2 (1951) pp. 284–305.

70. J-P. Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. vol. 58 (1953) pp. 258–294.

71. ———, *Quelques calculs de groupes d'homotopie*, C. R. Acad. Sci. Paris vol. 236 (1953) pp. 2475–2477.

72. G. C. Shephard, *On finite groups generated by reflections*, Enseignement Mathématique, to appear.

73. N. Steenrod, *The topology of fibre bundles*, Princeton Mathematical Series, no. 14.

74. E. Stiefel, *Ueber eine Beziehung zwischen geschlossenen Lie'schen Gruppen und diskontinuerlichen Bewegungsgruppen Euklidischer Räume und ihre Anwendung auf die Aufzählung einfacher Lie'scher Gruppen*, Comment Math. Helv. vol. 14 (1941–1942) pp. 350–380.

75. M. Sugawara, *On the homotopy groups of rotation groups*, Math. J. Okayama Univ. vol. 3 (1953–1954) pp. 11–21.

76. ———, *Some remarks on homotopy groups of rotation groups*, ibid. pp. 129–133.

77. J. Tits, *Étude géométrique d'une classe d'espaces homogènes*, C. R. Acad. Sci. Paris vol. 239 (1954) pp. 466–468.

78. ———, *Sur les R-espaces*, ibid. pp. 850–852.

79. H. C. Wang, *Closed manifolds with homogeneous complex structure*, Amer. J. Math. vol. 76 (1954) pp. 1–32.

80. ———, *Complex parallisable manifolds*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 771–776.

81. H. Weyl, *Theorie der Darstellung kontinuerlicher halb-einfacher Gruppen durch lineare Transformationen*, I, II, III, Math. Zeit. vol. 23 (1925) pp. 271–309, vol. 24 (1926) pp. 328–395.

82. Chih-Tah Yen, *Sur les polynomes de Poincaré des groupes de Lie exceptionnels*, C. R. Acad. Sci. Paris vol. 228 (1949) pp. 628–630.

Institute for Advanced Study

THE JUNE MEETING IN VANCOUVER

The five hundred fifteenth meeting of the American Mathematical Society was held at the University of British Columbia, Vancouver, Canada, on Saturday, June 18, 1955, following the meeting on Friday of the Pacific Northwest Section of the Mathematical Association of America. Attendance was 71, including 54 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Iacopo Barsotti of the University of Pittsburgh and University of Southern California addressed the Society on *Algebraic group-varieties*. He was introduced by Professor R. M. Winger. Professor T. E. Hull presided at the session for contributed papers.

On Friday evening before the meeting there was a joint dinner of the Society and the Association, at which the visitors were greeted by Dean W. H. Gage of the University of British Columbia.

Following are the abstracts of papers presented at the meeting, those whose numbers are followed by "*t*" having been read by title. Mr. John was introduced by Professor Casper Goffman, Mr. Kobayashi by Professor C. B. Allendoerfer, Mr. Montague by Mr. Dana Scott, Dr. Saworotnow by Professor Choy-tak Taam, and Professor Wasel by Reverend T. J. Saunders.

ALGEBRA AND THEORY OF NUMBERS

596. R. A. Beaumont: *Free R-modules and algebras over a noncommutative ring.*

Let ${}_mR_n$ be the set of m by n matrices over a ring R with identity. A matrix A in ${}_mR_n$ is a unit if there exists a matrix B in ${}_nR_m$ such that $AB = I_m$ and $BA = I_n$. The only rings for which units exist ($m \neq n$) are noncommutative rings which satisfy neither chain condition for right ideals or for left ideals. Every free R-module has a unique basis number if and only if there are no units in ${}_mR_n$ for every m, n, $m \neq n$. A necessary and sufficient condition is found for the equivalence of algebras over a noncommutative ring. This condition is in the form of a matrix identity involving the multiplication tables of the algebras. (Received May 5, 1955.)

597. D. G. Higman: *On integral representations of orders in separable algebras.*

J.-M. Maranda has recently established a very interesting set of results concerning representations of finite groups by groups of automorphisms of modules finitely generated over Dedekind rings (Canadian Journal of Mathematics vol. 5 (1953) pp. 344–355, and a second paper to appear soon in the same Journal). For representations Γ, Δ of the group ring $\mathfrak{O}$ by matrices in a Dedekind ring $\mathfrak{o}$, and $\mathfrak{a}$ an integral ideal in $\mathfrak{o}$, let $B(\Gamma, \Delta; \mathfrak{a})$ denote the $\mathfrak{o}$-module of binding systems, and let $B_s(\Gamma, \Delta; \mathfrak{a})$ denote the $\mathfrak{o}$-submodule of binding systems strongly equivalent to $\mathfrak{O}$ (see the first mentioned paper of Maranda for definitions). The present note is based on the observation that the essential property of the group ring for Maranda's work is the following: The annihilators in $\mathfrak{o}$ of the modules $B(\Gamma, \Delta; \mathfrak{a})/B_s(\Gamma, \Delta; \mathfrak{a})$ for all Γ, Δ, $\mathfrak{a}$, have a nonzero intersection—in fact the group order N is contained therein. Let A be a finite-dimensional algebra over the quotient field K of $\mathfrak{o}$, and let $\mathfrak{O}$ be an $\mathfrak{o}$-order in A. Assuming that $\mathfrak{O}$ has a linearly independent $\mathfrak{o}$-basis it is proved that the corresponding intersection is nonzero if and only if A is separable. This result suffices to extend Maranda's theory without essential change from finite groups to orders in separable algebras. (Received May 6, 1955.)

598*t*. D. G. Higman: *Remark on Frobenius algebras.*

A finite-dimensional associative algebra A with identity element over a field K is a Frobenius algebra if and only if there exists an invariant bilinear form on A, i.e., a bilinear for (a, b) on A with values in K which is nonsingular and such that $(ab, c) = (a, bc)$. If $a_1, \cdots, a_n$ is a basis of A, a dual basis $\hat{a}_1, \cdots, \hat{a}_n$ is defined by $(a_i, \hat{a}_j) = \delta_{ij}$. The following theorems are proved: *Theorem* 1. A representation Γ of A in K is a direct sum of principal indecomposable representations if and only if there exists a matrix X in K such that $\sum \Gamma(\hat{a}_i) X \Gamma(a_i) = I$, the identity matrix. *Theorem* 2. A is separable if and only if $c(A) =$ the center of A, where $c(a) = \sum \hat{a}_i a a_i$. (Received May 6, 1955.)

599*t*. S. A. Jennings: *Radical rings with nilpotent associated groups.*

Let R be an associative ring which is a radical ring in the sense of Jacobson. Then the elements of R form a group G under the multiplication $x * y = x + y + xy$. The group G is said to be associated with R, while the Lie ring $\Lambda = (R)_L$ formed by the elements of R under addition and the multiplication $x \circ y = xy - yx$ is the Lie ring associated with R. Relations between R, G, and Λ are studied. In particular, it is proved that if G is nilpotent then Λ is nilpotent, and conversely. (Received May 4, 1955.)

600*t*. S. A. Jennings and H. J. Zassenhaus: *Non-archimedean group valuations.*

Let ϕ be a non-archimedean valuation of a field F, let $\mathfrak{o}_\phi = \{\lambda \mid \phi(\lambda) \leq 1\}$, $\mathfrak{p}_\phi = \{\lambda \mid \phi(\lambda) < 1\}$ where $\lambda \in F$, and let χ be the characteristic of $\mathfrak{o}_\phi$ modulo $\mathfrak{p}_\phi$. For any group G, a real valued non-negative function Φ on Γ, the group ring of G over F, will be called a G-invariant extension of ϕ if Φ is a Kürschak valuation of Γ such that $\Phi(g-1) < 1$, $\Phi(1) = 1$, $\Phi(\lambda x) = \phi(\lambda)\Phi(x)$ for $g \in G$, $\lambda \in F$, $x \in \Gamma$. A function ψ on G will be called a χ-valuation of G over F if $0 \leq \psi(g) < 1$, $\psi(1) = 0$, $\psi(gh) \leq \max(\psi(g), \psi(h))$, $\psi(ghg^{-1}h^{-1}) \leq \psi(g)\psi(h)$, $\psi(g^n) = \psi(g)$ if $(n, \chi) = 1$, $\psi(g^\chi) \leq \max(\phi(\chi)\psi(g), \psi(g)^\chi)$ and $\psi(g^\chi) = \phi(\chi)\psi(g)$ if $\psi(g)^{\chi-1} < \phi(\chi)$, $g, h \in G$. If Φ is a G-invariant extension of ϕ, then $\psi(g) = \Phi^*(g) = \Phi(g-1)$ is a χ-valuation of G, while any χ-valuation ψ of G induces a G-invariant extension $\Phi = \psi^*$ of ϕ by taking for all cases in which x can be represented as $\sum \lambda_{i_1,\dots,i_r}(g_{i_1}-1) \cdots (g_{i_r}-1)$, $\psi^*(x) = \inf \max \phi(\lambda_{i_1,\cdots,i_r})\psi(g_{i_1}) \cdots \psi(g_{i_r})$ for $x = \sum \alpha_g g \in \Gamma$, and $\psi^*(x) = \phi(\alpha_1)$ otherwise. Relations between χ-valuations of G and G-invariant extensions of ϕ are studied. In particular, the subgroups $G_{\chi,\rho} = \{g \mid \Phi(g) \leq \rho\}$ are identified abstractly for certain discrete G-invariant extensions of the trivial valuation $\phi(\lambda) = 1$, $\lambda \neq 0$ of a prime field F of characteristic χ. (Received March 28, 1955.)

601. J. E. Maxfield (p), R. G. Selfridge, and J. S. Lomont: *Similarity classifications of complex matrices.*

The following theorems, among others, are proved: A necessary and sufficient condition that an n by n matrix that is similar to a diagonal matrix be similar to a matrix A having only rational integral entries is that the characteristic roots of A be algebraic integers and that the traces of the first n powers of A be rational integers. A necessary and sufficient condition that an n by n matrix C that is similar to a diagonal matrix be similar to a real matrix is that the traces of the first n powers of C be real. A necessary and sufficient condition that a matrix be similar to a real matrix is that it be similar to its complex conjugate. (Received April 19, 1955.)

602*t*. E. V. Schenkman: *On the Engel condition of order 2 for groups.*

Let G be a group in which $[[b, a]a]$ is the identity for all a and b in G. Then for arbitrary a, b, c, d in G, $[[[a, b]c]d]$ is the identity and $[[a, b]c]$ has order 3. Thus G is nilpotent of class 3. (Received May 26, 1955.)

ANALYSIS

603*t*. M. G. Arsove: *Normal families of subharmonic functions.*

Let $\mathcal{F}$ be a family of subharmonic (or more generally δ-subharmonic) functions w on a plane region Ω, let $\Phi_w^z(t)$ be the total variation of the mass distribution for w on the disc of radius t centered at z, let K be any compact subset of Ω, and let $\Psi_{\mathcal{F}}^K(r) = \sup_{w\in\mathcal{F}, z\in K}\int_0^r \Phi_w^z(t)\,d\log t$. Theorem: if the functions in $\mathcal{F}$ are locally uniformly bounded and (*) $\lim\sup_{r\to 0} \Psi_{\mathcal{F}}(r) < \infty$ for all compact subsets K of Ω, then (1) $\mathcal{F}$ is normal, (2) every $w\in\mathcal{F}$ is the difference of continuous subharmonic functions, and (3) the total variations of the mass distributions for functions in $\mathcal{F}$ are uniformly bounded on compact subsets of Ω. A sufficient condition for (*) is that the mass distributions for the functions w in $\mathcal{F}$ be given by density functions ρ_w forming a bounded subset of L^p, $1 < p \leqq \infty$. Under the norm $\|w\| = \|w\|_p + \|\rho_w\|_p$ the δ-subharmonic functions of finite norm comprise a Banach space, and for $p = 2$ normality considerations lead to the existence of a reproducing kernel analogous to the Bergman kernel function for L^2 spaces of analytic or harmonic functions. (Received May 11, 1955.)

604. Arthur Erdélyi and C. A. Swanson (p): *Asymptotic forms for Whittaker functions.* Preliminary report.

Uniform asymptotic representations for the Whittaker functions $M_{k,m}(4kx)$ and $W_{k,m}(4kx)$ are obtained for unrestricted real x and large values of $|k|$, under suitable restrictions upon arg k. The method used is an adaptation of T. M. Cherry's technique for treating differential equations with transition points; solutions of the confluent hypergeometric equation are compared with appropriate Airy functions and Bessel functions by means of (singular) Volterra integral equations, the solutions being identified by their asymptotic behavior as x approaches a singularity while k is fixed. Two asymptotic forms are obtained for each of the two functions $M_{k,m}(4kx)$ and $W_{k,m}(4kx)$. One of these involves Bessel functions of order $2m$ and holds in the interval $-\infty < x \leqq 1-\epsilon < 1$ as $k\to\infty$; the other involves Airy functions and holds in the interval $0 < \epsilon \leqq x < \infty$ as $k\to\infty$. In addition, the Bessel function approximation for $M_{k,m}$ holds also for fixed k as $x\to 0$, and the Airy function approximation for $W_{k,m}$ holds also for fixed k as $x\to\infty$. (Received May 5, 1955.)

605. R. T. Ives: *Semi-convexity and products of locally bounded spaces.*

A set U containing the origin of a linear space is called *semi-convex* iff there is a $\beta \in]0, 1]$ such that for all non-negative λ_1 and λ_2 with $\lambda_1+\lambda_2=\beta$, it is true that $\lambda_1 U+\lambda_2 U \subset U$. For such a β, U is called *β-convex*. If τ is an admissible quasi-norm for a locally bounded topological linear space E, and $U=\{x \in E: \tau(x)<1\}$, then U is β-convex iff $\beta\tau(x+y) \leqq \tau(x)+\tau(y)$ for all $x \in E$, $y \in E$. A topological linear space is called *locally semi-convex* iff it has a fundamental system of semi-convex neighborhoods of the origin. A space is locally semi-convex iff it is isomorphic to a subspace of a product of locally bounded spaces. The space of equivalence classes of real measurable functions on the unit interval, metrized by $\rho=[\int_0^1|f-g|]/[1+\int_0^1|f-g|]$, is not locally semi-convex; the same for an example of Bourgin. This answers a question of Klee [Trans. Amer. Math. Soc. vol. 78 (1955) p. 34]. (Received May 5, 1955.)

606t. Jacob Korevaar: *Distributions defined by fundamental sequences.* IV. *The integral of a product and Schwartz' theory.*

Let the distribution ϕ be integrable over $[a, b]$, and let the product ϕg be defined on $(a-\epsilon, b+\epsilon)$ for some $\epsilon>0$. Then $\int_a^b \phi g$ exists and may be computed by integration by parts. Example $(a<\tau<b)$: $\int_a^b \delta_\tau(t) g(t) dt=[U_\tau(t) g(t)]_a^b-\int_a^b U_\tau(t) g'(t) dt = g(\tau)$. Hadamard's finite part of a certain kind of divergent integral is the integral of a product ϕg. One also has $\int_a^b \phi g=\lim \int_a^b f_n g$ for integrable $\{f_n\}$ such that $f_n \rightarrow \phi$ on $(a-\epsilon, b+\epsilon)$. Yield: a result on the (ordinary) convergence of Fourier series. A result on substitution in a distribution is proved. Example: when $g'(t) \neq 0$, $g(\tau)=0$, then $\delta\{g(t)\}=|g'(\tau)|^{-1}\delta_\tau(t)$. Application is made to substitution in a definite integral. Let $\mathfrak{D}$ denote the linear space of all $g \in C^\infty[0, \infty)$ which vanish for $t \geqq c_g$ where c_g varies with g. In $\mathfrak{D}$ the relation $g_n \rightarrow g$ will mean that (i) for some $c>0$, $g(t)=g_n(t)=0$ for $t \geqq c$, $n=1, 2, \cdots$; (ii) for every $k \geqq 0$, $g_n^{(k)}(t) \rightarrow g^{(k)}(t)$ uniformly on $[0, c]$. The inner product of a distribution ϕ and a function $g \in \mathfrak{D}$ is defined as $\int_0^b \phi g$ $(b>c_g)$. It is shown that there is a one to one correspondence between the distributions ϕ on the half-line $t \geqq 0$ and the continuous linear functionals T on $\mathfrak{D}$ (Schwartz' distributions) such that if $\phi \leftrightarrow T$, then $(\phi, g)=T(g)$ for every $g \in \mathfrak{D}$. The correspondence is preserved under operations such as differentiation. (Received April 20, 1955.)

607t. Jacob Korevaar: *Distributions defined by fundamental sequences.* V. *Fourier series of distributions.*

Corresponding to every distribution ϕ there will be infinitely many Fourier series (1) $a_0/2+\sum_1^\infty (a_k \cos kt+b_k \sin kt)$ on the interval $[0, 2\pi]$. When ϕ is integrable over $[0, 2\pi]$, then one of the series is given by a_k $(b_k)=(1/\pi)\int_0^{2\pi} \phi(t) \cos (\sin) kt dt$. The following results are needed in applications. A series (1) is a Fourier series of a distribution if and only if there are constants M and m such that $|a_k|+|b_k|<Mk^m$, $k=1, 2, \cdots$. (This condition ensures convergence on $(0, 2\pi)$ in the sense of distributions.) Termwise differentiation of a Fourier series for ϕ gives a Fourier series for ϕ'. Every Fourier series of ϕ converges to ϕ on $(0, 2\pi)$. Some definitions follow. A sequence of integrable functions $f_n(t)$ converging to a distribution ϕ on $(0, 2\pi)$ is called F-admissible when (i) the limits $\lim (1/\pi)\int_0^{2\pi} f_n(t) \cos (\sin) kt dt=a_k$ (b_k) exist for $k=0, 1, \cdots$; (ii) the a_k and b_k satisfy a set of inequalities (2). Let $\{f_n\}$ be one of the infinitely many F-admissible sequences converging to a given ϕ on $(0, 2\pi)$. The Fourier series of $\phi\{f_n\}$ is the series (1) formed with the coefficients (i). It converges to ϕ on $(0, 2\pi)$. (Received April 20, 1955.)

608t. Erwin Kreyszig: *Partial differential equations with singular coefficients.*

Particular solutions (1) $u(z, z^*) = \int_{-1}^{1}(z/2)^n \exp\left(\sum_{\mu=0}^{m} q_\mu(z, z^*)t^\mu\right)(1-t^2)^{n-1/2}dt$ of partial differential equations (2) $u_{zz^*}+B(z, z^*)u_{z^*}+C(z, z^*)u=0$ satisfy also an ordinary differential equation (3) $\sum_{\kappa=0}^{k} b_\kappa(z, z^*)\partial^\kappa u/\partial z_1^\kappa=0$, $b_k=1$, $z=z_1+iz_2$, $z^*=z_1-iz_2$, of order $k=m+1$. See Bergman, Rec. Math. (Mat. Sbornik) vol. 2 (1937) p. 1169 and the previous notes of the author. All types of (2) corresponding to (1) can be determined. Properties of (1) can be obtained using the theory of ordinary differential equations. B and C might have singularities which then correspond to those of the coefficients b_κ of (3). The fact that B and C depend in a simple manner on q_0, q_1, and q_2 only is advantageous for obtaining relations between B and C and the quantities b_κ. Theorems: I. If B has a pole of order $p \geqq 1$ at a point $z=a$, then $b_\kappa=b_\kappa(m)$ has a pole of order pk, $k=m+1-\kappa$, at the same point. If B has in particular a pole of first order at $z=a$, then (3) is of Fuchs type at this point. II. If C has a pole of order $j>1$ at a point $z^*=c^*$, then $b_\kappa(m)$ has a pole of order $k(j+1)/2+(\delta_{0\kappa}+\delta_{1\kappa})(j-1)/2$ at the same point; $\delta_{0\kappa}$ and $\delta_{1\kappa}$ are Kronecker symbols. It is possible to generalize the methods for other types of particular solutions (1). (Received April 25, 1955.)

609t. Anneli L. Lax: *On Cauchy's problem for partial differential equations with multiple characteristics.*

Let $L[u]=0$ be a linear partial differential equation of order m for $u(x, t)$, a_{ij} the coefficient of $\partial^{i+j}/\partial x^i\partial t^j$ in $L[u]$. Associate $p(s)=a_{0,m}s^m+a_{1,m-1}s^{m-1}+ \cdots +a_{m,0}$ with the principal part of $L[u]$ and $q_k(s)=a_{0,k}s^k+a_{1,k-1}s^{k-1}+ \cdots +a_{k,0}$ with the homogeneous operator of order k in $L[u]$. For constant coefficients a_{ij}, Cauchy's problem for a noncharacteristic initial curve is well posed if and only if the roots of $p(s)$ are real, and, whenever $s=\lambda$ is an r-tuple root of $p(s)$, it is an $(r-s)$-tuple root of $q_{m-s}(s)$, $s=1, 2, \cdots, r-1$. For variable coefficients $a_{ij}(x, t)$ and one multiple characteristic, $L[u]$ is transformed so that the multiple characteristic coincides with the t-axis. Cauchy's problem is well posed if for the transformed equation, the roots of $p(s)$ are real functions and, whenever 0 is an r-tuple of $p(s)$, it is an $(r-s)$-tuple root of $q_{m-s}(s)$, $s=1, 2, \cdots, r-1$. The results for constant a_{ij} are proved by using an inequality from the theory of solutions by Fourier series and applying the theory of algebraic functions, those for variable a_{ij} by a reduction of $L[u]$ to an equivalent first order non-degenerate hyperbolic system. (Received April 25, 1955.)

610. A. E. Livingston (p) and Lee Lorch: *The zeros of certain sine-like integrals.*

Let (i) $f(t)\geqq 0$, $0\leqq t<1$, (ii) $f(t) \not\equiv 0$ on any subinterval of $[0, 1]$, (iii) $f(t+n) = (-1)^n f(t)$, $n=1, 2, \cdots$, (iv) $f(t)/t$ be Lebesgue integrable over $(0, 1)$. Let C be defined (uniquely) by the two conditions $2\int_0^C f(t)dt=\int_0^1 f(t)dt$ and $0<C<1$. Then the function $\int_x^\infty f(t)dt/t$ has precisely one zero in the interval $n<x<n+1$, $n=0, 1, \cdots$, say x_n, and we show that $x_n-n \downarrow C$. A by-product of the proof is that the function $tG(2t)=t[\psi(t+1/2)-\psi(t)]$ is completely monotonic, $0<t< \infty$, where $\psi(t)=\Gamma'(t)/\Gamma(t)$. (Received May 5, 1955.)

611. M. D. Marcus and B. N. Moyls (p): *Second order Hermitian systems of linear differential equations.*

Let A, B, C be Hermitian n-square matrices. The behavior of solutions of the vector-matrix differential equation $A\ddot{x}(t)+B\dot{x}(t)+Cx(t-h)u(t-h)=F(t)$ is determined by the roots of (1) $|As^2+Bs+\exp(-hs)C|=0$. $u(t-h)$ is the unit step function, $h\geqq 0$. Considered here is the more general (2) $|s^2A+sB+p(s)C|=0$. Let s_1, s_2, p_1, p_2 be respectively, the real and imaginary parts of s and $p(s)$. Results: Assume A, B, $C\geqq 0$. (i) Let $\inf p_2(s_1, s_2)s_2^{-1}=d$ for $s_2\neq 0$, $s_1>0$; $\inf p_1(s_1, 0)s_1^{-1}=k$ for $s_1>0$. If either $B+qC>0$ for $q=d$, k or $B+qC\geqq 0$ when $A>0$, then all roots of (2) lie in the negative half-plane $\operatorname{Re} s\leqq 0$. (ii) If $p=\exp(-hs)$, $h>0$, the hypotheses of (i) become $B-hC>0$, $B>0$ or $B-hC\geqq 0$, $A>0$. (iii) If $p=1$ the condition is $A>0$ or $B>0$. In case the hypotheses of (i) fail, vertical strips in the right half-plane are given in which the roots of (2) must lie. (Received April 18, 1955.)

612*t*. Leo Moser and Max Wyman: *Asymptotic formula for the Bell numbers.*

Properties of the Bell numbers G_n, defined by $e^{e^x-1}=\sum_{n=0}^{\infty} G_nx^n/n!$ have been studied by many authors. However, only L. F. Epstein [Journal of Mathematics and Physics vol. 18 (1939) pp. 166–182] has given a formula for the asymptotic behavior of G_n. His method is rather long and yields only the first term of the asymptotic formula. Moreover, his final result contains an error. In the present paper it is shown by an entirely different method that $G_n\sim(R+1)^{-1/2}e^{n(R+R^{-1}-1)}$ where R is the real solution of $Re^R=n$. Furthermore, the complete asymptotic expansion of G_n is obtained. (Received March 21, 1955.)

613*t*. Edgar Reich: *On schlicht functions with real coefficients.* Preliminary report.

Let S_r be the class of functions $f(z)=z+a_2z^2+\cdots$ (a_k real) which are regular and schlicht in the unit circle. It is desirable to obtain distortion theorems for S_r which become false when either the hypothesis that the a_k are real is dropped, or when S_r is replaced by the class of typically real functions. An example of a theorem of this type is $|\operatorname{Im}[1/f(z)]|\leqq|z|+1/|z|$ (obtained by integrating Tammi's differential equation [Ann. Ac. Sci. Fenn. AI, 1954, paper no. 173]); the result is sharp in the sense that the right-hand side cannot be replaced by any smaller function of $|z|$. (Received April 20, 1955.)

614*t*. P. P. Saworotnow: *On the imbedding of a right complemented algebra into Ambrose's H^*-algebra.* Preliminary report.

A right complemented algebra is a Banach algebra $\mathfrak{A}$ which is a Hilbert space and which has the property that every orthogonal complement of a right ideal is a right ideal. This notion was introduced in the author's dissertation (Harvard University, 1954) (see also Bull. Amer. Math. Soc. Abstract 61-2-286). We shall say that an element x in $\mathfrak{A}$ has a left adjoint x^l if $(xy, z)=(y, x^lz)$ holds for every y, z in $\mathfrak{A}$ (compare with W. Ambrose, Trans. Amer. Math. Soc. vol. 57, pp. 364–368). The author proves that every semi-simple right complemented algebra contains a dense set of elements having a left adjoint. It follows then that every semi-simple right complemented algebra is also left complemented. The other more interesting result: for every semi-simple right complemented algebra $\mathfrak{A}$ it is possible to construct an H^*-algebra $\tilde{\mathfrak{A}}$ such that $\mathfrak{A}$ is dense in $\tilde{\mathfrak{A}}$, i.e., every simple semi-simple right complemented algebra is of the type described in one of the examples given in the author's dissertation. (Received April 8, 1955.)

615t. J. B. Serrin: *On the Harnack inequality for second order elliptic differential equations.*

The author considers the linear elliptic differential operator in n variables, $L=\sum a_{ij}\partial^2/\partial x_i\partial x_j+\sum b_i\partial/\partial x_i+c$, where the coefficients are functions of $x=(x_1, \cdots, x_n)$. Concerning the coefficients he assumes merely the normalization det $(a_{ij})=1$ and the conditions (1) $\lambda(x)=$ least proper value of $(a_{ij})\geqq k>0$, (2) $|b_i|\leqq B<\infty$, (3) $c\leqq 0$, and (4) $|a_{ij}(x)-a_{ij}(y)|\leqq\phi(|x-y|)$, where $\int_0\phi(r)r^{-1}dr<\infty$. Let U be a non-negative function satisfying $LU=0$ in a bounded domain T, and let R be a closed region in T. Let x and y be any two points in R. Then there exists a positive constant M, depending only on R, T, k, B, and $\int_0^d\phi(r)r^{-1}dr$ ($d=$ diameter of T), such that $M^{-1}U(y)\leqq U(x)\leqq MU(y)$. The proof is elementary in character and based on the maximum principle; moreover, the constant M, although complicated, may be given explicitly (this is important for several applications we have in mind). It should be noted that previous proofs of Harnack's inequality for elliptic equations (cf. Lichtenstein, Rend. Circ. Mat. Palermo vol. 33 (1912), and Feller, Math. Ann. vol. 102 (1930)) involve considerably stronger assumptions on the coefficients, and resort to the existence of a Green's function; furthermore they supply only the existence of M, while we give an explicit value. (Received April 22, 1955.)

616t. A. D. Wasel: *An integral operator in the theory of compressible fluids.*

The author considers (1) $\psi^*_{z\bar{z}}+F\psi^*=0$ (Bergman, Trans. Amer. Math. Soc. vol. 62 (1947) p. 464) which arises in the theory of compressible fluids when studying flows in the pseudo-logarithmic plane, i.e., the plane whose coordinates are $\lambda=(z+\bar{z})/2$, $\theta=(z-\bar{z})/2i$. λ is a function of the speed, see (2.7) of the above paper, and θ the angle which the velocity vector forms with the positive x-axis of the physical plane. Theorem. Let $R_n(\lambda)$, $n=0, 1, \cdots$, satisfy $R'_{n+1}=R''_n+4FR_n$, $R_0(\lambda)=1$ ($R'_n=dR_n/d\lambda$), $R_n(-\infty)=0$, and let $\sum_{n=0}^{\infty}a_nz^n$ be the function element of $f(z)$ with radius of convergence r. Then $R^{[1]}(f)=\lim_{N\to\infty}\int_C\sum_{n=0}^{N}R_n(\lambda)(t^2z)^nf(t^2z)t^{-1}\exp(-1/(2t^2))dt$, C a simple path joining -1 and 1 and avoiding the origin, is a solution of (1). $R^{[1]}$ is defined in $D=[(|z|<2|\lambda|)\cap(|z|<r)]$. By substituting $u=t^2$ and taking residues, $R^{[1]}$ becomes $\lim_{N\to\infty}\pi i[a_0\sum_{n=0}^{N}(-)^nR_nz^n/(n!2^n)+a_1z\sum_{n=0}^{N}(-)^nR_nz^n/((n+1)2^{n+1})+\cdots]=\lim_{N\to\infty}\sum_{n=0}^{\infty}a_nS_n^{(N)}(z)$. A sequence of functions Q_n exists which dominate R_n (v. Mises and Schiffer, Adv. in Applied Mech., 1948, pp. 258–259). Further $|S_n^{(N)}|\leqq|S_0^{(N)}|$, $n=1, 2, \cdots$. These facts imply that $R^{[1]}$ converges in D. Direct substitution shows that each $\lim_{N\to\infty}S_n^{(N)}$ is a solution of (1) and the theorem follows. (Received April 8, 1955.)

617t. C. H. Wilcox: *A generalization of theorems of Rellich and Atkinson.*

An exterior region V is a region consisting of all points outside a closed bounded surface S. A function $u(r)$ is a radiation function for V if it is of class C^2 and satisfies the Helmholtz equation and Sommerfeld's radiation condition $\lim_{R\to\infty}\int_{r=R}|\partial u/\partial r-iku|^2dS=0$ in the closure of V, where $k\neq0$ is a complex number satisfying Im $k\geqq0$. It is shown that every radiation function $u(r)$ satisfies $\int_\Omega|u|^2d\Omega=O(1/r^2)$, $r\to\infty$, where r, θ, ϕ are spherical coordinates, Ω is the unit sphere and $d\Omega$ the element of solid angle. As a consequence it is shown that $u(r)=(e^{ikr}/r)\sum_{n=0}^{\infty}f_n(\theta, \phi)/r^n$ in $r\geqq c$, provided $r=c$ contains S, where the series converges absolutely and uniformly in $r\geqq c+\epsilon>c$. This result was proved by F. V. Atkinson [Philosophical Magazine vol.

40 (1949) pp. 645–651] under the additional hypothesis that $u(r)=O(1/r)$, $r\to\infty$. With the help of the expansion the following uniqueness theorem is proved. Let V be an exterior region bounded internally by a regular surface S (in the sense of Kellogg, *Foundations of potential theory*, p. 112) and let f be an arbitrary continuous function on S. Then there is at most one radiation function $u(r)$ for V which satisfies $u=f$ or $\partial u/\partial n=f$ on S. This result was proved by F. Rellich [Jber. Deutschen Math. Verein. vol. 53 (1943) pp. 57–65] on the assumption that $k>0$. (Received May 3, 1955.)

APPLIED MATHEMATICS

618*t*. Isadore Heller: *Neighbor relations on the convex of cyclic permutations.*

Two vertices of a polyhedron are called neighbors of order k when they have a face of dimension k, and none of lower dimension, in common. $K(P)$ denotes the maximum value of k for a given polyhedron P, and $[X]$ denotes the largest integer not exceeding X. For the convex hull (polyhedron) P_n of all permutations of n elements (represented by square matrices of order n and interpreted as points in n^2-space) it was shown (Bull. Amer. Math. Soc. (1955) and Proceedings of the January 1955 Symposium on Linear Programming) that $K(P_n)=[n/2]$, which is rather small as compared with dim $P_n=(n-1)^2$. For the convex hull Q_n of all cyclic permutations of n elements that leave no element fixed, H. Kuhn performed computations showing that any two vertices of Q_6, but not any two vertices of Q_7, are neighbors of order 1, which means that $K(Q_6)=1$ and $K(Q_7)>1$. The present note, dealing with general n, proves, for $n\geqq 8$: (1) $K(Q_n)=K(P_n)-1=n/2-1$ if $n=4m+2$; (2) $K(Q_n)=K(P_n)=[n/2]$ if $n\neq 4m+2$. For $n=1, 2, \cdots, 6, 7$, $K(Q_n)=0, 0, 1, 1, 1, 2, 2$ respectively. (Received May 5, 1955.)

619*t*. Erwin Kreyszig: *Inclusion of eigenvalues.*

Given a Hermitian matrix A and an iterative sequence of vectors x_0, $x_1=Ax_0, \cdots$, $x_\nu=A^\nu x_0$. To determine intervals of the real axis which contain at least one of the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A ("inclusion intervals"). According to an (unpublished) theory of H. Wielandt it is possible to use all of the given vectors for this purpose. The accuracy of the inclusion depends on the number of the vectors used; but the complexity of the computation increases with the number of the vectors used. For practical purposes the "two step method" (i.e. the determination of inclusion intervals by aid of three vectors $x_{\nu-2}$, $x_{\nu-1}$, x_ν) seems to be the optimum: The inclusion is much better than those obtained by classical methods (Weinstein, Proc. Nat. Acad. Sci. U.S.A. vol. 20 (1934) p. 529; Collatz, *Eigenwertaufgaben*, Leipzig, 1949); on the other hand the numerical computation involved is not too complicated. It is possible to determine an inclusion interval of minimal length. One can always determine three inclusion intervals the intersections of which are only the end points. The analytic procedure involves differences of great numbers; the accuracy of the numerical computation can be improved by aid of orthogonalization methods similar to those used by Karush (Pacific Journal of Mathematics vol. 1 (1951) p. 233). (Received April 25, 1955.)

620*t*. Erwin Kreyszig: *Improved two step inclusion methods for eigenvalues.*

The inclusion of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of an Hermitian matrix A by aid of two step methods (see the previous note) can be improved using special information, e.g. on eigenvalue-free intervals, perhaps in the case of positive definite A or on bounds for certain eigenvalues. This information can be derived from physical properties of the problem or using general methods (Collatz, Zeits. Angew. Math. Mech. vol. 19 (1939) p. 224). There is no principal difference between the two step method and classical methods, but the inclusion is much better than in the case of the latter. Two step methods can be used also in the case of ordinary differential equations and integral equations. It is possible to develop geometrical procedures which yield a general picture of the inclusion intervals. When analytical processes for obtaining the inclusion intervals are used approximation methods are advantageous for lowering the number of computations. Orthogonalization can be used in the same manner as in the case without special information. (Received April 25, 1955.)

621. W. M. Stone: *On the probability of detection with a postdetection filter.*

The theory of Kac and Siegert (Journal of Applied Physics vol. 18 (1947) pp. 383–397) for radio receivers has been extended to realistic systems involving first and second order filters. Surprisingly simple formulas for probability of detection are obtained if signal amplitude is assumed to be Rayleigh-distributed. This is possible because certain infinite products and infinite sums involving the zeros of Bessel functions may be evaluated in closed form. The case of constant amplitude signal may be treated by a Gram-Charlier type of expansion since general expressions for the central standard moments of the distribution of output are readily obtainable. (Received May 9, 1955.)

GEOMETRY

622. V. L. Klee, Jr.: *Strict separation of convex sets.*

The convex subsets A and B of a topological linear space are said to be *strictly* separated by a hyperplane H provided A is contained in one of the *open* half-spaces determined by H and B is contained in the other. This paper gives several results related to strict separation of convex sets. The basic theorem is (I): *For a locally compact closed convex subset A of a locally convex topological linear space E, the following two assertions are equivalent:* (i) *Whenever H is a hyperplane in E which supports A, then $A \cap H$ is linearly bounded.* (ii) *Whenever B is a closed convex subset of $E \setminus A$ which can be separated from A, then B can be separated from A by a hyperplane which misses A.* Among the corollaries is (II): *If A and B are disjoint closed convex subsets of E^n and neither contains a ray in its boundary, then A and B can be strictly separated by a hyperplane.* (Received May 2, 1955.)

623t. Shôshichi Kobayashi: *Affine transformation group of Riemannian manifolds.*

Let M be a complete irreducible Riemannian manifold. Then every transformation of M which preserves the affine connection associated with the metric is necessarily an isometry, except in the case where M is the 1-dimensional Euclidean space. The proof is divided into two cases: (1) If the transformation has no fixed point, then it is an isometry without exception and (2) if it has a fixed point, then it is an isometry except in the case mentioned above. (Received April 26, 1955.)

624. T. G. Ostrom: *$n+1$ curves, dualities, and Desargues' Theorem.*

An $n+1$ curve in a finite projective plane is a set of $n+1$ points, no three of which are collinear. Here $n+1$ denotes the number of points on a line. The following aspects of $n+1$ curves are investigated: (1) properties of collineations which leave an $n+1$ curve invariant. (2) ways in which an $n+1$ curve resembles a conic in a Desarguesian plane. (3) the relation between polarities and Fano's configuration when n is even. (4) applications to cyclic planes. (Received May 2, 1955.)

LOGIC AND FOUNDATIONS

625*t*. Richard Montague: *Well-founded relations; generalizations of principles of induction and recursion.*

For formalization see abstract 627. If α is a variable and $\phi(\alpha)$ a formula, then $\mathbf{O}\alpha\phi(\alpha)$ abbreviates $\mathbf{V}\alpha[\phi(\alpha)\wedge\boldsymbol{\Lambda}\beta[\phi(\beta)\rightarrow\beta=\alpha]]$, where β is the first variable different from α and not occurring in $\phi(\alpha)$. Generalizations of principles (I) and (II) of abstract 627 can be obtained using the notion of a well-founded relation (see Zermelo, Fund. Math. vol. 25 (1935)). (I) (*The general principle of induction*). *If the formula $\phi(x)$ does not contain the variable y, then the following is a theorem: R is well-founded* $\wedge\boldsymbol{\Lambda}y[y\in$ *field* $R\wedge\boldsymbol{\Lambda}x[xRy\rightarrow\phi(x)]\rightarrow\phi(y)]\rightarrow\boldsymbol{\Lambda}y[y\in$ *field* $R\rightarrow\phi(y)]$. (II) (*A general principle of definition by recursion*). *If the term ζ does not contain the variable f, then the following is a theorem: R is well-founded* $\rightarrow\mathbf{O}f$ [*f is function* $\wedge$ *domain* $f=$ *field* $R\wedge\boldsymbol{\Lambda}x[x\in$ *field* $R\rightarrow f(x)=\zeta(\{y|\mathbf{V}z[zRy\wedge y=f(z)]\})]$. Recursion theorems for natural numbers, transfinite ordinals, and metamathematics, as well as (II) of abstract 627, can be obtained immediately from (II). (III) There is a term ζ not containing the variable f such that the converse of the theorem mentioned in (II) is a theorem. Using (II) an isomorphism theorem for well-founded relations can be obtained; a special case was proved in Mostowski, Fund. Math. vol. 36 (1949) pp. 143 ff. (IV) *R is well-founded*$\rightarrow\mathbf{V}x[R$ *is isomorphic to* $\mathbf{E}x]$. ($\mathbf{E}x$ is the membership relation restricted to the set x.) (Received May 6, 1955.)

626*t*. Dana Scott: *Definitions by abstraction in axiomatic set theory.*

For terminology and formalization see abstracts 627 and 628. The following is obtained without using the axiom of choice: (I) *If the variables x, y, and z are not bound in the formula $\phi(x, y)$, and the formulas $[\phi(x, y)\rightarrow\phi(y, x)]$ and $[\phi(x, y)\wedge\phi(y, z)\rightarrow\phi(x, z)]$ are theorems, then there is a term $\zeta(x)$ (whose free variables are those variables other than y which are free in $\phi(x, y)$) such that the following are theorems*: (i) $\mathbf{V}y\phi(x, y)\rightarrow\mathbf{V}y\,y\in\zeta(x)$, (ii) $\boldsymbol{\Lambda}y[y\in\zeta(x)\rightarrow\phi(x, y)]$, (iii) $\mathbf{V}z\phi(x, z)\rightarrow[\zeta(x)=\zeta(y)\leftrightarrow\phi(x, y)]$; in fact, the term obtained by eliminating defined symbols from $\{y|\phi(x, y)\wedge\sim\mathbf{V}z[\phi(x, z)\vee\rho z\in\rho y]\}$ will serve as $\zeta(x)$. Applications: It is customary to identify the cardinal number of a set with the least ordinal with which the set is cardinally equivalent; thus the axiom of choice is required to show that every set has a cardinal. But let $\phi(x, y)$ be a formula expressing the cardinal equivalence of x and y. Define the cardinal of x as $\zeta(x)$, where $\zeta(x)$ is the term mentioned in (I). The properties corresponding to (i) and (iii) are sufficient for an adequate notion of cardinal number. Thus the axiom of choice is unnecessary in this context. Similar remarks apply to relation types and isomorphism types. (Anthony P. Morse has introduced these latter notions into axiomatic set theory using the axiom of choice.) (Received May 6, 1955.)

627*t*. Alfred Tarski: *General principles of induction and recursion in axiomatic set theory.*

For logical symbols see Tarski, Mostowski, Robinson, *Undecidable theories*. Present remarks apply to formalized Zermelo-Fraenkel set theory, but extensions to other forms of axiomatic set theory are possible. Two kinds of expressions are distinguished—terms and formulas. The simplest terms are variables; if ζ, η are terms, then $\zeta \in \eta$, $\zeta = \eta$ are formulas; if $\phi(\alpha)$ is a formula and α a variable, then $\{\alpha | \phi(\alpha)\}$ is a term (denoting the set of sets α which satisfy $\phi(\alpha)$, if there is such a set, and the empty set otherwise); if ϕ, ψ are formulas, so are $[\phi \vee \psi]$, $\sim\phi$, etc. $\mathbf{C}x$ is the transitive closure of the set x; see Bernays, J. Symbolic Logic vol. 7 (1942) p. 136. Using transitive closure, one proves the set-theoretical principles of induction (I) and definition by recursion (II). (I) *If the formula* $\phi(x)$ *does not contain the variable* y, *then* $\bigwedge y[\bigwedge x[x \in y \rightarrow \phi(x)] \rightarrow \phi(y)] \rightarrow \bigwedge y \phi(y)$ *is a theorem.* (II) *If* $\zeta(x)$ *is a term not containing the variables* y *or* z, *then there is a term* $\eta(x)$ *with the same free variables as* $\zeta(x)$ *such that* $\bigwedge x \eta(x) = \zeta(\{y | \bigvee z[z \in x \wedge y = \eta(z)]\})$ *is a theorem.* Under the hypotheses of (II) it can also be shown there is a term $\lambda(x)$ where $\bigwedge x \lambda(x) = \zeta(\{y | \bigvee z[z \in \mathbf{C}x \wedge y = \eta(z)]\})$ is a theorem. If two terms $\eta(x)$ and $\eta'(x)$ satisfy the conclusion of (II), then $\bigwedge x \eta(x) = \eta'(x)$ is a theorem. (II) was formulated but not proved by the author; independent proofs were found by Richard Montague and Dana Scott. (Received May 6, 1955.)

628*t*. Alfred Tarski: *The notion of rank in axiomatic set theory and some of its applications.*

For terminology and formalization see abstract 627. By (II) of that abstract, recursive definitions of the form $Fx = \zeta(\{y | \bigvee z[z \in x \wedge y = Fz]\})$ can be replaced by explicit definitions. The rank of a set (introduced by a more involved method in Bernays, J. Symbolic Logic vol. 13 (1948) p. 67) can be defined recursively: $\rho x = \mathbf{C}\{y | \bigvee z[z \in x \wedge y = \rho z]\}$ or equivalently $\rho x = \{y | \bigvee z[z \in \mathbf{C}x \wedge y = \rho z]\}$. Application of rank: (I) *The theory of ordinals can be developed within the theory of ranks.* The following definition is equivalent to usual definitions of ordinals: x *is ordinal* $\leftrightarrow \rho x = x$. The ordinal sum of arbitrary sets x and y, defined recursively as follows: $x + y = x \cup \{w | \bigvee z[z \in y \wedge w = x + z]\}$, reduces, when x, y are ordinals, to the usual ordinal addition. (II) *If the formula* $\phi(x)$ *does not contain the variable* y, *then there is a term* ζ *(whose free variables are those variables other than* x *which are free in* $\phi(x)$*) such that the following are theorems*: (i) $\bigvee x \phi(x) \rightarrow \bigvee x x \in \zeta$, (ii) $\bigwedge x[x \in \zeta \rightarrow \phi(x)]$, (iii) $\sim \bigvee y \bigwedge x[x \in y \leftrightarrow \phi(x)] \rightarrow \zeta$ *is infinite.* (III) *It can be shown that, for any set* x, *the set of all sets* y *such that* $\rho y \in \rho x$ *exists*; let $\mathbf{R}x$ be this set. Every set is an element of some set $\mathbf{R}x$, where x is an ordinal; further, the sets $\mathbf{R}x$ are well-ordered by the inclusion relation. Thus the universe is, so to speak, effectively covered by a well-ordered sequence of sets. (Received May 6, 1955.)

STATISTICS AND PROBABILITY

629. Melvin Dresher: *A moment inequality.*

In a previous paper [*Moment spaces and inequalities*, Duke Math. J. vol. 20 (1953) pp. 261–271] the author described a general procedure for obtaining integral inequalities by analyzing the boundary of the convex hull of a given curve. The present paper applies these techniques to derive the inequality max $[(\mu_1 - \mu_2)/(1 - \mu_1), \mu_1 - \sigma, 1 - 2\mu_1] \leq \text{Median} \leq \min [\mu_2/\mu_1, \mu_1 + \sigma, 2\mu_1]$, where μ_1, μ_2, σ are the first moment, second moment, and standard deviations, respectively, of the distribution function on the interval (0, 1). (Received May 4, 1955.)

630*t*. P. W. M. John: *On the Feller-Lundberg phenomenon in birth and death processes.*

For a stationary birth and death process let t_∞ be the time at which the population is infinite. Then t_∞ is a random variable with expected value $\bar{t}_\infty$. The Feller-Lundberg phenomenon asserts that under certain conditions the probability $p[t_\infty < \infty] > 0$. It is shown here that $p[t_\infty < \infty] > 0$ implies $p[t_\infty < \infty] = 1$. Indeed, an expression is found for $\bar{t}_\infty$, from which the above result, as well as a new proof of the Feller-Lundberg Theorem, extended to birth and death processes, is derived. The latter result differs in detail from one obtained by Reuter and Ledermann. (Proc. Cambridge Philos. Soc. vol. 49 (1953) pp. 256–262.) (Received March 28, 1955.)

TOPOLOGY

631t. Ernest Michael: *On a theorem of Kuratowski.*

The following theorem, due to Kuratowski in the separable case, has been proved by J. Dugundji (unpublished). If X and Y are metric spaces, if Y is LC^n, and if $A \subset X$ is closed with (covering-) dim $(X-A) \leq n+1$, then every continuous $f: A \to Y$ can be extended continuously over an open $U \supset A$; if Y is also C^n, then one can even take $U = X$. Similar methods, together with the relevant known extension theorems without dimensional assumptions (cf. O. Hanner [Ark. Mat. vol. 1 (1951) pp. 375–382], and C. H. Dowker [Ark. Mat. vol. 2 (1952) pp. 307–313]) imply that this result remains true for all normal (resp. collectionwise normal) X if and only the metric space Y is complete and separable (resp. complete). (In general, one need only require that dim $(C) \leq n+1$ for every subset C of $X-A$ which is closed in X; for metric X, this of course implies that dim $(X-A) \leq n+1$). (Received April 6, 1955.)

V. L. KLEE, JR.
Associate Secretary

BOOK REVIEWS

Some basic problems of the mathematical theory of elasticity. By N. I. Muskhelishvili. 3d, rev. and augmented ed., trans. from the Russian by J. R. M. Radok. Groningen, Noordhoff, 1953. 31+704 pp. 38.00 florins.

The three-dimensionality of space is one of those things the applied mathematician has to put up with. It is not merely that three is greater than two and so offers more complications; the trouble seems to lie rather in the oddness of three which hinders an effective algebraic approach. Thus after one is tired of playing with the equations of three-dimensional elasticity in the best available notation, the tensorial, and having little to show in the way of problems actually solved, one turns with relief to plane problems in which the complex variable may be used; here we have not only notational compactness; we have access to a rich store of mathematical knowledge accumulated for other purposes.

The application to elasticity of the theory of the functions of complex variables has been largely the work of Russian mathematicians, and this book gives us an authoritative account, written by one who has himself contributed much. As the title of the book indicates, it is not a treatise on the whole body of elasticity but a discussion of certain problems, which are plane problems or problems which can be reduced to such.

The book had its origin in lectures delivered in 1931–32 in the Seismological Institute of the Academy of Sciences of the U. S. S. R. (but it contains nothing about elastic vibrations) and the Physico-Mathematical Institute of Mathematics and Mechanics at the University of Leningrad. The first edition appeared in 1933, the second in 1935, and the third (of which the book under review is a translation) in 1948, the general design of exposition remaining unchanged (the author says) throughout the several editions. The preface by Academician A. N. Krylov to the first edition is included; this ends as follows:

"There only remains to express the wish that in future editions, which without doubt will be required, the author illustrate the general deductions and formulae by numerical examples, by diagrams and by indications as to the number of ordinates or subdivisions required for approximate integration in order to ensure accuracy within, say, ½%. He will thereby render a great service to engineers and make his excellent book more accessible to those people who will apply its deductions to the solution of the purely practical problems of the building industry."

Happily (in the opinion of the reviewer) the author does not seem to have taken this advice too seriously. The book is a mathematician's book, using consistently a powerful mathematical method (complex variable) and, although there are many practical applications, one might say that the engineer is expected to bring his own cup to the well instead of having it piped to him.

The historical notes show, at times, a rather petulant insistence on the importance of the work of Russians. One does not question this; the elegant and systematic application of complex variable theory to elasticity seems to be their achievement. But it is perhaps a little too much to describe Love's famous Treatise as "in many respects obsolete," particularly as the author makes no fewer than nineteen references to it.

However this is a minor matter. It does not mar what is indeed an excellent book, well printed, with formulae generously spaced and easy to read, and (what is more important) written in an uncramped style which carries the reader along smoothly without leaving in his mind a trail of half-formulated queries. It is a clear book. Paper has not been parsimoniously saved, and one judges that the translator has done his work well.

There are seven Parts, three Appendixes, authors' index, references, and subject index. The last is of course too brief; they always are.

Part I covers the usual three-dimensional theory of stress, strain, and Hooke's Law. It is done in the traditional clumsy notation, but this is no fault, because this three-dimensional theory is only presented to make the book self-contained. In Part II we pass to the general formulae of plane theory (plane strain and generalized plane stress, essentially one mathematically) with the biharmonic stress function and its complex representation, fundamental boundary value problems and their reduction to problems in complex function theory. Multivalued displacements (dislocations) are discussed and thermal stresses. Also conformal transformations with examples, well illustrated by diagrams.

Part III gives solutions by power series (circle and circular hole, concentrated forces, circular ring, etc.) and solutions by conformal mapping. Parts IV and V deal with Cauchy integrals and their application to boundary value problems, in particular for regions mapped on a circle by rational functions (many particular cases considered); also semi-infinite regions.

Part VI treats regions with cuts in them under the general heading of the problem of linear relationship, alias the Riemann problem or (as the author would prefer) the Hilbert problem, the problem being to find a sectionally holomorphic function $F(z)$ with a line of discontinuity L, the boundary values of which from the left and from the right satisfy the condition

$$F^+(t) = G(t)F^-(t) + f(t) \text{ on } L$$

(except at the ends), where $G(t)$ and $f(t)$ are functions given on L and $G(t) \neq 0$ everywhere on L.

Part VII becomes a little more three-dimensional, dealing with extension, torsion and bending of homogeneous and compound bars.

This is a book to be recommended in the highest terms to every serious student of the mathematical theory of elasticity. And to engineers also, for though they may find some of the work too purely mathematical for their taste, they will be rewarded by solutions of definite problems completely worked out. The reviewer was tickled by the enlivenment imparted to the solution of the torsion problem for a circular cylinder reinforced by an eccentric bar; one part of the solution is due to the cylinder itself and the other part to the "indignation" aroused by the presence of the reinforcement!

J. L. SYNGE

Vorlesungen über Differential- und Integralrechnung. Vol. III. *Integralrechnung auf dem Gebiete mehrerer Variablen.* By A. Ostrowski. Basel, Birkhäuser, 1954. 475 pp., 36 figs. 78 Swiss fr.; paper bound 73.85 Swiss fr.

This is the third and concluding volume of Professor Ostrowski's comprehensive text on calculus. For the reviews of volumes I and II, both by the present writer, the reader is referred to Bull. Amer. Math. Soc. vol. 52 (1946) pp. 798–799 and vol. 58 (1952) pp. 513–515. In this third volume there are seven chapters, all of them dealing to some extent with integration. There is an index for volumes II and III, a glossary of symbols for these two volumes, and a short list of corrections for volumes I and II.

Chapter I deals with the technique of integration. The topics are: complex numbers, partial fractions and integration of rational functions, integration of algebraic and transcendental functions, and the transcendence of e. The discussion of partial fractions is complete, with proofs.

Chapter II, on the definition of multiple integrals, is actually one third devoted to "the general case" of integrals of functions of one variable. The author first discusses sets of zero Jordan content, hereafter called ***null sets***. He then defines the double integral of f over R, where R is a closed rectangle with sides parallel to the axes and f is continuous in R except on a null set, and is bounded. Such a function is called integrable, and the existence of its integral is proved. The same method is used for single and triple integrals. The author explains in the preface his reasons for this choice of a basic definition of integration for this particular book. Integrals over other bounded regions are defined by enclosing the region G in a rectangle R and extending the function by zero values in $R-G$. The boundary of G is required to be a null set. There follows the theory of evaluating multiple integrals by iterated integrals.

Chapters III and IV are concerned with line integrals, surface integrals, the divergence theorem, Stokes's theorem, exact differentials, change of variable in multiple integrals, and applications of multiple integrals to the finding of volumes, centers of mass, and so on. The treatment is careful and thorough. There is, for example, a proof that any plane region whose boundary consists of a finite number of smooth arcs can be cut up into a finite number of regions of "canonical" form relative to the axes (i.e. regions for which the intersection of the interior of the region with a line parallel to an axis is at most a single segment). It is shown that the divergence and curl, as expressed in rectangular coordinates, are invariants relative to rotations of axes.

Improper integrals are considered in Chapter V (functions of one variable) and Chapter VI (functions of several variables). Here also we find the discussion of important special integrals, a good deal about the gamma function, and consideration of the double limit situations (e.g. change of order of integration in iterated improper integrals) so crucial in many calculations. Chapter VI concludes with a demonstration of the integral inequality of Pauli, which can be used to express the uncertainty principle of quantum mechanics. The treatment of improper multiple integrals was very interesting to the reviewer, for this is a subject on which most books at this level are very sketchy. If G is any open set in the plane, the integral of f over G is defined in such a way that, if it exists, then the integral of $|f|$ over G also exists. Here, as so often elsewhere throughout these volumes of Ostrowski, there is much profit for the teacher of calculus and analysis.

The final chapter deals with Fourier series and integrals. The convergence theory is disposed of neatly and briskly (for functions differentiable from right and left at a point, when defined there by the limit from the corresponding side). Among the further topics are:

Parseval's formula, Poissons's summation formula, and the Rayleigh-Plancherel formula. The exercises contain many important results, including Fejér's theorem on summation of Fourier series by arithmetic means. Contrary to the standard practice, the author defines complex-valued functions f and g to be orthogonal if the integral of fg (instead of $f\bar{g}$) is 0. The discussion of Bessel's inequality and Parseval's formula, however, is solely for real-valued functions.

As in the first two volumes, in the third also we find a large and valuable collection of exercises. They occupy more than one fifth of the space in the book. Teachers of analysis will find a perusal of them rewarding.

The book as a whole is an admirable exposition of the fundamentals of calculus in a thoroughgoing way at a reasonably elementary level. The author does not hesitate to take time and space for unusually thorough discussion of matters which are often glossed over in elementary texts and ignored in more advanced works.

It is too bad that the book is so expensive. The cost of the three volumes is 184.10 Swiss francs, or about $43.00 at the current rate of exchange. This is formidable to the point of putting the book completely out of reach of most European students. As I was told in one German university town, a student can live for a month on less than it takes to buy the three volumes of this book.

ANGUS E. TAYLOR

The elements of probability theory and some of its applications. By Harald Cramér. New York, Wiley, 1955. 281 pp. $7.00.

This book may properly be called the junior students' *Mathematical methods of statistics*, which has won the author wide recognition. It fills a need for an introductory text for a class whose main interest is statistics. More than half (roughly from Chapter 8 on) of the material belongs to conventional statistics rather than conventional probability; in fact the latter part of the book is a small compendium on sampling and testing. Statisticians will, however, find this part rather old-fashioned: the Neyman-Pearson theory is barely touched upon while Wald's sequential analysis and decision theory are only mentioned. It would seem that the elements of such modern theories have interesting probability content and are no less amenable to an elementary discussion than some of the topics chosen here. As a probability text for the general mathematics student the book will be found somewhat lacking in attractions, although quite adequate and very respectable. It goes as far as the Bernoulli and De Moivre theorems and a statement of the central limit theorem and some of its ramifications. From there on it is the χ^2, t and F distributions and correlation and regression in more than one dimension. The (moment) generating function is introduced but its use is not exploited. Whatever material is covered is, however, handled expertly and neatly and there are some instructive examples offering a few glimpses into more advanced topics (e.g. §6.5, §7.5). Students of sciences whose main interest in probability is not the handling of data will find the applications in this book biased, as recognized in the preface. For example, the simplest gambling or random walk problems (which invariably appeal to the novice and are growing in theoretical and practical importance) are illustrated by only one or two examples (§4.3). Admittedly at this junior level it is impossible to present many interesting results of probability (Feller in his well-known book has nevertheless succeeded in doing a lot of this with only slightly more advanced techniques); it may be argued whether some of the descriptive statistics cannot be sacrificed to make room for other topics. Another arguable point is the frequency definition of probability entailing a merely operative definition of random variable. It may be mentioned that even in his larger book the author tried to smooth the way for the reader by offering an axiomatic definition of random variable, which has been criticized both here and abroad. (In his 1937 Cambridge tract he took an impeccable and austere approach.) The realities of an elementary textbook may be different, but even if we should agree that the frequency definition is the only one which can be put over at this level, there is still the question whether education should follow the way of least resistance.

The book is written in lucid style and with uniform care. A specially good feature is the inclusion of abundant and varied exercises with solutions, which should add a great deal to the usefulness of the book as a text.

K. L. CHUNG

Analytische Geometrie. By Gunter Pickert. (Mathematik und ihre Anwendungen in Physik und Technik, series A, vol. 24.) Leipzig, Geest and Portig, 1953. 10+397 pp. 26 DM.

A welcome innovation in this textbook is its division into three parts: I. Affine geometry, II. Euclidean geometry, III. Projective geometry. In Part I, real affine geometry of n dimensions is derived from six axioms concerning the primitive concepts *point* and *vector*. But the development is mainly algebraic. There are several chapters on linear equations, matrices and determinants. In Part II, the Euclidean metric is introduced by means of four further axioms concerning length and orthogonality. These yield a definition for the inner (or scalar) product $\mathfrak{x}\mathfrak{y}$ of two vectors $\mathfrak{x}$ and $\mathfrak{y}$. (American readers may at first be confused by the author's use of a dot when a vector is multiplied by a scalar but *not* when two vectors are multiplied together!) The other topics in this part include congruent transformations, complex geometry, and quadrics. In Part III, projective n-space is derived from affine $(n+1)$-space by identifying the points of the former with the classes of parallel vectors in the latter. After discussing cross-ratio, collineations and correlations, polarities, and the projective theory of quadrics, the author shows how the affine and metrical geometries can be derived from projective geometry. It is unfortunate that he failed to take full advantage of his division of the book into three parts. Content (i.e., area, volume, etc.) is considered in Part II, and barycentric coordinates in Part III, whereas both these subjects properly belong to affine geometry. The text and the 76 figures are clear and accurate. The book ends with a short bibliography, a full index, and a useful list of symbols.

H. S. M. COXETER

Rational fluid mechanics, 1687–1765. By C. Truesdell. Editor's introduction to vol. II, 12 of Euler's works. Offprint from Leonhardi Euleri Commentationes Mechanicae ad Theoriam corporum fluidorum pertinentes (Euleri Opera Omnia, Series II, vol. 12.) Zürich, Füssli, 1954. 125 pp.

The volumes of the great Euler edition have in recent years been provided with excellent introductions, which help to clarify the astonishing achievements of this eighteenth century mathematician. Professor Truesdell has written one of these introductions; it is the preface to the first volume of Euler's contributions to the theory of fluid motion, and an interesting piece of work it is. We find in his essay not only an extensive account of Euler's main papers on hydrodynamics, but also a report on the achievements of those authors who, from Archimedes on, prepared the way to Euler's theory. We thus have in this introduction an accurate and comprehensive account of an important field of early science, a field never before so carefully investigated; a valuable contribution to the history of science in general and to the appreciation of Euler's hydrodynamical work in particular.

We owe to Euler what we may call "classical" hydrodynamics, the theory underlying the science of our present textbooks, which he established so thoroughly that all authors before him can safely be classified as "prehistoric." However, in contrast to Euler's mechanics of points and rigid bodies, his hydrodynamics has not come to us in the form of a textbook, but only in a series of papers. The principal four papers, republished in this tome II 12, and analyzed by Professor Truesdell in his introduction, were written between 1752 and 1755. The oldest of them, the "Principia motus fluidorum," was the last to see the light; we find it in the Commentaries of the St. Petersburg Academy published in 1761. The three other essays, the "Principes généraux de l'état d'équilibre des fluides," with two sequels, form a consecutive set in vol. II of the Memoirs of the Berlin Academy of 1755, published in 1757. Students of Euler's work will be encouraged when they see that these Berlin papers are written in French. This also helped Euler's contemporaries and students, so that his hydrodynamics became best known through these articles, with their 145 pages long enough to form a book by itself. The St. Petersburg paper is in Latin, but this need not discourage too many souls; Euler was no Cicero and his Latin easily gives up its secrets to any one of good will with some high school knowledge of the language and a dictionary.

The St. Petersburg paper has often been quoted because it contains the continuity equation and the dynamical equations for ideal compressible fluids. It also contains the "Laplace" equation $\Delta V=0$ and the "Killing" equation for locally and instantaneously rigid motion. The theory in full maturity can be found in the three Berlin papers; the editor calls their appearance a turning point in the history of physics. Indeed, it is here that a definite break is made with the principle that mechanics is necessarily a theory of little material particles: Euler's hydrodynamics is a field theory of continuous matter, and as such the prototype of all later aether, electric, magnetic and caloric theories. It also led Euler to the composition of that other masterpiece, his theory of rigid bodies, with its famous "Euler equations." The four papers on fluid bodies have also other important traits; for instance, they offer for the first time an exact separation of the kinematical and dynamical equations of the theory of continua. Another feature is the lucid formulation of the ancient field of hydrostatics.

Some more of Euler's papers, mostly later ones, are analyzed in this book, by which we can see how Euler dealt with such concepts as resistance, convection currents, and the influence of temperature. Of special interest is also Professor Truesdell's analysis of the works of Euler's predecessors and contemporaries, among whom the Bernoullis and D'Alembert stand out. The reader has a chance to get an idea of what Daniel Bernoulli's famous "Hydrodynamica" of 1738 was about

—the book in which we find the kinetic derivation of Boyle's law. Here entertainment is waiting, because of the edifying spectacle of Daniel's father John, the Groningen professor, publishing a special book to tear his son's claims down; a book with some of the choicest bombast ever written by a loving father. However, as Professor Truesdell points out, the book's moral aspects should not blind us to its actual merits: John Bernoulli's "Hydraulics" of 1743 had one thing his son's "Hydrodynamica" lacked: method. In the words of the editor: "For the first time, fluid mechanics appeared in its proper station in the great system of classical mechanics." Euler recognized this fact in a nice Latin note to John Bernoulli, which is printed on p. 1 of tome II 12, and which the editor partly reproduces in translation.

This is enough to show that Professor Truesdell has given us a special treat; it is enhanced by the generous way in which his essay is published—the way all volumes of this magnificent edition are published. Let no student of the history of mathematics believe that he need not study this book because it contains only mechanics, that is, physics. With Euler, as with all great mathematicians before the middle of the last century, applied and theoretical science are so interrelated that separation is impossible. For instance, those who only deal with what officially is called mathematics, may well miss the fact that "Laplace's equation" $\Delta V=0$ appears first in Euler's hydrodynamics, and that "Lagrangian multipliers" are equally found in Euler's work on fluids. The "Cauchy-Riemann" equations also occur first in a paper on hydrodynamics, this time written by D'Alembert, and duly quoted by Professor Truesdell.

At the end of the book is a chapter on the hydrodynamical theory in Euler and Lagrange's researches on the theory of sound. It is in a paper by Euler of 1759 (printed in tome III 1) that we first find the wave equation as the expression of aerial propagation in one dimension. Euler's starting point was his hydrodynamical theory, and Professor Truesdell remarks: "It is the first example of a precise *analogy* in mathematical physics." In a subsequent paper of 1759 Euler returns to the wave equation, now in three dimensions, and tackles it for cylindrical and spherical waves. It is near this point that we are warned against the use of the terms "Lagrangian" versus "Eulerian" when we make the "material" or "spatial" choice of coordinates in describing the motion of a continuum, a discrimination which goes back to Dirichlet. However, Euler has claim to both methods, as already Riemann explained to Hankel. This is just another example of how little Euler was—and still is—known.

Essays like this one by Professor Truesdell can help to overcome this lack of information. It is not just a question of historical piety and correct assignment of priority. Euler actually is good reading, and we must consider Professor Truesdell's introduction as an invitation to read him, like Christopher Morley's introduction to Shakespeare. After all, as Jacobi already said: "Today it is quite impossible to swallow a single line by D'Alembert, while we still can read most of Euler's works with delight."

D. J. STRUIK

Algèbre locale. By P. Samuel. (Mémorial des Sciences Mathématiques, no. 123.) Paris, Gauthier-Villars, 1953. 76 pp. 950 fr.

The object of this monograph is a presentation of the theory of local rings and their generalizations, the semilocal rings and the M-adic rings. Those aspects of the theory are dealt with which are valid in rings of arbitrary dimension. Thus the special properties of one-dimensional rings such as rings of p-adic integers or of power series in one variable are not included.

All rings considered are commutative and have an identity element. *Local rings* were introduced some fifteen years ago by Krull; they are Noetherian rings with a single maximal ideal. More generally, a *semilocal ring*, in the sense of Chevalley, is a Noetherian ring A with only a finite number of maximal ideals. If M is their intersection, then $\bigcap_{n=1}^{\infty} M^n = (0)$, and the sequence of ideals $\{M^n\}$ defines a Hausdorff topology on A. More generally, a Noetherian topological ring in which the topology is Hausdorff and is defined by the powers M^n of some ideal M is called *M-adic*. A *Zariski ring* is an M-adic ring in which every ideal is a closed set. M-adic rings and Zariski rings were introduced by Zariski (who, however, called the latter type *generalized semilocal*). The semilocal rings of Chevalley are Zariski rings, as are all complete M-adic rings. The more elementary properties of these rings are considered in Chapter I. Here are discussed their completions, homomorphisms, quotient rings, direct decompositions, and finite extensions.

In Chapter II we are concerned with a semilocal ring A and a defining ideal V of A—that is, an ideal V in A such that $M^t \subset V \subset M$, t being some integer and M the product of the maximal ideals of A. It is then proved that the length of A/V^n as an A-module is a polynomial of $P_V(n)$ for n sufficiently large. The degree d of this polynomial is independent of V and is, in fact, the minimum number of generators in any defining ideal. It is called the dimension of A and thus coincides with the notion of dimension of a local ring in the sense of Chevalley and Krull. The polynomial $d!P_V(n)$ has integer coefficients and its leading coefficient $e(V)$ is called the ***multiplicity*** of V. This generalizes a similar notion of Chevalley, who used it to define intersection multiplicities in algebraic geometry. The contents of this chapter, which include various properties of $P_V(n)$ and of $e(V)$, constitute a generalization and simplification of some previous results of the author.

Chapter III is devoted to the study of *geometric local rings*, a class of rings which includes the local rings of points on algebraic variety. Let K be a field and let $N(n, K)$ be the quotient ring of the polynomial ring $K[X_1, \cdots, X_n]$ with respect to the ideal $(X_1, \cdots, X_n)$. The geometric local rings are the members of the smallest class of rings containing all $N(n, K)$ and closed under the operations of completion and formation of quotient ring or residue class ring with respect to a prime ideal. These rings G are more tractable than local rings in general for they are rings with nucleus—i.e., they contain a nucleus N and an intermediate ring B $(N \subset B \subset G)$ such that B is finite over N, G is a quotient ring of B with respect to a maximal ideal. A *nucleus* is hereby a ring of the form $N(n, K)$ or $N(n, m, K)$, the latter being the quotient ring of the formal power series ring $K\{X_1, \cdots, X_n\}$ with respect to the ideal $(X_{m+1}, \cdots, X_n)$. An important relation exists between the multiplicities in a geometric local ring A and its completion A^*. Namely, if Q is primary in A and Q^* is a primary component of A^*Q, then Q and Q^* have the same length. It follows that if Q is prime then A^*Q is an intersection of prime ideals. Moreover, if P is the radical of the primary ideal Q, and if P^* is a prime divisor of A^*P, then $e(A_PQ) = e(A^*_{P^*}Q)$. This "theorem of transition" allows passage in proofs from a ring to its completion, for example in the proof of the following theorem. Let A be a geometric local ring, Q an ideal generated by a set of parameters, V an ideal generated by a subset of these parameters, and $\{P_i\}$ the minimal prime divisors of V; then

$$e(Q) = \sum_i e((Q + P_i)/P_i)e(A_{P_i}V).$$

This is the associativity formula, an all-important one for the theory of multiplicities. The contents of this chapter are due to Chevalley, with some improvements by the author. In the definition of nucleus, Chevalley and the present book assume for certain technical reasons that $[K:K^P]$ is finite, P being the characteristic. However, Nagata has shown that this restriction can be eliminated [Sûgaku, vol. 5 (1954) pp. 229–238 (in Japanese)].

The material of Chapter IV, on the structure of complete local rings, is due to the reviewer. Let A be a complete local ring, M its maximal ideal, ϕ the canonical homomorphism of A onto A/M. If A and A/M have equal characteristics p, then A contains a field K such that $\phi(K)=A/M$. If $p=0$, then this can be readily derived from "Hensel's Lemma," the statement of which for complete local rings is just what one expects; in the more difficult case $p>0$, the proof follows from the lifting theorem stated below. (A particularly simple proof in this equal-characteristic case has been recently given by A. Geddes [J. London Math. Soc. vol. 29 (1954) pp. 334–341].) Suppose now that A/M has characteristic $p>0$ (which is certainly the case if this characteristic is different from that of A, on which we make no assumption), let B be a complete discrete valuation ring of characteristic zero whose maximal ideal is generated by p, and assume τ a homomorphism of B onto A/M (existence of B and τ is well known). The lifting theorem then asserts the existence of a homomorphism σ of B into A such that $\phi\sigma=\tau$. From this can be deduced that every complete local ring is the homomorphic image of a power series ring over K or B, and also is (in the absence of zero divisors) a finite module over such a power series ring. Some further theorems on structure are proved and some applications made to the ideal theory in complete local rings.

In Chapter V, it is proved that if a ring with nucleus lacks zero divisors and is integrally closed then its completion also possesses these properties. This is a generalization of Zariski's theorem on the analytical irreducibility and analytical normality of normal varieties. A form of the Weierstrass preparation theorem is given and with its aid unique prime factorization is proved in the power series rings described in the preceding paragraph.

In Chapter VI there is defined a Kronecker product for two M-adic rings. Some other questions are briefly considered.

I. S. COHEN

EDITORIAL NOTE: The preceding review was found among Professor Cohen's papers after his death. Attached notes indicate that a final paragraph was to have mentioned the historical notes at the end of the chapters; the extensive bibliography (98 items); and the existence of a number of misprints, some merely typographical, others actual mistakes (pp. 22, 35, 37, 51). The mistake which Cohen considered most serious appears to be on p. 35, fourth line from the bottom.

Funzioni ipergeometriche confluenti. By F. G. Tricomi. Rome, Cremonese, 1954. 16+312 pp. 3500 Lire.

This book on confluent hypergeometric functions differs quite considerably from the same author's *Lezioni sulle funzioni ipergeometriche confluenti* reviewed recently in this *Bulletin* (vol. 60 (1954) pp. 185–189), and is to all intents and purposes a new book. Almost the only common points with the *Lezioni* are: the notation for confluent hypergeometric functions (which differs somewhat from that most commonly adopted), some of the applications discussed in the book, and, of course, the general point of view. In every other respect the differences are much more important than the similarities. The earlier work gave an introduction to confluent hypergeometric functions and was self-contained to a remarkable extent; the present work is a treatise on the subject, demanding from the reader both more knowledge, and more willingness to look up references. The author did not aim at encyclopedic completeness, yet he endeavored to give, and fully succeeded in giving, a comprehensive picture of the theory of confluent hypergeometric functions and of their applications in physics, engineering, and probability theory. Readers of earlier books by the same author need not be told that the presentation is excellent, the arrangement well planned, and the style both lucid and readable.

The so-called special functions present a somewhat peculiar problem of exposition in that books on them are read alike by physicists, engineers, and professional mathematicians; by beginners and experts; by specialists and casual users; shortly by a wide circle of readers with vastly differing mathematical training and interests. A monograph giving detailed proofs is apt to become repetitious and boring as well as exceedingly bulky, and one giving outlines of proofs only is likely to become so sketchy that it will amount to little more than a collection of results. The first kind of monograph will be an invaluable work of reference, but it is unlikely to be readable; the second kind may, with consummate skill on the part of the author, be made into a serviceable work of reference, but it will be singularly unenlightening as to methods and trends. Both kinds will be useful tools in the hands of an expert, and neither of them will be a favorite with the general reader or the beginner. It takes all the author's well-known skill to steer a reasonable middle course, and he must be complimented on having produced one of the best monographs of this kind in recent years. All basic results, and many other results, are proved in full detail, some other results are stated with references to the literature where proofs can be found, yet others are merely mentioned, again with references to existing literature. The book is well documented without overwhelming the reader with a multitude of references. Papers are referred to in footnotes throughout the book,

$$\Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \Phi(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c} \Phi(a-c+1, 2-c; x), \tag{6}$$

the limiting form of (6) for integer c follow, and the transformation $\Psi(a, c; x) = x^{1-c}\Psi(a-c+1, 2-c; x)$ is deduced from (6). The connection with Whittaker's functions $M_{k,m}(x)$ and $W_{k,m}(x)$ and with Bessel functions is also discussed. This chapter contains further integral representations involving Bessel functions, hypergeometric functions, Legendre functions, integral representations of the Mellin-Barnes type, and some miscellaneous results including results on the derivatives of confluent hypergeometric functions with respect to the parameters a, c.

In Chapter III the author discusses the asymptotic behavior of confluent hypergeometric functions, the zeros of these functions, and gives a description of their graphs for real a, c, x. This chapter is unusually well organized. Although full proofs could not be given everywhere, the reader obtains a very complete picture, including some recent results by the author and others.

In Chapter IV the author turns to special confluent hypergeometric functions. He starts with the incomplete gamma functions

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt, \qquad \Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt$$

whose investigation is based on the function

$$\gamma^*(\alpha, x) = \frac{\gamma(\alpha, x)}{x^\alpha \Gamma(\alpha)} = \Phi^*(\alpha, \alpha+1; -x).$$

The properties of these functions are largely derived from properties of Φ, Ψ, but here again the complete description of the asymptotic behavior, zeros, and graphs in the real domain, deserves special praise. Error functions and related functions, and the exponential integral and related functions are special incomplete gamma functions and presented in this context. Parabolic cylinder functions, which arise when $c = 1/2$ or $3/2$, are also included in this chapter.

The fifth and last chapter contains some examples of applications of confluent hypergeometric functions in various fields. The examples selected by the author, the two body problem of wave mechanics, bending of elastic plates, resultant of a large number of random vectors, water waves, the distribution of integers which can be represented as the sums of two kth powers, and the reflection of electromagnetic waves on a parabolic cylinder, show the wide range of applications of these functions.

A table of Laplace transform pairs involving confluent hypergeometric functions, the list of references mentioned above, and an index conclude this excellent volume.

The printing is excellent, and very few misprints were noted by this reviewer. Tricomi's book is the first volume in a new series of mathematical monographs sponsored by the Italian *Consiglio Nazionale della Ricerche*. It is a promising beginning, and in wishing the new venture every success, one can hardly do better than express the hope that future volumes of the series will be as useful and as readable as Tricomi's book.

A. ERDÉLYI

The language of taxonomy—An application of symbolic logic to the study of classificatory systems. By J. R. Gregg. New York, Columbia University Press, 1954. 9+70 pp. $2.50.

The subtitle and almost every page of this little book hold forth promise to mathematicians on the alert for radical new applications of mathematics, but I fear that they will be rather disappointed. The specific subject matter seems dry and relatively unimportant for mathematicians and biologists alike; the method chosen by the author seems ineffective and tedious; there are scarcely any interesting arguments or deductions presented or alluded to; and the final results are meager.

The general problem of classifying a vast set of objects has many aspects of intellectual and perhaps of mathematical interest, especially when the objects are organisms, with their philogenetic connections. This book is confined to discovering a description of the formal, set theoretic, structure of the biological taxonomic systems in actual use, completely abstracted from how these systems depend on the structure and philogeny of organisms. For example, it is within the scope of the book to say that a species is a set of organisms and a genus a set theoretic union of species but not to say when two organisms should be assigned to a common species or genus.

As the author says, biologists have over the centuries evolved a remarkably accurate and effective language for describing the forms of organisms, but they have as yet no special language for talking about taxonomic systems, as opposed to talking taxonomy. The author seeks to supply the missing language with the aid of symbolic logic. I believe he would have done far better to look to a powerful, accurate, flexible, living language that, like biological nomenclature, has been evolving over the centuries, namely the everyday language of mathematical publication. For example, the author presents with some fanfare the following definition attributed to Woodger:

(3.1) (HR = the set in which any relation z is a member if and only if ($z \in$ One-Many) and ($\breve{z}$“UN = “$\aleph_0$” $\{\mathrm{B}‘z\}$)).

Translation: An HR (hierarchical relation) is the relation of immediate-predecessor-to in a partial ordering in which there is a unique initial element and in which every other element has a unique immediate predecessor.

Again, consider the definition of *taxonomic classificatory system*, **TCS**, from which many theorems are said to flow.

(5.6) (**TCS** = the set in which any relation z is a member if and only if ($z \in$ HR) and there is some u such that ($u \subset$ FIELD‘z) and ($z =$ IC(u)) and ($u \subset$ G) and, for all x, all y, and all w, if (($x;y$) $\in$ IC(u)) and (($x;w$) $\in$ IC(u)) and ($y \neq w$) then (($y \cap w$) = EM)).

This seems to mean the relation of immediate inclusion in a class of sets of organisms, $A, B, C, \cdots$, such that this relation is hierarchical and such that if $A \cap B \neq 0$ then $A \subset B$ or $B \subset A$. The author says in a footnote that since formulating (5.6) he has discovered a better definition, which he couches in language much more like ordinary mathematical language.

Two justifications of the author's very formal terminology are implicit in his statement that (3.1) is "a notation more favorable for calculating and for methodological discourse in general." Now there is almost no calculation or formal deduction in the book and little is called for, because the subject matter is logically so simple. As for the second justification, the notation does not seem to me nearly as favorable as ordinary mathematical language. The author himself almost always provides some translation into that language, and without it the book would be practically unintelligible.

I think it was a minor methodological step in the wrong direction for the author to adopt the theory of types, which mathematicians have long preferred to live without. Incidentally, he apparently forgets that he has adopted it in formulating several of his definitions, e.g., (1.15), (2.8), (2.17). He does encounter one difficulty in which the theory of types might have been put to some use. Even if, for example, a genus has only one species, taxonomists do not like to say that the genus and the species are the same. One solution would be to admit that, though in one sense a genus and a species are both sets of organisms, there is another sense in which a genus is a set of species. Personally, I think the difficulty is even better resolved by reference to intentional and extensional definition.

I have said that the results are meager. Indeed (5.6) and its exploration seem to be the high point of the book.

There are few errors, of which the most interesting is this. To illustrate the concept of incompatible relations, it is asserted that "if x is father of y then x is not cousin [of any order] of y, and vice versa." Actually, it is incestuous (nephew-aunt mating) for a man to beget a first cousin, but not impossible; moreover, I conjecture that most fathers and their sons are distant cousins.

To sum up, the book attacks a problem of minor interest with some success, success that would have been greater had the author been able to interest mathematicians in his project while it was under way.

LEONARD J. SAVAGE

Approximations for digital computers. By C. Hastings, Jr., assisted by J. T. Hayward and J. P. Wong, Jr. Princeton University Press, 1955. 8+201 pp. $4.00.

The theory of "best approximation," i.e. of minimizing the maximum of the absolute difference between a given function and its approximating function, has been studied extensively ever since it was initiated by Chebyshev, but practical approximations have in the past usually been based on other, analytically more manageable, methods. It appears from this book (implicitly, since no motivation is given explicitly) that Chebyshev approximation is just what is required in work with digital computing machines. Apparently one gives the machine a simple polynomial or rational approximation, which it can easily evaluate, in preference to an exact transcendental function. It also appears that almost none of the elaborate theory of Chebyshev approximation is of any practical use, and that the construction of useful approximations is still much more of an art than a science. Indeed, it is only rarely that the "best" approximation can be found explicitly, and the computer must usually be satisfied with some other approximation which is still uniformly close enough for his needs. The first part of the book consists of the reproduction of a strip film, with running commentary, describing the art of concocting approximations as practiced by an acknowledged expert. The second part presents 76 numerical approximations to 23 varied functions, each with a carefully drawn graph of the error. The book is evidently addressed to computer technicians rather than to mathematicians.

Mathematicians (by whom I mean people who produce, rather than consume, mathematics, whether pure or applied) may well be depressed by the realization of how ineffective even a well-developed theory can be in the face of practical requirements. On the other hand, a study of some of the forms which have been found useful might well suggest new lines of research. The level of the exposition is implicitly a disheartening commentary on the present level of mathematical education of computer personnel.

R. P. BOAS, JR.

BRIEF MENTION

Leitfaden der Nomographie. By W. Meyer zur Capellen. Berlin, Springer, 1953. 4+178 pp., 203 figs. 17.40 DM.

Nomographie. By M. W. Pentkowski. Trans. from the Russian by M. Peschel. Berlin, Akademie-Verlag, 1953. 16+268 pp., 140 figs. 15 DM.

These two books contain much useful information not elsewhere available and present it in stimulating manner. Both pay considerable attention, for example, to the question of errors. In the first, the discussion proceeds by numerous varied examples taken from engineering practice. The types are listed in an easily usable index. The sources form a bibliography of 68 titles. The theory is only lightly treated. With sharply contrasting method Pentkowski's book discusses few applications to specific equations from engineering. Most of the types considered are among those given in d'Ocagne's treatise. A mathematical classification of these types and their relations is not attempted. The object is rather to discuss how data should influence type, size and disposition if the nomogram is to have specified accuracy. This is done at great length and with a high degree of success.

J. M. THOMAS

An introduction to the theory of numbers. By G. H. Hardy and E. M. Wright. 3d ed. Oxford University Press, 1954. 16+420 pp. $6.75.

The first edition was reviewed in this Bulletin vol. 45 (1939) p. 507. A number of changes have been made, among them the inclusion of an elementary proof of the prime number theorem.

Mathematical foundations of quantum mechanics. By J. von Neumann. Trans. by R. T. Beyer. Princeton University Press, 1955. 12+445 pp. $6.00.

The German edition appeared in 1932 and was reviewed in this Bulletin vol. 39, p. 493. The translation was revised by von Neumann.

Die Briefwechsel von Johann Bernoulli. Ed. by the Naturforschende Gesellschaft in Basel. Vol. 1. Basel, Birkhäuser, 1955. 531 pp. 60 Swiss fr.

This magnificently produced volume is the first of 20 or more in which the Naturforschende Gesellschaft in Basel intends to publish the works and correspondence of the Bernoullis (and of the Basel mathematician Jakob Hermann (1678–1733)). The editor, O. Spiess, has provided this volume with everything that could be desired in the way of introductions, notes, and appendices. Vol. 1 contains correspondence between Johann Bernoulli and his brother Jakob, and some miscellaneous correspondence, but its greater, and by far the most interesting, part consists of Johann Bernoulli's correspondence with the Marquis De Lhospital (as he himself signed his name), who is remembered not only for his "rule" about indeterminate forms but as the author of the first calculus textbook. This correspondence, hitherto unpublished, indicates that the Marquis' originality has been much overestimated, and that even his famous rule should properly be attributed to Johann Bernoulli.

R. P. BOAS, JR.

RESEARCH PROBLEMS

21. Richard Bellman: *Dynamic programming.*

Given N coins, k of which are defective, either lighter or heavier than the other $N-k$ coins which are assumed to be of equal weight, and a balance, determine the weighing procedures which minimize the number of weighings required to separate the defective coins from the ordinary coins. Consider the following two cases

a. The k defective coins are all of the same weight, heavier or lighter than the regular coins.

b. The k defective coins are all of different weight.

Also determine the weighing procedures which minimize the expected time required to determine the defectives.

This is a particular case of the general "sorting" problem where an individual element of a set may be characterized by a number of properties and we have a number of testing devices for determining these properties. (Received May 31, 1955.)

22. Richard Bellman: *Analysis.*

The derivative of the gamma function satisfies the recurrence relation

$$\Gamma'(x+2) = (2x+1)\Gamma'(x+1) + x^2\Gamma'(x),$$

for $x>0$. Can one derive from this equation a convergent continued fraction expansion for $\Gamma'(x)/\Gamma'(x+1)$, or a related expression, which can be used either

a. To obtain a rapid method for computing $\Gamma'(1)$, the negative of Euler's constant, or

b. To obtain some results concerning the arithmetic character of Euler's constant? (Received May 31, 1955.)

23. Richard Bellman: *Number theory.*

There are a number of numerical techniques available for determining the maximum over the x_i of the linear form, $L(x) = \sum_{i=1}^{N} a_i x_i$, subject to the linear constraints $\sum_{j=1}^{N} b_{ij} x_j \leqq c_i$, $i=1, 2, \cdots, M$, whenever it exists. Can one obtain a usable algorithm for the cases where we impose additional constraints of the form

a. $x_i = 0$ or 1, for $i = 1, 2, \cdots, N$, or

b. x_i is zero or a positive integer? (Received May 31, 1955.)

24. Sherman Stein: *Number theory.*

Let a be a positive rational fraction with odd denominator and $u_n = (2n+1)$, $n = 1, 2, \cdots$. Let b_1 be the smallest of the u_i satisfying $a-(u_i)^{-1} \geqq 0$. Having defined $b_1, b_2, \cdots, b_n$, define b_{n+1} as the smallest u_i, $u_i > b_n$, with $a-(b_1)^{-1} - \cdots - (b_{n+1})^{-1} \geqq 0$. Is the sequence $b_1, \cdots, b_n, \cdots$ finite for each a? (Received May 23, 1955.)

25. Sherman Stein: *Geometry.*

Let $J \subset R_3$ be a rectifiable Jordan curve, with the property that for each rotation R, there is a translation T, depending on R, such that $(TRJ) \cap J$ has a nonzero length. Must J contain the arc of a circle? (Received May 23, 1955.)

RUSSIAN TRANSLATION PROJECT

The project for translation of selected Russian papers on mathematics is being carried forward under a grant from the National Science Foundation. A new series of translations prepared under this grant will be published in the near future. Instead of being issued as separate reprints as in the past, the papers selected under the current grant will be published in three cloth-bound volumes with contents as shown below.

VOLUME 1

Aleksandrov, P. S. The present status of the theory of dimension. Uspehi Mat. Nauk (N.S.) 6, no. 5(45), 43–68 (1951).

Kreĭn, M. G. On an application of the fixed-point principle in the theory of linear transformations of spaces with an indefinite metric. Uspehi Mat. Nauk (N.S.) 5, no. 2(36), 180–190 (1950).

Šilov, G. E. On decomposition of a commutative normed ring in a direct sum of ideals. Mat. Sbornik N.S. 32(74), 353–364 (1953).

Gel'fand, I. M., and Fomin, S. V. Geodesic flows on manifolds of constant negative curvature. Uspehi Mat. Nauk. (N.S.) 7, No. 1(47), 118–137 (1952).

Suvorov, G. D. Prime ends of a sequence of plane regions converging to a nucleus. Mat. Sbornik N.S. 33(75), no. 1, 73–100 (1953).

Pontryagin, L. S. On zeros of some transcendental functions. Izv. Akad. Nauk SSSR. Ser. Mat. 6, 115–134 (1942).

Gihman, I. I. On the theory of differential equations of random processes. I, II. Ukrain. Mat. Ž. 2, no. 4, 37–63 (1950); 3, no. 3, 317–339 (1951).

Kreĭn, M. G. On certain problems on the maximum and minimum of characteristic values and on the Lyapunov zones of stability. Prikl. Mat. Meh. 15, 323–348 (1951).

Staržinskiĭ, V. M. Survey of works on conditions of stability of the trivial solution of a system of linear differential equations with periodic coefficients. Prikl. Mat. Meh. 18, 469–510 (1954).

Reĭzin', L. È. Behavior near to a singular point of integral curves of a system of three differential equations. Latvijas PSR Zinātnu Akad. Vēstis 1951, no. 2(43) 333–346 (1951).

Gel'fand, I. M., and Levitan, B. M. On the determination of a differential equation from its special function. Izv. Akad. Nauk SSSR. Ser. Mat. 15, 309–360 (1951).

VOLUME 2

Mal'cev, A. I. On some classes of infinite soluble groups. Mat. Sbornik N.S. 28(70), 567–588 (1951).

Kulikov, L. Ya. On direct decompositions of groups. Ukrain. Mat. Ž. **4**, 230–275, 347–372 (1952).

Golovin, O. N. Nilpotent products of groups. Mat. Sbornik N.S. **27**(69), 427–454 (1950).

Golovin, O. N. The metabelian products of groups. Mat. Sbornik N.S. **28**(70), 431–444 (1951).

Golovin, O. N. On the isomorphism of nilpotent decompositions of groups. Mat. Sbornik N.S. **28**(70), 445–452 (1951).

Naimark, M. A. Description of all irreducible unitary representations of the classical groups. Doklady Akad. Nauk SSSR (N.S.) **84**, 883–886 (1952).

Gel'fand, I. M., and Graev, M. I. Unitary representations of the real unimodular group (principal nondegenerate series). Izv. Akad. Nauk SSSR. Ser. Mat. **17**, 189–248 (1953).

Gel'fand, I. M., and Šapiro, Z. Ya. Representations of the group of rotations in three-dimensional space and their applications. Uspehi Mat. Nauk (N.S.) **7**, no. 1(47), 3–117 (1952).

VOLUME 3

Zykov, A. A. The spectrum problem in the extended predicate calculus. Izv. Akad. Nauk SSSR. Ser. Mat. **17**, 63–76 (1953).

Faddeev, D. K. Simple algebras over a field of algebraic functions of one variable. Trudy Mat. Inst. Steklov. **38**, 321–344 (1951).

Nečaev, V. I. Waring's problem for polynomials. Trudy Mat. Inst. Steklov. **38**, 190–243 (1951).

Linnik, Yu. V., and Malyšev, A. V. Applications of the arithmetic of quaternions to the theory of ternary quadratic forms and to the decomposition of numbers into cubes. Uspehi Mat. Nauk (N.S.) **8**, no. 5(57), 3–71 (1953).

Val'fiš [Walfisz], A. Z. On the representation of numbers by sums of squares. Asymptotic formulas. Uspehi Mat. Nauk (N.S.) **7**, no. 6(52), 97–178 (1952).

The volumes will be printed by photo-offset from vari-typed copy. The initial edition will be stocked and sold by the American Mathematical Society in the same manner as its other publications. The first volume will appear in the fall of 1955, the other two during the winter. The anticipated prices for the three volumes are, respectively, $3.60, 3.70, 3.30 (25% discount to members of the Society). Orders should be sent to the Society.

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STRUCTURE AND REPRESENTATION OF NONASSOCIATIVE ALGEBRAS[1]

R. D. SCHAFER

1. Introduction. By a nonassociative algebra is meant a vector space which is equipped with a bilinear multiplication. If the multiplication is associative, we have the familiar notion of an associative algebra. Also Lie algebras are an essential tool for the study of Lie groups, as is well known. However, we shall not be particularly concerned with associative or Lie algebras today, except as models of what well-behaved nonassociative algebras should be.

The study of algebras which are not associative is not a recent development. The 8-dimensional algebra of Cayley numbers was known as early as 1845 [26].[2] However, it is within the last fifteen years that the study has received its greatest impetus. With a few notable exceptions—and always excepting Lie algebras of course—there were only isolated results before that time. By now a pattern is emerging, for certain finite-dimensional algebras at least, and this paper is an exposition of some of the principal results achieved recently in the structure and representation of finite-dimensional nonassociative algebras.

A. A. Albert has been the prime mover in this study. The depth and scope of the results, at least for one class of nonassociative algebras, may be judged by the fact that N. Jacobson will give the Colloquium lectures at the Summer Meeting of the Society on the topic of Jordan algebras. Except for this reference to these two men, I shall not attempt to identify the authors of any theorems in this talk. The bibliography of the published paper will speak for itself.

2. The associative and Lie theories as models. Let F be an arbitrary field and A be a finite-dimensional associative algebra over F. It is well known that there is an ideal N, called the radical of A, which is the unique maximal nilideal of A (that is, the maximal ideal consisting entirely of nilpotent elements). Furthermore, N is nilpotent in the sense that there is an integer t with the property that any product $z_1z_2 \cdots z_t$ of t elements from N is zero; hence N is also the

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An address delivered before the Brooklyn meeting of the Society on April 15, 1955, by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors May 31, 1955.

[1] This paper was supported in part by a grant from the National Science Foundation.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

unique maximal nilpotent ideal of A. Modulo this radical the algebra is semisimple; that is, the difference algebra A/N has radical equal to zero. Moreover, any semisimple associative algebra is uniquely expressible as a direct sum $S_1 \oplus \cdots \oplus S_r$ of simple two-sided ideals (where an algebra is simple provided it has no proper ideals and is not a 1-dimensional algebra in which all products are zero). Any simple associative algebra S is the Kronecker product $F_s \times D$ (over F) of the total matric algebra F_s of dimension s^2 and a division algebra D over F, where s is unique and D is uniquely determined up to isomorphism. Hence (up to a determination of all division algebras D over F) the structure of any semisimple associative algebra over F is known.

Let A/N be separable (that is, the center of each simple component is a separable field over F; this would always be the case if F were of characteristic 0). Then A has a Wedderburn decomposition $A = S + N$ where S is a subalgebra of A isomorphic to A/N and $S+N$ is a vector space direct sum. This decomposition is unique up to an inner automorphism of A in the following strong sense. Let z be any nilpotent element in A. Then, even though there is no identity element 1 in A, the meaning of $(1-z)a(1-z)^{-1}$ is clear, and the mapping G_z: $a \rightarrow (1-z)a(1-z)^{-1}$ is an inner automorphism of A. Suppose that A has Wedderburn decompositions $A = S+N = S_1+N$. Then there is an element z of the radical N such that the corresponding inner automorphism G_z maps S_1 onto S (and of course leaves N invariant) [73].

This of course has been but the briefest of sketches of the associative structure theory, and omits many important features. It is astonishing, however, how closely the structure of Lie algebras of characteristic 0 parallels the associative theory up to this point.

Let F be a field of characteristic 0 and A be a finite-dimensional Lie algebra over F. Then the radical N of A is not the maximal nilideal (since A itself is a nilalgebra, the square of every element being zero), nor is it in general the maximal nilpotent ideal of A. It is an ideal between these two. Define $B^{(1)} = B$, $B^{(i+1)} = (B^{(i)})^2$. Then B is solvable in case there is an integer k such that $B^{(k)} = 0$, and the radical N of A is the unique maximal solvable ideal of A. With this definition of radical, the difference algebra A/N is semisimple and is uniquely expressible as a direct sum of simple two-sided ideals. If F is algebraically closed, the classification of simple Lie algebras into four great classes and five exceptional algebras is well known. This leads to a determination of the simple Lie algebras over arbitrary F of characteristic 0 which by now is almost complete, and in this sense we can say that all semisimple Lie algebras over F are known.

Since F is of characteristic 0, there is no question of separability involved. Any finite-dimensional Lie algebra A over F has a Levi decomposition $A = S+N$ where S is a subalgebra isomorphic to A/N and $S+N$ is a vector space direct sum. Moreover, we have a similar strict conjugacy of semisimple components in such decompositions. Let z be any element of the maximal nilpotent ideal N_0 of $A(N_0 \subseteq N)$, and let $D_z = \mathrm{ad}\, z$ (the right multiplication of A determined by z) be the nilpotent inner derivation of A corresponding to z. Then $G_z = \exp D_z$ is an inner automorphism of A. If $A = S+N = S_1+N$, then there is an element z in N_0 such that G_z maps S_1 onto S [37].[3]

Lest we be too taken up with the similarities between these two theories, we should perhaps recall one of the important tools for the study of associative algebras; namely, the Peirce decomposition relative to an idempotent. Let e be an idempotent ($e^2 = e \neq 0$) in an associative algebra A over an arbitrary field F. Then A may be written as the vector space direct sum $A = A_{11}+A_{10}+A_{01}+A_{00}$ of spaces A_{ij} which consist of those elements a_{ij} in A satisfying $ea_{ij} = ia_{ij}$, $a_{ij}e = ja_{ij}$ $(i, j = 0, 1)$. The properties of this decomposition are essential to the proofs of the associative theorems I have mentioned. On the other hand there are clearly no idempotents in Lie algebras, and the proofs of the parallel theorems rely on other methods, notably on a trace argument which breaks down for fields of characteristic $p > 0$.

3. Arbitrary nonassociative algebras. Turning now to arbitrary nonassociative algebras, I should mention first that the concepts of subalgebra, ideal, homomorphism, isomorphism, simple algebra, difference algebra, and direct sum do not involve associativity in any way. If a nonassociative algebra can be written as the direct sum $S_1 \oplus \cdots \oplus S_r$ of simple two-sided ideals S_i, it is easy to see that the simple summands in such a decomposition are uniquely determined. If we hope to use the associative and Lie theories as models at all, then we are forced to define a finite-dimensional nonassociative algebra to be semisimple in case it is a direct sum of simple two-sided ideals. (For some purposes it may be desirable to restrict the notion of simplicity, but that is a refinement which I can take up later.) The characteristic property of the radical then is not that it is the maximal nil-, solvable, or nilpotent ideal, but that it is the minimal ideal N such that A/N is zero or a direct sum of simple two-sided ideals [4]. Thus any nonassociative algebra A has a radical N, and it is easy to see that the difference algebra A/N is semisimple in the original sense that its radical is zero. Also A/N is uniquely expressible as a direct sum of simple two-sided ideals.

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[3] It follows from the Campbell-Hausdorff formula that, if $G = G_{z_1} G_{z_2} \cdots G_{z_k}$ for $z_i \in N_0$, then there exists $z \in N_0$ such that $G = G_z$.

Guided by the associative and Lie theories, one is then led to ask: what is the nature of the radical N, what are the simple algebras, what is the nature of A as an extension of N? For an arbitrary algebra A, the answer is that anything can happen.

It is possible to construct a nonassociative algebra with almost any undesirable property. Most of these constructions are based on the use of a multiplication table. Let $u_1, \cdots, u_n$ be a basis for A as an n-dimensional vector space over F. Then $u_iu_j = \sum_k \gamma_{ijk}u_k$ for uniquely determined elements γ_{ijk} in F. Conversely, any n^3 equations of this type (that is, with n^3 multiplication constants γ_{ijk} arbitrary in F) determine a nonassociative algebra of dimension n over F, since the definition of multiplication is extended to products of arbitrary pairs in A by bilinearity.

Let A be a 2-dimensional algebra[4] with basis u,v and with multiplication table $u^2=u$, $uv=v^2=v$, $vu=0$. The 1-dimensional ideal N with basis v is the only proper ideal of A. Since A cannot be the direct sum of two proper ideals, A is not semisimple. Then, since $A/N=F\bar{u}\cong F$ is semisimple, N is the radical of A. But v is idempotent, so $N=Fv\cong F$ is a field. This is the extreme of being unlike a radical (in the nil-, solvable, or nilpotent sense).

Again let A be a 2-dimensional algebra with basis u,v and this time with multiplication table $u^2=u+v$, $uv=v$, $vu=v^2=0$. Again N with basis v is the only proper ideal of A and, since $\bar{u}^2=\bar{u}$, $A/N=F\bar{u}\cong F$, N is the radical of A. Since N is a zero algebra, this radical is well-behaved. However, there cannot be any Wedderburn decomposition for A. For there is no idempotent e in A, as there would have to be for A to contain a subalgebra $S\cong A/N\cong F$.

The literature contains other unpleasant examples, some not nearly as easily constructed as these. They serve to put limits on our expectations for a general theory, and focus our attention on classes of algebras for which associative algebras and Lie algebras of characteristic 0 serve as reasonable models. They also increase our appreciation of those techniques which are applicable to arbitrary nonassociative algebras.

Let a be an element in a nonassociative algebra A over F, and R_a be the right multiplication of A determined by a; that is, R_a is the mapping $x\rightarrow xa=xR_a$ for every x in A. Similarly for the left multiplication L_a: $x\rightarrow ax=xL_a$. The right and left multiplications are elements of F_n, the total matric algebra of all linear transformations on the vector space A of dimension n. We denote by E the associative subalgebra of F_n generated by the right and left multiplications of A. The structural properties of A are reflected in those of its associative multiplication algebra E [**41**; **3**]. If A is semisimple, then so is E. If A is simple, then so is E, and E is indeed a total matrix algebra over its center. The radical of A may be characterized in terms of the radical of E [**4**; **57**].

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[4] This example is a modification of the 3-dimensional example in [4].

The center of a nonassociative algebra is the set of elements c which not only commute with every element of A but associate with every pair of elements a,b in A. If A is simple, the center of A is either 0 or a field. In the latter case A may be regarded as an algebra over its center, and as such is central simple (that is, remains simple under scalar extension of the ground field). In any event the center of E is a field, the multiplication center of A, and A is a central simple algebra over its multiplication center [**41**; **3**; **44**]. A nonassociative algebra is separable (that is, is semisimple over any scalar extension of the ground field) if and only if the multiplication centers of its simple summands are separable fields.

Solvability of an algebra ($B^{(k)}=0$) is a nonassociative notion. It turns out that the radical of an algebra always contains the maximal solvable ideal [**5**] (although we have seen an example showing that it can contain much more). Nilpotence of an algebra can be turned into a nonassociative notion by demanding that there is an integer t such that any product $z_1z_2 \cdots z_t$ of t elements, no matter how associated, is zero. Clearly any nilpotent algebra is solvable. Moreover, any ideal N_0 of A is nilpotent if and only if the subalgebra of E generated by the right and left multiplications corresponding to elements of N_0 is a nilpotent associative algebra [**5**; **33**].

There are also relationships between arbitrary nonassociative algebras and Lie algebras. A derivation D of a nonassociative algebra A is a linear transformation on A having the property of differentiation: $(xy)D=x(yD)+(xD)y$. The set of all derivations of A is a Lie algebra (with product the ordinary commutator $[D,D']=DD'-D'D$), and this is called the derivation algebra of A [**42**]. The Lie enveloping algebra of the right and left multiplications of A (that is, the algebra generated by addition and commutation, rather than addition and multiplication) is called the Lie multiplication algebra L of A [**88**; **47**]. If A is semisimple of characteristic 0, and if A has an identity element 1, then every derivation D of A is inner in the sense that D is in L [**47**]. For associative and Lie algebras inner derivations have the familiar forms $D=R_x-L_x$ and $D=R_x$ ($=$ ad x) respectively.

4. Alternative and Jordan algebras. Let us turn now to consideration of two classes of algebras whose behavior, according to the norms we have set, is excellent. These are the well-known alternative and Jordan algebras. It turns out that they are intimately interrelated.

An algebra A is called alternative in case the identities $ax^2=(ax)x$, $x^2a=x(xa)$ are satisfied in A. Clearly all associative algebras are alternative. It turns out that the only simple alternative algebras which are not associative are 8-dimensional algebras over their centers which are generalizations of the original algebra of Cayley numbers [109; 110; 83]. Hence all semisimple alternative algebras are known.

Any alternative algebra is power-associative (that is, if $x^1=x$, and $x^{i+1}=x^i x$, we have $x^i x^k=x^{i+k}$ so that all products of m factors, each equal to x, are equal and we can write this common element as x^m). Hence nilpotence of an element is unambiguously defined. The radical N of A is the maximal nilideal, and N is nilpotent [112; 87]. If A/N is separable, then A has a Wedderburn decomposition $A=S+N$ where $S\cong A/N$ [87]. All of this is for an arbitrary field F, and the proofs rely heavily on the fact that there is a Peirce decomposition of A whose properties are identical with those of the special case of associative algebras [109; 112].

If F is of characteristic 0, we can prove the strict conjugacy of semisimple summands in Wedderburn decompositions. If F is of characteristic p, the question is still open, and the solution depends on arriving at a suitable form for the inner automorphisms of A. The derivations are more tractable [88]. If $A=S+N=S_1+N$ is of characteristic 0, then there is a nilpotent derivation D of A in the radical of E such that the automorphism $G=\exp D$ of A maps S_1 onto S [90]. Moreover, any alternative algebra of characteristic 0 is semisimple if and only if its derivation algebra is semisimple or 0 [90].[5]

Jordan algebras are also defined by identities: $xa=ax$, $(x^2a)x=x^2(ax)$. Since the second of these identities is a weak form of associativity, any commutative associative algebra is a Jordan algebra. Let us assume henceforth that the characteristic of F is not two. The most easily realizable Jordan algebras are those obtained from associative algebras as follows. Let E be any associative algebra with multiplication denoted by $x\cdot y$. Consider a new algebra B^+ which is the same vector space as B, but in which multiplication is defined by $xy=(x\cdot y+y\cdot x)/2$. B^+ is clearly commutative, and it is easy to verify that the Jordan identity is also satisfied. Any nonassociative algebra A over F which is isomorphic to a subalgebra of some B^+ is called a special Jordan algebra. The special Jordan algebras are not defined by identities, since a homomorphic image of a special Jordan algebra need not be special [28].

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[5] This generalizes the associative theorem in [38]. The analogue for Jordan algebras is proved in [89].

Any Jordan algebra (of characteristic $\neq 2$) is power-associative. The radical N is the maximal nilideal, and N is nilpotent [6]. Every simple Jordan algebra has an identity element 1, and is therefore central simple over its center. With a possible exception which can occur only if the characteristic is $p>0$, all central simple Jordan algebras are known. The degree of a central simple algebra is the unique number q of the pairwise orthogonal primitive idempotents e_i in $1=e_1+\cdots+e_q$ over any algebraically closed extension Ω of F. It is an open question whether any central simple Jordan algebras of degree one and dimension greater than one exist. If they do, they are of characteristic $p>0$, and they are exceptional in the sense that they are not special Jordan algebras [13]. With this possible exception, there are four great classes of central simple Jordan algebras (all of which are special)—algebras of degree two defined by quadratic forms and three classes of algebras of self-adjoint elements in certain involutorial associative algebras [61; 5; 6; 62; 45; 40; 13]—and one class of exceptional algebras which are 27-dimensional generalizations of the algebra of 3-rowed hermitian matrices with elements which are Cayley numbers [1; 85; 12; 13].

If A/N is a known separable Jordan algebra,[6] then A has a Wedderburn decomposition $A=S+N$ where $S\cong A/N$ [80; 20]. If $A=S+N$ $=S_1+N$ is of characteristic 0, then there is a nilpotent derivation D of A in the radical of E such that the automorphism $G=\exp D$ of A maps S_1 onto S [49].

The proofs of these theorems depend on the decomposition of a Jordan algebra relative to an idempotent [61; 6]. This decomposition is quite evident in case $A=B^+$, for e is idempotent in B^+ if and only if e is idempotent in B. Then $B=B_{11}+B_{10}+B_{01}+B_{00}$ and b_{ij} in B_{ij} give $b_{ij}e=(b_{ij}\cdot e+e\cdot b_{ij})/2=(i+j)b_{ij}/2$ and $A=A_1+A_{1/2}+A_0$ where $A_1=B_{11}$, $A_{1/2}=B_{10}+B_{01}$, $A_0=B_{00}$. The same decomposition $A=A_1+A_{1/2}+A_0$, with $a_ie=ia_i$ for a_i in A_i, holds for any idempotent e in a general Jordan algebra A, and the properties of this decomposition are fundamental in the structure theory.

5. Power-associative algebras. All of the algebras which we have considered have been power-associative (that is, each single element generates an associative subalgebra). It is natural to try to see how far the results can be extended to power-associative algebras [7; 8]. We are always assuming now that the characteristic of F is not two.

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[6] The proof in [80] is valid for F of characteristic $\neq 2$ in case each simple summand of (the separable algebra) A/N is a known simple Jordan algebra.

Let us begin with commutative power-associative algebras. Relative to any idempotent e there is a decomposition $A = A_1 + A_{1/2} + A_0$ which enjoys most of the properties of this decomposition for Jordan algebras [8; 13]. We modify the definition of a simple algebra, demanding that the algebra not be a nilalgebra as well as not having any proper ideals. The radical of A is then the minimal ideal N such that A/N is a direct sum of such simple algebras. We can either further restrict the characteristic of F to be $\neq 3, 5$, or assume that A is strictly power-associative in the sense that A_K is power-associative for every scalar extension K of F. Then N is the maximal nilideal of A. Every simple algebra has an identity element 1. Moreover, the only simple algebras of degree greater than 2 are the known simple Jordan algebras [13; 15; 69]. An interesting feature of the technique used in obtaining these results is that the result on simple algebras is used to prove that the maximal nilideal is the radical.

If F is of characteristic $p > 5$, there do exist examples of simple commutative power-associative algebras of degree 2 which are not Jordan algebras [68]. However, it is an open question[7] whether such examples exist for F of characteristic 0. The strongest theorem for algebras of degree 2 is that A is a Jordan algebra if and only if $A_i A_{1/2} \subseteq A_{1/2}$ $(i = 0, 1)$ for every idempotent e in A [18]. If F is of characteristic 0, then any simple commutative power-associative algebra of degree 1 is 1-dimensional over its center [13].

Whether or not the new definition of simplicity is an actual restriction is not known. That is, whether or not commutative power-associative nilalgebras without proper ideals exist (except of course for the zero algebra of dimension 1) is still an open problem. Whether or not the radical is nilpotent (or solvable) is a related question. To my knowledge nothing is known about Wedderburn decompositions for commutative power-associative algebras.

The situation with respect to noncommutative algebras is rather chaotic. However, pleasing results are known for some classes of algebras. The most effective tool is the passage from A to the attached commutative algebra A^+ in which multiplication is defined by $x \circ y = (xy + yx)/2$ where xy is the product in A [8]. Whether or not the method is successful depends on how much is lost in transition from A to A^+. For example, if A is anticommutative $(ax = -xa)$, then everything is lost because $x \circ y = 0$.

Noncommutative Jordan algebras [94] are defined by the Jordan identity $(x^2a)x = x^2(ax)$ and an identity which is a consequence of this if A has an identity element: $(xa)x = x(ax)$. The latter is called the flexible law [8], and is satisfied in any commutative (or anticommutative) algebra. Noncommutative Jordan algebras are power-associative, and they include Jordan, alternative, and—trivially—Lie algebras. They also include the quasiassociative algebras A (over some scalar extension K of F there is an associative algebra B and a scalar $\lambda \neq 1/2$ in K such that $A_K \cong B(\lambda)$, the algebra which is the same vector space as B but in which multiplication is defined by $xy = \lambda x \cdot y + (1-\lambda) y \cdot x$ where $x \cdot y$ is the product in B) [8]. The attached algebra A^+ of any noncommutative Jordan algebra A is a (commutative) Jordan algebra. If the characteristic of F is 0, then A is trace-admissible according to the following definition. There is a linear function $\tau(x)$ on A (namely, the trace of the right multiplication of A^+ corresponding to x) with the following properties: $\tau(xy) = \tau(yx)$, $\tau((xy)z) = \tau(x(yz))$, $\tau(e) \neq 0$ for any idempotent e of A, $\tau(x) = 0$ if x is nilpotent. But for any trace-admissible algebra, if the definition of simplicity includes not being a nilalgebra, it is known [10] that the radical is the maximal nilideal and is the set of all z in A satisfying $\tau(xz) = 0$ for every x. The radical is not nilpotent, or even solvable, since every Lie algebra is its own radical by this definition (in which the simple algebras are taken to be not nilalgebras). The simple algebras are the simple (commutative) Jordan algebras, the simple flexible algebras of degree two, and the simple quasiassociative algebras. One can construct an example[8] to show that noncommutative Jordan algebras do not in general have Wedderburn decompositions (even in case $N^2 = 0$).

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[7] L. A. Kokoris has subsequently announced (April 22, 1955) that every simple commutative power-associative algebra of degree 2 and characteristic 0 is a Jordan algebra.

Right alternative algebras are defined by the identity $ax^2 = (ax)x$. If A is right alternative, then A^+ is a special Jordan algebra. By a modification of the trace-admissibility technique, it can be shown that for F of arbitrary characteristic $\neq 2$ the radical of A is its maximal nilideal, and that every semisimple right alternative algebra is actually alternative [9; 19].

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[8] Let A be the 5-dimensional noncommutative Jordan algebra with basis e_{11}, e_{12}, e_{21}, e_{22}, z and multiplication table $e_{11}^2 = e_{12}e_{21} = e_{11}$, $e_{11}e_{12} = e_{12}e_{22} = e_{12}$, $e_{21}e_{12} = e_{22}^2 = e_{22}$, $e_{11}e_{21} = e_{21}e_{22} = e_{22}z = ze_{11} = z$, $e_{21}e_{11} = e_{22}e_{21} = e_{21} - z$, all other products zero. The radical N is spanned by z, $N^2 = 0$, and $A/N \cong F_2$ with basis $\bar{e}_{11}$, $\bar{e}_{12}$, $\bar{e}_{21}$, $\bar{e}_{22}$. Suppose there were a subalgebra $S \cong A/N$. Then S would have a usual matrix basis g_{11}, g_{12}, g_{21}, g_{22}, and there would be an automorphism H of A/N such that $\bar{g}_{ij} = \bar{e}_{ij}H$. But any automorphism of A/N is inner, as there is a nonsingular element $y = \alpha\bar{e}_{11} + \beta\bar{e}_{12} + \gamma\bar{e}_{21} + \delta\bar{e}_{22}$ $(\Delta = \alpha\delta - \beta\gamma \neq 0)$ in A/N such that $\bar{g}_{ij} = y\bar{e}_{ij}y^{-1}$. It follows that $g_{11} = \Delta^{-1}(\alpha\delta e_{11} - \alpha\beta e_{12} + \gamma\delta e_{21} - \beta\gamma e_{22}) + \epsilon z$, $g_{12} = \Delta^{-1}(-\alpha\gamma e_{11} + \alpha^2 e_{12} - \gamma^2 e_{21} + \alpha\gamma e_{22}) + \kappa z$, $g_{21} = \Delta^{-1}(\beta\delta e_{11} - \beta^2 e_{12} + \delta^2 e_{21} - \beta\delta e_{22}) + \lambda z$, $g_{22} = \Delta^{-1}(-\beta\gamma e_{11} + \alpha\beta e_{12} - \gamma\delta e_{21} + \alpha\delta e_{22}) + \mu z$. Equating coefficients of z in $g_{11}g_{12} = g_{12}$, $g_{22}g_{11} = g_{11}g_{22} = 0$, $g_{22}g_{21} = g_{21}$, $g_{21}g_{12} = g_{22}$ yields equations in $\alpha, \beta, \cdots, \mu$ which force $\Delta = 0$, a contradiction.

Every finite strictly power-associative division ring of characteristic $\neq 2$ is a finite field [17]. This generalizes Wedderburn's celebrated theorem on finite associative division rings. The proof appeals directly to the structure theory for the attached commutative power-associative algebra A^+ over the prime field F.

If A is a simple power-associative algebra, then there is no reason why the commutative power-associative algebra A^+ need be simple. If, however, A^+ is simple, then it is clear that A must be simple. If A^+ is central simple of degree greater than 2, then A^+ is a known Jordan algebra and (at least over a scalar extension K of F) the multiplication in A can be given by an explicit construction [14; 15], the details of which I do not have time to relate.

6. Modules and representations. A very valuable technique which has been introduced in the last few years is the use of modules for nonassociative algebras (or bimodules, since the concept is a generalization of the familiar two-sided module for associative algebras). An equivalent notion is that of a representation (or birepresentation).

Let C be a class of nonassociative algebras over F. If A is in C, and if M is a vector space over F, let ma and am in M be two bilinear compositions for a in A, m in M. Then the direct sum $A+M$ of the vector spaces A and M is turned into a nonassociative algebra over F (the semidirect sum, or split null extension) by defining multiplication in $A+M$ by $(a_1+m_1)(a_2+m_2)=a_1a_2+(m_1a_2+a_1m_2)$. If $A+M$ is in C, then M is called a module for A in C. For example, if C is the class of all nonassociative algebras over F, then no conditions other than the originally assumed bilinearity are imposed on the compositions am and ma. If C is the class of all associative algebras, then the compositions in M must satisfy $(ma)b=m(ab)$, $(am)b=a(mb)$, $(ab)m=a(bm)$ for a, b in A, m in M; that is, M is a two-sided A-module for the associative algebra A [38]. If C is any class of nonassociative algebras defined by multilinear identities $I_\nu(a_1, a_2, \cdots, a_{n_\nu})=0$, then M is a module for A in C in case all of the identities, obtained by replacing any single a_j in A in the identities $I_\nu(a_1, a_2, \cdots, a_{n_\nu})=0$ by m in M, are satisfied [31].

If M is a module for A, then the mappings $S_a: m\rightarrow ma$ and $T_a: m\rightarrow am$ are linear transformations on M, and $a\rightarrow S_a$, $a\rightarrow T_a$ are linear mappings of A into the space of all linear transformations on M. A pair (S, T) of linear mappings $a\rightarrow S_a$, $a\rightarrow T_a$ of A into the space of all linear transformations on a vector space M is called a representation of A in C in case M, equipped with the compositions $ma=mS_a$,

$am = Tm_a$, is a module for A in C. Any algebra A in a class C defined by identities has the regular representation (R, L) where M is A itself, and R_a and L_a are right and left multiplications in A. An important simplification is possible if C contains only commutative (or anticommutative) algebras. For then $ma = am$ (or $ma = -am$) implies $S_a = T_a$ (or $S_a = -T_a$), so that effectively only one mapping S: $a \to S_a$ is involved instead of the pair (S, T). This simplification can be made for Jordan and Lie algebras (for Lie algebras the notion of representation is the usual one).

The defining identities for alternative and Jordan algebras may be linearized.[9] Modules and representations for alternative and Jordan algebras[10] are then easily defined, and these have already proved to be very effective in the study of subalgebras of these algebras [49; 90; 53].

The analogues for alternative and Jordan algebras of the first Whitehead lemma [38; 49; 90] depend on representation theory, as do the proofs of the theorems concerning the strict conjugacy of semisimple components in Wedderburn decompositions which I have already mentioned. Representations have also been used to prove that, if B is an alternative algebra with an identity 1 and if A is a Cayley subalgebra containing 1, then B is the Kronecker product $B = A \times Z$ (over F) where Z is the center of B. Similarly, if B is a Jordan algebra with identity 1 and if A is a known (27-dimensional) exceptional simple subalgebra containing 1, then B is also the Kronecker product $B = A \times Z$ [54].

Just as the multiplication algebras E and L (which I have already mentioned) are generated by the elements R_a and L_a of the regular representation, there are associative (and Lie) enveloping algebras for any representation (S, T). For alternative and Jordan algebras A one can define a universal associative algebra U for the representations, universal in the sense that any representation of A is obtained from a representation (in the usual sense of homomorphism) of the associative algebra U [53]. Furthermore, U is finite-dimensional for finite-dimensional alternative and Jordan algebras A [49; 90].

If A is a separable alternative algebra or a known separable Jordan algebra, then every module for A is completely reducible. The irreducible modules for these algebras have been determined [53].

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[9] The multilinear identities obtained are equivalent to the original ones if F is of characteristic $\neq 2$ for alternative algebras and of characteristic $\neq 2, 3$ for Jordan algebras.

[10] Included among the representations of Jordan algebras are the special representations defined by $S_{ab} = S_aS_b + S_bS_a$ [22; 40].

There are intimations of cohomology in some of these results on alternative and Jordan algebras [49; 90; 91; 53]. One can hope for the development of a cohomology theory, not only for these algebras, but one for general nonassociative algebras which will provide a measure of the deviation of nonassociative algebras from the norms of good behavior which are gradually appearing.

7. Conclusion. In this short space of an hour I have had to omit many topics. I have made no mention of rings or infinite-dimensional algebras [102; 99; 16; 63; 24; 67; 55; 56; 100; 101; 78; 70; 11; 64]. Moreover, I have not touched on the interesting relationships between nonassociative algebras and the exceptional simple Lie algebras [43; 27; 35; 105]. Nor has there been time to explain how some of these algebras arise in other branches of mathematics [29; 75; 36; 95; 98; 104; 103; 39] and in the formalism of quantum mechanics [58; 59; 60] and genetics [32; 33; 34; 86]. I hope, however, that I have been able to give the nonspecialist some insight into the results and problems of a rich and active field of research.

BIBLIOGRAPHY

1. A. A. Albert, *On a certain algebra of quantum mechanics*, Ann. of Math. vol. 35 (1934) pp. 65–73.

2. ———, *Quadratic forms permitting composition*, Ann. of Math. vol. 43 (1942) pp. 161–177.

3. ———, *Non-associative algebras*, I, Ann. of Math. vol. 43 (1942) pp. 685–707; II, ibid. pp. 708–723.

4. ———, *The radical of a non-associative algebra*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 891–897.

5. ———, *On Jordan algebras of linear transformations*, Trans. Amer. Math. Soc. vol. 59 (1946) pp. 524–555.

6. ———, *A structure theory for Jordan algebras*, Ann. of Math. vol. 48 (1947) pp. 546–567.

7. ———, *On the power-associativity of rings*, Summa Brasiliensis Mathematicae vol. 2 (1948) pp. 21–32.

8. ———, *Power-associative rings*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 552–593.

9. ———, *On right alternative algebras*, Ann. of Math. vol. 50 (1949) pp. 318–328.

10. ———, *A theory of trace-admissible algebras*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 317–322.

11. ———, *Absolute-valued algebraic algebras*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 763–768; *A note of correction*, Bull. Amer. Math. Soc. vol. 55 (1949) p. 1191.

12. ———, *A note on the exceptional Jordan algebra*, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 372–374.

13. ———, *A theory of power-associative commutative algebras*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 503–527.

14. ———, *New simple power-associative algebras*, Summa Brasiliensis Mathematicae vol. 2 (1951) pp. 183–194.

15. ———, *Power-associative algebras*, Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, Providence, American Mathematical Society, 1952, vol. 2, pp. 2–32.

16. ———, *On simple alternative rings*, Canadian Journal of Mathematics vol. 4 (1952) pp. 129–135.

17. ———, *On nonassociative division algebras*, Trans. Amer. Math. Soc. vol. 72 (1952) pp. 296–309.

18. ———, *On commutative power-associative algebras of degree two*, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 323–343.

19. ———, *The structure of right alternative algebras*, Ann. of Math. (2) vol. 59 (1954) pp. 408–417.

20. V. G. Askinuze, *A theorem on the splittability of J-algebras*, Ukrain Mat. Zurnal. vol. 3 (1951) pp. 381–398 (Russian).

21. E. A. Behrens, *Nichtassoziative Ringe*, Math. Ann. vol. 127 (1954) pp. 441–452.

22. G. Birkhoff and P. M. Whitman, *Representations of Jordan and Lie algebras*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 116–136.

23. R. H. Bruck, *Some results in the theory of linear non-associative algebras*, Trans. Amer. Math. Soc. vol. 56 (1944) pp. 141–199.

24. R. H. Bruck and E. Kleinfeld, *The structure of alternative division rings*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 878–890.

25. H. E. Campbell, *An extension of the "principal theorem" of Wedderburn*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 581–585.

26. A. Cayley, *On Jacobi's elliptic functions, in reply to the Rev. Brice Bronwin; and on quaternions*, Philosophical Magazine vol. 26 (1845) pp. 210–213.

27. C. Chevalley and R. D. Schafer, *The exceptional simple Lie algebras F_4 and E_6*, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 137–141.

28. P. M. Cohn, *On homomorphic images of special Jordan algebras*, Canadian Journal of Mathematics vol. 6 (1954) pp. 253–264.

29. L. E. Dickson, *On quaternions and their generalization and the history of the eight square theorem*, Ann. of Math. vol. 20 (1919) pp. 155–171.

30. R. Dubisch and S. Perlis, *On the radical of a non-associative algebra*, Amer. J. Math. vol. 70 (1948) pp. 540–546.

31. S. Eilenberg, *Extensions of general algebras*, Annales de la Société Polonaise de Mathématique vol. 21 (1948) pp. 125–134.

32. I. M. H. Etherington, *Genetic algebras*, Proceedings of the Royal Society of Edinburgh vol. 59 (1939) pp. 242–258.

33. ———, *Special train algebras*, Quart. J. Math. Oxford Ser. vol. 12 (1941) pp. 1–8.

34. ———, *Non-associative algebra and the symbolism of genetics*, Proceedings of the Royal Society of Edinburgh vol. 61 (1941) pp. 24–42.

35. H. Freudenthal, *Oktaven, Ausnahmengruppen und Oktavengeometrie*, Utrecht, 1951.

36. M. Hall, *Projective planes*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 229–277. Correction to *Projective planes*, ibid. vol. 65 (1949) p. 474.

37. Harish-Chandra, *On the radical of a Lie algebra*, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 14–17.

38. G. P. Hochschild, *Semi-simple algebras and generalized derivations*, Amer. J. Math. vol. 64 (1942) pp. 677–694.

39. H. Hopf, *Ein topologischer Beitrag zur reellen Algebra*, Comment. Math. Helv. vol. 13 (1941) pp. 219–239.

40. F. D. Jacobson and N. Jacobson, *Classification and representation of semi-simple Jordan algebras*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 141–169.

41. N. Jacobson, *A note on non-associative algebras*, Duke Math. J. vol. 3 (1937) pp. 544–548.

42. ———, *Abstract derivation and Lie algebras*, Trans. Amer. Math. Soc. vol. 42 (1937) pp. 206–224.

43. ———, *Cayley numbers and normal simple Lie algebras of type G*, Duke Math. J. vol. 5 (1939) pp. 775–783.

44. ———, *Structure theory of simple rings without finiteness assumptions*, Trans. Amer. Math. Soc. vol. 57 (1945) pp. 228–245.

45. ———, *Isomorphisms of Jordan rings*, Amer. J. Math. vol. 70 (1948) pp. 317–326.

46. ———, *The center of a Jordan ring*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 316–322.

47. ———, *Derivation algebras and multiplication algebras of semi-simple Jordan algebras*, Ann. of Math. vol. 50 (1949) pp. 866–874.

48. ———, *Lie and Jordan triple systems*, Amer. J. Math. vol. 71 (1949) pp. 149–170

49. ———, *General representation theory of Jordan algebras*, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 509–530.

50. ———, *Representation theory for Jordan rings*, Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, Providence, American Mathematical Society, 1952, vol. 2, pp. 37–43.

51. ———, *Une généralisation du théorème d'Engel*, C. R. Acad. Sci. Paris vol. 234 (1952) pp. 579–581.

52. ———, *Operator commutativity in Jordan algebras*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 973–976.

53. ———, *Structure of alternative and Jordan bimodules*, Osaka Math. J. vol. 6 (1954) pp. 1–71.

54. ———, *A Kronecker factorization theorem for Cayley algebras and the exceptional simple Jordan algebra*, Amer. J. Math. vol. 76 (1954) pp. 447–452.

55. N. Jacobson and C. E. Rickart, *Jordan homomorphisms of rings*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 479–502.

56. ———, *Homomorphisms of Jordan rings of self-adjoint elements*, Trans. Amer. Math. Soc. vol. 72 (1952) pp. 310–322.

57. W. E. Jenner, *The radical of a non-associative ring*, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 348–351.

58. P. Jordan, *Über eine Klasse nichtassoziativer hyperkomplexer Algebren*, Nachr. Ges. Wiss. Göttingen, 1932, pp. 569–575.

59. ———, *Über Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik*, Nachr. Ges. Wiss. Göttingen (1933) pp. 209–214.

60. ———, *Über die Multiplikation quantenmechanischer Grössen*, Zeitschrift für Physik vol. 80 (1933) pp. 285–291.

61. P. Jordan, J. von Neumann, and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. of Math. vol. 35 (1934) pp. 29–64.

62. G. K. Kalisch, *On special Jordan algebras*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 482–494.

63. I. Kaplansky, *Semisimple alternative rings*, Portugaliae Mathematica vol. 10 (1951) pp. 37–50.

64. ———, *Infinite-dimensional quadratic forms permitting composition*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 956–960.

65. E. Kleinfeld, *Alternative division rings of characteristic 2*, Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951) pp. 818–820.

66. ———, *Right alternative rings*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 939–944.

67. ———, *Simple alternative rings*, Ann. of Math. vol. 58 (1953) pp. 544–547.

68. L. A. Kokoris, *Power-associative commutative algebras of degree two*, Proc. Nat. Acad. Sci. U.S.A. vol. 38 (1952) pp. 534–537.

69. ———, *New results on power-associative algebras*, Trans. Amer. Math. Soc. vol. 77 (1954) pp. 363–373.

70. A. Kurosch, *Non-associative free algebras and free products of algebras*, Rec. Math. (Mat. Sbornik) N.S. vol. 20 (62) (1947) pp. 239–262 (Russian).

71. ———, *The present status of the theory of rings and algebras*, Uspehi Matematičeskih Nauk (N.S.) vol. 6, no. 2 (42) (1951) pp. 3–15 (Russian).

72. W. G. Lister, *A structure theory of Lie triple systems*, Trans. Amer. Math. Soc. vol. 72 (1952) pp. 217–242.

73. A. Malcev, *On the representation of an algebra as a direct sum of the radical and a semisimple subalgebra*, C. R. (Doklady) Acad. Sci. URSS vol. 36 (1942) pp. 42–45.

74. ———, *On a representation of nonassociative rings*, Uspehi Matematičeskih Nauk (N.S.) vol. 7, no. 1 (47) (1952) pp. 181–185 (Russian).

75. R. Moufang, *Alternativkörper und der Satz vom vollständigen Vierseit* (D_9), Abh. Math. Sem. Hamburgischen Univ. vol. 9 (1933) pp. 207–222.

76. ———, *Zur Struktur von Alternativkörpern*, Math. Ann. vol. 110 (1935) pp. 416–430.

77. W. H. Mills, *A theorem on the representation theory of Jordan algebras*, Pacific Journal of Mathematics vol. 1 (1951) pp. 255–264.

78. B. H. Neumann, *Embedding non-associative rings in division rings*, Proc. London Math. Soc. (3) vol. 1 (1951) pp. 241–256.

79. L. J. Paige, *A theorem on power-associative loop algebras*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 279–280.

80. A. J. Penico, *The Wedderburn principal theorem for Jordan algebras*, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 404–420.

81. C. M. Price, *Jordan division algebras and the algebras* $A(\lambda)$, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 291–300.

82. R. L. San Soucie, *Right alternative division rings of characteristic* 2, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 291–296.

83. R. D. Schafer, *Alternative algebras over an arbitrary field*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 549–555.

84. ———, *Concerning automorphisms of non-associative algebras*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 573–583.

85. ———, *The exceptional simple Jordan algebras*, Amer. J. Math. vol. 70 (1948) pp. 82–94.

86. ———, *Structure of genetic algebras*, Amer. J. Math. vol. 71 (1949) pp. 121–135.

87. ———, *The Wedderburn principal theorem for alternative algebras*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 604–614.

88. ———, *Inner derivations of non-associative algebras*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 769–776.

89. ———, *A theorem on the derivations of Jordan algebras*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 290–294.

90. ———, *Representations of alternative algebras*, Trans. Amer. Math. Soc. vol. 72 (1952) pp. 1–17.

91. ———, *The Casimir operation for alternative algebras*, Proc. Amer. Math. Soc. vol. 73 (1953) pp. 444–451.

92. ———, *A generalization of a theorem of Albert*, Proc. Amer. Math. Soc. vol. 73 (1953) pp. 452–455.

93. ———, *On the algebras formed by the Cayley-Dickson process*, Amer. J. Math. vol. 76 (1954) pp. 435–446.

94. ———, *Noncommutative Jordan algebras of characteristic* 0, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 472–475.

95. L. A. Skornyakov, *Natural domains of Veblen-Wedderburn projective planes*, Izvestiya Akad. Nauk SSSR vol. 13 (1949) pp. 447–472 (Russian). Amer. Math. Soc. Translation no. 58.

96. ———, *Alternative fields*, Ukrain. Mat. Zurnal vol. 2 (1950) pp. 70–85 (Russian).

97. ———, *Right-alternative fields*, Izvestiya Akad. Nauk SSSR Ser. Mat. vol. 15 (1951) pp. 177–184 (Russian).

98. ———, *Projective planes*, Uspehi Matematičeskih Nauk (N.S.) vol. 6 (1951) pp. 112–154 (Russian). Amer. Math. Soc. Translation no. 99.

99. M. F. Smiley, *The radical of an alternative ring*, Ann. of Math. vol. 49 (1948) pp. 702–709.

100. ———, *Application of a radical of Brown and McCoy to non-associative rings*, Amer. J. Math. vol. 72 (1950) pp. 93–100.

101. ———, *On the ideals and automorphisms of non-associative rings*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 138–143.

102. ———, *Some questions concerning alternative rings*, Bull. Amer. Math. Soc. vol. 57 (1951) pp. 36–43.

103. E. Stiefel, *On Cauchy-Riemann equations in higher dimensions*, Journal of Research of the National Bureau of Standards vol. 48 (1952) pp. 395–398.

104. O. Taussky, *An algebraic property of Laplace's differential equation*, Quart. J. Math. vol. 10 (1939) pp. 99–103.

105. M. L. Tomber, *Lie algebras of type F*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 759–768.

106. J. von Neumann, *On an algebraic generalization of the quantum mechanical formalism*, Mat. Sbornik vol. 1 (1936) pp. 415–482.

107. F. B. Wright, *Absolute valued algebras*, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 330–332.

108. J. L. Zemmer, Jr., *On the subalgebras of finite division algebras*, Canadian Journal of Mathematics vol. 4 (1952) pp. 491–503.

109. M. Zorn, *Theorie der alternativen Ringe*, Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1930) pp. 123–147.

110. ———, *Alternativkörper und quadratische systeme*, Abh. Math. Sem. Hamburgischen Univ. vol. 9 (1933) pp. 395–402.

111. ———, *The automorphisms of Cayley's non-associative algebra*, Proc. Nat. Acad. Sci. U.S.A. vol. 21 (1935) pp. 355–358.

112. ———, *Alternative rings and related questions* I: *Existence of the radical*, Ann. of Math. vol. 42 (1941) pp. 676–686.

University of Connecticut

ASYMPTOTIC PHENOMENA IN MATHEMATICAL PHYSICS

K. O. FRIEDRICHS

The problems I intend to speak about belong to the somewhat undefined and disputed region at the border between mathematics and physics. The fields of physics from which these problems originate are rather classical; the mathematical questions involved are also rather classical. That does not mean that these problems belong to the past. On the contrary, they are quite alive today and—I am convinced—they will remain so for some time.

The problems concern what may be called asymptotic phenomena. Instead of explaining in general terms what I mean by asymptotic phenomena, I prefer to single out at first one class of such phenomena: *discontinuities*. A typical discontinuity of the kind I have in mind is the boundary of the *shadow* which appears when a light wave passes an object. Now, the propagation of light is governed by a partial differential equation which has continuous solutions. How then is it possible that a discontinuity arises? Of course, actually there is no sharp discontinuity at the shadow boundary; there is a transition from light to dark which takes place across a very narrow strip along the shadow boundary. Nevertheless, it is remarkable enough that the differential equations of wave motion have solutions which involve such quick transitions—in fact, most differential equations of physics possess such solutions—and it is an interesting task to study those features of these equations which make such quick transitions possible.

Discontinuities and quick transitions occur in various branches of physics. A striking example of a discontinuity is the *shock* in gas motion. Quick transitions occur frequently in situations in which one perhaps would not speak of a discontinuity. A case in point is Prandtl's ingenious conception of the *boundary layer*. This is a narrow layer along the surface of a body, traveling in a fluid, across which the flow velocity changes quickly. Prandtl's observation of this quick transition was the starting point for his theory of fluid resistance. Other cases, closely related to the boundary layer phenomenon, are the so-called *edge effect* in the deformation of elastic plates and shells and the *skin effect* in the flow of electric currents. A number of other such effects will be described in the later parts of this lecture. All

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these effects may be regarded as typical asymptotic phenomena.

Without attempting to give a precise definition of this term, I shall simply call asymptotic all those phenomena which show discontinuities, quick transitions, nonuniformities, or other incongruities resulting from approximate description.

In the mathematical treatment of such phenomena, physicists have to a certain degree relied on their intuition, and very effectively so; but they have also developed or employed systematic mathematical procedures.

In such a systematic approach one may develop an appropriate quantity with respect to powers of a parameter, ϵ. This expansion is to be set up in such a way that the quantity is continuous for $\epsilon > 0$ but discontinuous for $\epsilon = 0$. Naturally, a series expansion with this character must have peculiar properties. A most remarkable property is that in general these series *do not converge.*

No doubt, divergent series are very useful; it has even been said that they are more useful than convergent ones. However this may be, if a divergent series is useful it must be meaningful.

The use of a series which does not necessarily converge is a typical instance of a "formal procedure" and I should perhaps say a word about the role of *formal procedures* in mathematical physics.

Those who employ mathematics as a tool have rarely been inhibited by the fear of divergence; they have always been confident that, somehow or other, formal procedures are valid. A mathematician may be inclined to frown on this attitude as a superstition; but, on second thought he will yield and try to show that formal procedures—I mean those used by good physicists—indeed are valid if only the meaning of validity is properly interpreted.

There are numerous instances of justification of formal procedures by re-interpretation. I need only refer to the generalizations of the notions of function and differential operator, which have been very effective, in particular, in recent years. It would certainly be interesting to trace the effect of these generalizations in mathematical physics; but I do not intend to do so.

The present talk will be solely concerned with the formal expansions used in the analysis of asymptotic phenomena. The idea of giving validity to these formal series is classical: essentially it goes back to Poincaré.

Poincaré advanced this idea in his work on ordinary differential equations in 1886. Before that time many formal series solutions of such equations had been developed and it was found that they did not converge—in general. Poincaré proved that these formal series solutions represent *asymptotic expansions of actual solutions*. Thus it became clear in which way formal series solutions may be regarded as "valid."

Let me explain the meaning of the phrase "formal series solution" and its "asymptotic" character in connection with an elementary differential equation, namely the *differential equation of the second order*,

$$\epsilon u'' + au' + bu = 0. \tag{1}$$

Here u is a function of a variable z, which in the present context may just as well be taken as real. Furthermore, a and b are analytic functions of z and ϵ is a parameter. Note that this parameter occurs in such a way that the order of the differential equation is reduced for $\epsilon = 0$.

We are interested not in solutions of this differential equation for each fixed value of the parameter ϵ, but in the dependence of such solutions on this parameter, in particular, in the neighborhood of $\epsilon = 0$.

The formal series which we shall consider are not simply power series in ϵ; they are rather of the form

$$e^{S(z)/\epsilon} \sum_n \epsilon^n v_n(z), \tag{2}$$

which will be referred to as the "standard" form. One may try to find solutions of the differential equation (1) which admits such a series expansion. To this end one tries to determine the functions S and v_n by inserting this series into the differential equation and setting the coefficient of every power of ϵ equal to zero. For the functions S and v_n one then finds simple differential equations which are easily solved. The equation to be satisfied by the function S, the so-called "characteristic equation," is

$$(S')^2 + aS' = 0. \tag{3}$$

Here S' is the derivative of S. Inserting the functions S and v_n thus found into the series (2) one obtains a *formal series solution* of the differential equation. In general, though, this series does not converge.

A formal series of the type described is said to represent the *asymptotic expansion* of a function $u(z, \epsilon)$ if the remainder of the terms up to the Nth order is of the order $N+1$; precisely, if

$$e^{-S(z)/\epsilon} u(z, \epsilon) = \sum_{n=0}^{N} \epsilon^n v_n(z) + O(\epsilon^{N+1}),$$

uniformly in an appropriate z-interval.

The problem treated by Poincaré was a little different from the one just described since he did not consider expansions with respect to powers of ϵ but with respect to powers of z^{-1}. Nevertheless, it was to be expected that the analogue of what he proved also holds in the case considered here. That is, each formal series of the standard type should be the asymptotic expansion of an actual solution. That this is so for equations of the second order was proved as early as 1899 by Horn. The corresponding general theory for equations of the nth order, developed in 1908 by Birkhoff, initiated an extensive literature in this and related fields.

These results may be used to answer questions concerning the behavior of specific solutions of the differential equation as the parameter ϵ tends to zero. I should like to discuss one such question, which is extremely elementary, but nevertheless leads in a natural way to the boundary layer phenomenon.

Let us *prescribe boundary values* for the solution of our differential equation at two points, $z=0$ and $z=z_1$, and ask how the solution of this boundary value problem behaves as $\epsilon \rightarrow 0$. Note that for $\epsilon=0$ the differential equation reduces to an equation of the first order. One may therefore wonder whether the solution of the equation of the second order approaches a solution of the equation of the first order. Now a solution of the first order equation is already determined by one boundary condition; one cannot expect that both conditions will be satisfied in the limit. One *boundary condition*—at least—*will get lost*. The question is, which one?

This question and related questions can easily be answered with the aid of two solutions possessing a standard expansion. The answer is that under appropriate conditions the solution of the boundary value indeed does converge to a solution of the first order equation. This solution assumes one of the two boundary values but not the other one. Which boundary value is lost depends on the sign of a/ϵ. Let us assume that the lost boundary value is the one prescribed at $z=0$.

The process of losing a boundary value takes place through *non-uniform convergence*. If the parameter ϵ is small, the solution will run near the limit solution except in a small segment at the end point $z=0$ where it changes quickly in order, as it were, to retrieve the boundary value about to be lost.

Thus a "quick transition" is found to occur. It must occur since a boundary condition is about to be lost; and this loss in turn is necessary since the order of the differential equation is about to drop. To be sure, the *reduction of the order of a differential equation* combined with the *loss of a condition such as a boundary condition* is the most characteristic mathematical feature of asymptotic phenomena.

The next step in the asymptotic analysis of our problem consists in a detailed *description* of the solution of the boundary value problem *within the transition layer*. To this end we introduce a new independent variable by *stretching* the original variable in an appropriate manner. Specifically, we introduce the ratio

$$\zeta = z/\epsilon$$

as a new variable. We then consider the quantity u as a function of ζ, in addition to ϵ, and ask whether or not this new function approaches a limit as $\epsilon \to 0$. This is indeed the case. The limit process is now uniform even near $z=0$. Therefore the new limit function may serve as an approximate description of the quick change of u in the transition layer.

The new limit function is defined for all $\zeta \geqq 0$; in fact it approaches a definite value as $\zeta \to \infty$. Remarkably enough, this value of u at $\zeta = \infty$ is exactly equal to the value which the limit function in the first "direct" process assumes at $z=0$.

This peculiar phenomenon may at first sight appear a little paradoxical, but actually, it is quite natural. Evidently, any fixed z-neighborhood of the point $z=0$ corresponds to an arbitrarily large part of the ζ-axis if only ϵ is made sufficiently small. It is therefore clear that a connection of the two limit functions must involve the behavior of the direct limit function at $z=0$ and the behavior of the second limit function at infinity. The phenomenon just described will be referred to as "identification phenomenon."

The results discussed in connection with the simple equation of the second order are rather typical and they may frequently serve as a guide in understanding other asymptotic phenomena.

As an example, let us consider Prandtl's *boundary layer theory*. This theory was developed in order to solve the problem of fluid resistance, which had caused great difficulties since the time of d'Alembert. It was known that the resistance is due to the viscosity of the fluid; for, it was known that nonviscous fluids do not exert a force on bodies through it. Still, for fluids with low viscosity the assumption of absence of viscosity led to a very satisfactory description of the flow around the body, although it did not yield a resistance.

Prandtl in 1904 resolved this dilemma by advancing the hypothesis that the effect of viscosity is concentrated in a narrow layer near the surface of the body. On the basis of this hypothesis he was able to give a detailed description of the flow in it. With unfailing intuition he appraised the order of magnitude of the various terms of the governing differential equation and rejected those that he judged to be insignificant. The simplified equations thus obtained could then be solved.

There was never any doubt that the boundary layer theory gave a proper account of physical reality, but its mathematical aspects remained a puzzle for some time. Only when this theory is fitted into the framework of asymptotic analysis, does its mathematical structure become transparent.

Viscous fluid flow—in two dimensions, for simplicity—is governed by a partial differential equation of the fourth order. If the viscosity vanishes, the equation reduces to one of the third order. The expansion of viscous fluid flow in the neighborhood of inviscid fluid flow thus appears as an asymptotic expansion, the viscosity being the parameter. A viscous fluid sticks to the wall; hence two boundary conditions are imposed on the viscous fluid flow: namely the conditions that the tangential and normal velocity components vanish. An inviscid fluid is permitted to slide; hence only one condition is imposed on it. Thus one boundary condition gets lost when the viscosity becomes zero.

It is now clear that, before the boundary condition is lost, a quick change must take place across a thin layer near the boundary. This layer, of course, is Prandtl's boundary layer.

Prandtl's detailed description of the flow in the boundary layer can be re-derived by a stretching procedure similar to the one described above. The new stretched variable must be so chosen that with respect to it the boundary layer does not shrink to zero as the viscosity tends to zero.

The approach to the boundary layer theory outlined leads to a definite clarification of the issue but it does not yield a rigorous justification of this theory. The main reason for this difficulty is the nonlinearity of the problem.

The situation is similar in many other nonlinear asymptotic problems. Methods for approximate solutions of such problems are frequently suggested by the facts discussed in connection with the simple ordinary differential equation of the second order.

The boundary layer effect discussed is not the only form of breakdown of uniform convergence. Such a breakdown may also happen in the interior of the domain. A most remarkable such occurrence, in its mathematical aspects even more striking than the boundary layer phenomenon, is the phenomenon discovered by *Stokes* in 1857.

This phenomenon may be explained in connection with the simple differential equation of the second order (1) discussed before. In doing this it is preferable to assume that the coefficients and the solution are analytic functions of the complex variable z.

It was mentioned above that every formal series solution of the standard type is the asymptotic expansion of an actual solution of equation (1), but it was not stated where this asymptotic expansion is valid. In fact, it may happen that this expansion is valid in only a part of the domain in which the function $u(z, \epsilon)$ is defined. That is, it may happen that the function u admits the standard asymptotic expansion in only a part of the z-plane. In other parts it then will possess a completely different asymptotic expansion. The lines which separate subregions of different expansions are called "Stokes lines." The change of the asymptotic expansion on crossing these lines is the "Stokes phenomenon."

In short one may say: ***the Stokes phenomenon*** obtains at a line ***if the asymptotic expansion of the analytic continuation*** of u across this line ***is not given by the analytic continuation*** of the terms ***of the asymptotic expansion.***

If a Stokes phenomenon is present, the leading term of the asymptotic expansion of the solution u changes its character on crossing the Stokes line; one may therefore say that this term is discontinuous across the Stokes line. Suppose the function u stands for a physical quantity and suppose this quantity is approximately described by the leading term of the expansion. If this term is discontinuous, the physical quantity is approximately described as being discontinuous, although actually it is continuous. Thus we have encountered the possibility of describing continuous quantities as discontinuous ones by describing them asymptotically. This possibility is of great significance. To be sure, a large class of ***discontinuity phenomena in mathematical physics may be interpreted as Stokes or boundary layer phenomena.***

For an ordinary linear differential equation it is easy to locate lines at which a Stokes phenomenon occurs. One can always find such lines near a "turning point." A ***turning point*** or ***transition point*** is a point z at which two roots $S'(z)$ of the characteristic equation (3) coalesce: $S_1'(z) = S_2'(z)$. Stokes phenomena then may occur at certain rays through the turning point. For equations of the type here considered, these rays are curves on which the real parts of $S_1'(z)$ and $S_2'(z)$ agree: Re $S_1'(z) =$ Re $S_2'(z)$.

For the differential equation (1), in particular, there are four such rays issuing from a turning point. At which ones of these rays the Stokes phenomenon occurs depends on the solution considered.

Specifically, one can find a solution which possesses a standard asymptotic expansion in the open region R generated by three of the four sectors that are formed by the four rays. It possesses an asymptotic expansion also in the fourth sector, at the two rays bounding the fourth sector, and at the turning point; but all three expansions differ from each other and from the standard expansion in the region R. Clearly, a Stokes phenomenon is present; the two rays bounding the fourth sector are Stokes lines.

Of course, one wants to know how to find these different expansions.

Before indicating how one may attack this "continuation" problem I should mention that such a turning point problem was first treated by Jeffreys in 1923 in connection with a somewhat different differential equation.

One possible approach to a solution of the continuation problem consists in reducing it to the problem of finding the asymptotic expansion *at* the turning point. This problem will be referred to as the "connection problem."

I shall briefly indicate a formal procedure by which this connection problem may be attacked.

In this approach one again employs the "method of stretching." For equation (1), in particular, one introduces

$$\zeta = z/\epsilon^{1/2},$$

instead of z, as new independent variable, assuming the turning point to be at $z=0$. It would not be difficult to motivate the choice of $\epsilon^{1/2}$ as stretching factor instead of ϵ. Again, any fixed neighborhood of the turning point will eventually cover the whole ζ-plane. The quantity u, when considered as a function of ζ, now possesses an expansion with respect to powers of $\epsilon^{1/2}$. The terms of this expansion, defined in the whole ζ-plane, can be determined by identifying their behavior at $\zeta=\infty$ with the behavior of the terms of the direct expansion at $z=0$. In other words, one may employ an identification procedure similar to the one discussed in connection with the boundary value problem. The expansion of the solution u at the turning point can then be found.

As mentioned before, this approach is only a formal procedure; naturally, one will ask: does it yield correct results? For differential equation (1), in particular, it is not too difficult to prove that this is so. Such a proof is not so easy, however, in more complicated cases;

for example if an additional singularity is present, or if the equation is of higher order.

The first decisive step in developing a rigorous turning point theory was taken by Langer in 1934. Langer, in fact, treated at first a problem which is not as simple as the one discussed here. Subsequently, a considerable amount of work on the turning point problem has been done, and is being done today.

There are many interesting problems of *mathematical physics* in which a *turning point* analysis plays a role.

Wentzel, Brillouin, and Kramers in 1926 used asymptotic approximations and turning point considerations in solving eigenvalue problems in quantum theory.

A very remarkable problem which requires a turning point analysis is the problem of the *stability of viscous fluid* and the onset of turbulence. Quite a number of aerodynamicists and mathematicians have worked on this somewhat controversial question. Early theoretical investigations led to the prediction that such an instability should occur under peculiar circumstances. This prediction should perhaps have been believed by aerodynamicists, but it was not generally accepted at first. Eventually, the prediction was confirmed by experiment with surprising accuracy.

The pertinent mathematical situation was definitely clarified only in recent years by Wasow through a rigorous turning point analysis.

The asymptotic phenomena of ordinary differential equations which I have described up to now involve linear equations; of course such phenomena have also been studied in connection with *nonlinear* equations. An interesting problem concerns periodic solutions of a differential equation of the form

$$\epsilon u'' = f(u', u).$$

The question is what happens with these periodic solutions as $\epsilon \to 0$, in particular if the limit equation

$$f(u', u) = 0$$

has no periodic solution. Of course there could be no boundary layer effect in the strict sense since there is no boundary. What happens is that the limit function—if it exists—satisfies the equation $f(u', u) = 0$ except at certain points where the derivative u' has a jump discontinuity.

A problem of this type was first treated by van der Pol, 1927, who in this way explained the occurrence of certain jerky oscillations in electric networks, which he called "relaxation oscillations." Subsequently much work was done on electrical and mechanical oscillations of this kind, as well as on the purely mathematical aspects of the problem. Strong results on asymptotic periodic solutions have been obtained by Levinson since 1942.

Another, rather spectacular, case of a discontinuity which may take place in the interior of the domain and not at a boundary is the gas dynamical *shock*. The shock may also be interpreted as the limit of a quick transition and be treated by an asymptotic analysis which involves the drop of the order of a differential equation. The same may be said about the closely related phenomena of *explosion* or *detonation*.

Let me turn to *partial differential equations* and first consider the *hyperbolic* equation

$$u_{tt} - \Delta u = 0, \tag{4}$$

called the "wave equation." Here u is a function of t, x, y, z and Δ is the Laplacian.

The propagation of electromagnetic and acoustic waves is governed by this equation; but these processes are frequently treated in a different manner, in the manner of geometrical optics. One is led to this second treatment in a natural way by asking for formal solutions of the "standard" form

$$u = e^{S/\epsilon} \sum_{n} \epsilon^n v_n. \tag{5}$$

Here S and v_n are functions of t, x, y, z.

For these functions simple equations are found and readily solved. These equations are of the first order; the drop in order, so typical for asymptotic problems, is thus apparent. The equation for S, the characteristic equation

$$S_t^2 = (\nabla S)^2, \tag{6}$$

is nonlinear.

The formal series (5) is similar to that used for ordinary differential equations. There is a slight difference, however, since the parameter ϵ entering it does not occur in the differential equation. If a concrete problem is to be solved by using this formal solution, the parameter will have to be identified with one of the data of the problem, such as a wave length or a pulse width.

In accordance with the principle of Poincaré, one expects that there exist actual solutions having these formal solutions as asymptotic expansions. That this is so has apparently not yet been proved. One should think, however, that the available methods of proving the existence of solutions of hyperbolic equations would be strong enough for this purpose. But, the primary interest of the asymptotic theory of the wave equation seems to lie in the asymptotic expansion of the solutions of specific problems.

In many cases one can describe a wave process with the aid of the leading term of the above expansion (5). This description now leads to *geometrical optics*. The function S is the eiconal, and the equation $S=\text{const.}$ describes the motion of a wave front. It has been known for a long time that the transition of wave optics to geometrical optics involves asymptotic expansion; but little attention was paid to the fact that this expansion enables one also to determine the propagation of the amplitude. A systematic exploitation of this possibility was started only about ten years ago by Luneburg.

The most interesting asymptotic phenomenon of wave motion occurs when the eiconal S develops singularities on certain surfaces, called *caustics*. Such a singularity may occur since the differential equation satisfied by S is nonlinear.

Suppose now the wave function u possesses a standard asymptotic expansion on one side of the caustic, then its expansion on the other side will be a different one. In other words, a *Stokes phenomenon appears at the caustic.*

The situation is similar at a *shadow boundary*. The *transition from light to shadow is also a Stokes phenomenon*. For, a shadow boundary is just a line across which the asymptotic expansion changes, in other words, a Stokes line. To determine the asymptotic expansion in the shadow region is an interesting problem which has not yet been solved completely.

In connection with the interpretation of the shadow as a Stokes phenomenon I may perhaps make a general remark about *the role of discontinuities in the description of nature*. On the one hand, discontinuities appear to play a secondary role, namely when they are considered as approximate descriptions of continuous phenomena involving quick transitions. On the other hand, discontinuities play a primary role. For, the experimental description of nature and the theoretical description based on it involves objects with more or less sharp outlines. Therefore, nature could not be described in this way if natural objects did not possess sharp outlines, i.e. discontinuities. In other words, the quantities employed to describe nature could not even be defined if discontinuities did not occur. In this sense, discontinuities appear to play a primary role.

It may be debated whether or not this situation involves a vicious or a nonvicious circle. In any case, one may say that asymptotic description is not just a matter of imperfection, but is an essential element in the mathematical description of nature.

The next subject for discussion naturally would be *elliptic partial differential equations*. Various important mathematical results have been obtained in this field. I need only refer to the classical asymptotic theory of eigenvalues developed by Weyl, Courant, and Carleman.

Instead of discussing these results I prefer to discuss a few problems from *mechanics* which involve elliptic equations.

Let us first turn to problems of *elasticity*. One such problem arises when a thin circular disk is subjected to lateral pressure applied along the edge. The disk will deflect if this pressure is large enough. This problem was investigated by Stoker and myself in 1940. The main question was what happens if the lateral pressure is increased indefinitely or—what is equivalent—if one lets the thickness of the plate shrink to zero. The answer was quite unexpected to us.

The deflection w and the stress function ϕ, considered as functions of x and y, satisfy a pair of differential equations

$$h^2\Delta^2 w = f, \qquad \Delta^2\phi = g$$

in which Δ^2 is the biharmonic operator, and f and g are quadratic functions in the second derivatives of w and ϕ; furthermore, h is the thickness of the plate.

The equations which result when one sets $h = 0$ imply a constant distribution of the pressure in the plate. The question then arose, what is the value of this pressure? Is it the value prescribed at the edge?

One really had no right to expect this, since the order of the system of equations drops if one sets $h = 0$, and one must face the possibility that at least one boundary condition gets lost. This might be the boundary condition concerning the pressure. If so, there should be a thin boundary layer across which internal and external pressure are connected.

The answer to this question could be derived from a boundary layer analysis of the type described before with the aid of the method of stretching. The answer was that the interior limit pressure indeed is not equal to the external pressure; but in addition it was found that this pressure is negative, that is represents *tension*. Thus tension should prevail over most of the plate in spite of the fact that a compression is applied at its edge.

This result was very surprising and we wondered whether we were not misled by having employed a boundary layer analysis heuristically. However, the validity of this procedure was proved rigorously in this case; in fact, the present case is one of the few involving nonlinear equations in which this was possible.

Quick transitions of stresses and strains at the boundary of a deformed elastic body have been observed in many cases. Instead of a boundary-layer phenomenon one then speaks of an "edge effect." A number of edge effects closely related to the one discussed have been treated in the last ten years. The first detailed mathematical analysis of an edge *effect* was given by H. Reissner in 1912 in his theory of shells.

A rather famous edge effect was observed much earlier. This is the effect in the *bending of thin plates with free edges*. The differential equation for the deflection w of such a plate is the biharmonic equation $\Delta^2 w = 0$. Two boundary conditions should be imposed on w on mathematical grounds, but three conditions were strongly favored on physical grounds. A pair of two very peculiar boundary conditions were proposed by Kirchhoff in 1850 and later on justified by Kelvin and Tait by qualitative arguments which essentially involved a boundary layer. But only recently was this problem treated by a consistent asymptotic analysis.

In such an analysis, all significant quantities must first be developed with respect to the thickness of the plate. A stretching technique of the type discussed earlier leads to a description of the stresses in the boundary layer. In this way one finds that Kirchhoff's conditions indeed are correct, but, in addition, one can clearly understand in detail how the third boundary condition gets lost.

Asymptotic approximation of quantities defined in thin layers may lead to strange phenomena. I should like to mention one such phenomenon in connection with a problem in *fluid dynamics;* namely the problem of determining the flow of a layer of fluid over a bottom surface under the influence of gravity. This flow is described by a potential function.

There exists a very effective approximate treatment of such flow based on the assumption that the layer of fluid is very thin. This is the so-called "shallow water theory."

A peculiar feature of this approximation is that in it the motion is governed by a hyperbolic differential equation, while originally it is described by a potential function, the solution of an elliptic equation. Small disturbances, which in the original description would affect the whole flow instantaneously would be propagated with a finite speed according to the *shallow water theory*. How is this possible?

This discrepancy is a typical symptom of asymptotic approximation. The shallow water theory results from the leading term of an expansion with respect to the average thickness of the layer. In deriving this expansion one must stretch the vertical variable y and keep the horizontal variable x. The potential equation then goes over into the equation

$$h^2\phi_{xx} + \phi_{\eta\eta} = 0$$

where $\eta = y/h$. Clearly, for $h = 0$ the elliptic character of the differential equation is lost. One may perhaps hesitate to destroy the potential equation and to spoil the advantage of working with potential functions. Still, it is appropriate to do so. It has been suggested to call this procedure the "method of spoiling."

Incidentally, it was recently shown by Hyers and myself that a similar *method of spoiling* is the clue to a rigorous treatment of the "solitary wave," i.e. a steady shallow water wave with a single hump.

One more problem from fluid dynamics should be discussed, Prandtl's theory of the *airfoil of finite span*. If a thin wing of finite span travels through the air, a vortex sheet will develop at the trailing edge. A precise treatment of the resulting airflow offers insurmountable difficulties, but Prandtl gave an approximate treatment derived from rather intuitive arguments. He replaced the wing by a line, called "lifting" line at which the flow is assumed to have an appropriate singularity and made other simplifications. This procedure was strikingly effective in the case of normal flight, i.e. flight in the direction perpendicular to the wing, but the method breaks down when applied to a wing in yaw or a swept back wing.

This difficulty was quite recently overcome by a systematic asymptotic treatment of the problem. The wing was imbedded in a set of wings with the same span and similar cross-section profiles. When the chord ϵ of the profile approaches zero, the wing shrinks to a line, the "lifting line." The potential function describing the airflow past the wing is now developed with respect to ϵ about $\epsilon = 0$. The leading term in this expansion describes exactly Prandtl's approximate flow; it can easily be given explicitly as soon as the circulation around the lifting line is known.

In two dimensional airfoil theory the circulation can be deduced from the shape of the profile by Kutta and Joukowski's theory. In the present treatment the shape of the airfoil has disappeared in the limit. To find the missing circulation this shape must be recovered. That can be done by the method of stretching. The new, stretched variables may be so chosen that the profile remains fixed; but then the length of the span tends to infinity as $\epsilon \to 0$. With respect to the new variables the flow approaches a new limit. The new limit flow is essentially a two dimensional flow around an airfoil with infinite span. Such flow is well determined if its behavior at infinity is known. Now, infinity in the new variables corresponds to the neighborhood of the lifting line in the original variables. By identifying the behavior of the terms in the direct expansion at the lifting line with the behavior at infinity of the terms of the expansion after stretching one is able to find the missing circulation.

In this way, one can retrieve Prandtl's results for a wing flying in the head-on direction, and, in addition, one can treat *wings in yaw, and swept back wings*, which up to now appeared not to be amenable to an approach employing a lifting line.

In most problems discussed so far quite *similar methods of asymptotic analysis* were employed. These methods led to success in a number of cases, but still their scope is limited. That applies, in particular, to the simple method of stretching which played such a prominent part in our discussion.

Although the scope of the method of stretching is rather limited, the general idea of employing appropriate *transformations of the independent variable*, depending on the parameter, seems to be very fruitful. The importance of this idea, which occurs already in Poincaré's work, was strongly emphasized by Lighthill and various specifically adapted transformations were employed by him and others with remarkable success. One of the goals which may be attained in this way is uniformity. That is, one desires an approximation which is uniformly valid on the boundary and off the boundary in a problem of the boundary layer type, or on the Stokes line and off the Stokes line when a Stokes phenomenon is involved.

There are innumerable other asymptotic problems in mathematical physics more or less related to those discussed. An important field of such questions is concerned with physical processes which do or do not approach a *steady limit* as time goes on indefinitely. In fact, the occurrence of *stability and instability* may be regarded as an asymptotic phenomenon; and the decision between stability and instability therefore requires an asymptotic analysis.

It should not be forgotten that the foundations of *statistical mechanics* originated by Boltzmann and Gibbs abound with asymptotic problems of great significance and great difficulty.

It is not my intention to speak about these various fields. There is only one field of physics in which asymptotic problems occur to which I should like to refer: *quantum theory*.

It would have been tempting to speak about the *quantum theory of fields*. Here physicists have developed formal series expansions with great ingenuity. These series, to say the least, do not converge, and yet, to an amazing degree, they make sense physically. To be sure, to justify these formal procedures is a challenge to the mathematician. However, I want to confine myself to discussing one or two problems in quantum theory which are well understood mathematically.

The first of these problems concerns the differential equation

$$\epsilon i \frac{d}{ds} \psi = H(s)\psi$$

in which ψ is an element of a *Hilbert space* which depends on the variable s and H is a self-adjoint operator, which is also assumed to depend on s. One is interested in the asymptotic behavior of a solution for small values of ϵ.

The equation is essentially of the same type as the ordinary differential equation considered before, the only difference being that the function ψ is an element of a Hilbert space. For this reason, a few technical difficulties must be overcome; but the idea of asymptotic analysis is exactly the same as for ordinary differential equations.

The interest in this problem arises in connection with the "adiabatic theorem" in quantum theory. The differential equation is the *Schrödinger equation* for the state ψ of the system if s/ϵ is regarded as the time. The operator $H(s)$ is then a Hamiltonian which varies "slowly" if ϵ is small. The *adiabatic theorem* now states: if the state ψ was an eigenstate of H originally, $t=0$ say, it will remain approximately an eigenstate if t increases provided ϵ is small enough.

This theorem now results from the leading term of the asymptotic expansion of the solution ψ. In addition, however, the asymptotic analysis enables one to determine terms of higher order in the expansion and thus to estimate the "probability of nonadiabatic transition."

This is of particular interest in the case in which *two eigenvalues* of the operator H *coalesce* at some time during the process and the question has been asked what happens in such a case. Now, if one looks at the problem as an asymptotic problem, one need only realize that coalescence of eigenvalues corresponds to *a turning point*. A turning point analysis then gives a simple answer.

I should like to *summarize* some of the ideas presented. I have tried to show that a great number of asymptotic problems in mathematical physics have important features in common: in particular the drop of the order of the differential equation and the loss of a continuity or boundary condition. I have tried to show that many of these problems can be attacked by similar methods. These methods involve asymptotic expansion, and the analysis of the regions of nonuniformity by stretching or adjustment of the independent variables combined with an appropriate identification procedure.

Furthermore, as I had mentioned in connection with the problem of the shadow, *asymptotic description* is not only a convenient tool in the mathematical analysis of nature, it *has a* more *fundamental significance.*

This fact is also apparent in the relationship between classical and quantum mechanics; a few words may be said about this relationship.

When wave mechanics was discovered it was immediately recognized that the relationship between wave mechanics and classical mechanics is essentially the same as that between wave optics and geometrical optics. That is to say, classical mechanics results from the leading term in the asymptotic expansion of quantum mechanics and in this sense classical mechanics plays a secondary role. On the other hand, as has been stated frequently in discussions of the foundations of quantum-theory, it is impossible to define and explain the basic notions of quantum mechanics without reference to classical mechanics. In this sense then, classical mechanics plays a primary role.

Thus we meet again the same circular situation which we had discussed in connection with the problem of the shadow. Indeed, *the relationship between classical and quantum mechanics affords a striking illustration of the fundamental role which asymptotic description plays in the mathematical description of nature.*

SELECTED REFERENCES

From the vast literature on asymptotic phenomena we shall present only a small selection. Additional references will be found in the publications quoted; but even with these additions the bibliography would be far from complete. No references will be given to work on various subjects which were only slightly touched upon in the lecture.

In particular, no reference will be made to the extensive literature on asymptotic series (e.g. the work of van der Corput) and on methods of asymptotic expansion of integrals and special functions (such as the saddle point method), except the résumé which Stokes gave on the phenomenon named after him,

G. G. Stokes, *On the discontinuity of arbitrary constants that appear as multipliers of semi-convergent series*, Acta Math. vol. 26 (Abel Centenary Volume) (1902) pp. 393–397. Reprinted in *Mathematical and Physical Papers*, vol. V, pp. 283–287.

ORDINARY LINEAR DIFFERENTIAL EQUATIONS

Among the *classical work on asymptotic expansion* of solutions of ordinary differential equations we mention[1] that of Poincaré (1886), Horn (1899), Birkhoff (1908), Noaillon (1912), Tamarkin (1917), Perron (1918). Among more *recent work* we mention that of Trjitzinski (1934), Turittin (1936), and Hukuhara (1937). For the application of this theory to *eigenvalue problems* see the work of Tamarkin (1917 and 1927) and Birkhoff (1923); for the application to *boundary value problems* see Wasow (1944).

References to this work will be found in most *expository presentations* such as

G. D. Birkhoff and R. E. Langer, *The boundary problems and developments associated with a system of ordinary linear differential equations of the first order*, Proceedings of the American Academy of Arts and Sciences vol. 58 (1923) pp. 51–128. Reprinted in *Collected mathematical papers*, vol. I, pp. 347–424.

M. Hukuhara, *Sur les propriétés asymptotiques des solutions d'un système d'équations différentielles linéaires contenant un paramètre*, Kyushu Imperial University, Faculty of Engineering Memoirs, vol. 8, 1936–1940, pp. 249–280.

H. L. Turittin, *Asymptotic expansions of solutions of systems of ordinary linear differential equations containing a parameter*, Contributions to the Theory of Nonlinear Oscillations, vol. II, Annals of Mathematical Studies, no. 29, Princeton University Press, 1952, pp. 81–116.

W. Wasow, *Introduction to the asymptotic theory of ordinary linear differential equations*, Report of the National Bureau of Standards, 1954, pp. 54–56.

For the work of Jeffreys (1923), Langer (1934), Cherry (1949), and others on the *turning point problem* see the

Review of the Literature, *Asymptotic solutions of differential equations with turning points*, Department of Mathematics, California Institute of Technology, Pasadena, 1953.

This is referred to as "Cal. Tech. Review" in the following.

NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

For a summary of earlier work see

M. H. Dulac, *Points singuliers des équations différentielles*, Memorial des Sciences Mathématique, no. 61, 1934.

For the later work by Hukuhara (1935), Nagumo (1939), Malmquist (1940) and others see

W. Wasow, *Singular perturbations of boundary value problems for nonlinear differential equations of the second order*, Communications on Pure and Applied Mathematics, vol. 9 (1956) no. 1.

For the work of Lighthill (1949) and others on *uniform expansions* see

H. S. Tsien, *The PLK method*, Advances in Applied Mechanics, vol. IV, Academic Press Inc., New York, 1955.

W. Wasow, *On the convergence of an approximation method of M. J. Lighthill*, Journal of Rational Mechanics and Analysis, to appear in 1955.

See also the work by Bromberg and Latta referred to below.

For the work by various authors on "singular perturbations" of solutions of *boundary or initial value problems* and of *periodic solutions* see

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[1] The year indicated frequently refers only to the first of a number of publications.

N. Levinson, *Perturbation of discontinuous solutions of nonlinear systems of differential equations*, Acta Math. vol. 82 (1950) pp. 71–106

and also

A. B. Vasilieva, Mat. Sbornik N.S. vol. 31 (1952) pp. 587–644.

RELAXATION OSCILLATIONS

For van der Pol's relaxation oscillations and the asymptotic theory of Haag (1943), Dorodnitsyn (1947), and others see

J. J. Stoker, *Nonlinear vibrations*, New York, Interscience, 1950, Chap. V. 6 and Appendix IV.

STABILITY

For a treatment of problems of stability see

R. Bellman, *Stability theory of differential equations*, New York, McGraw-Hill, 1953

and the survey article

V. V. Nemitzki, Uspehi Matematičeskih Nauk N.S. vol. 9 (1954) pp. 39–56.

PARTIAL DIFFERENTIAL EQUATIONS

The asymptotic theory of the equation

$$\epsilon\Delta u = Lu$$

in which Lu is an expression of the first order was treated by Wasow (1944) and

N. Levinson, Ann. of Math. vol. 51 (1950) pp. 428–445.

Asymptotic properties of the Navier-Stokes equation and related equations were investigated by Lagerstrom, Cole, Latta and others; see

P. A. Lagerstrom and J. D. Cole, *Examples illustrating expansion procedures for the Navier-Stokes equations*, Guggenheim Aeronautical Laboratory, California Institute of Technology, Pasadena, California, May, 1955.

WAVE EQUATION AND GEOMETRICAL OPTICS

R. K. Luneburg, Lecture Notes, Brown University, 1944, and New York University, 1947–1948.

F. G. Friedlander, Proc. Cambridge Philos. Soc. vol. 43 (1946) pp. 284–286.

F. G. Friedlander and J. B. Keller, *Asymptotic expansions of solutions of* $(\nabla^2+k^2)u=0$, New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Research Report No. EM-67, September, 1954.

ELASTICITY

For the edge effect in shells and plates see

H. Reissner, *Spannungen in Kugelschalen (Kuppeln)*, Festschrift H. Mueller, Breslau, Leipzig, 1912, pp. 181–193.

O. Blumenthal, Archiv der Mathematik und Physik (3) vol. 19 (1912) pp. 136–174.

E. Reissner, Quarterly of Applied Mathematics vol. 10 (1952) pp. 167–173.

For the Kirchhoff's boundary conditions see a paper by R. Dressler and the author to appear in the Communications on Pure and Applied Mathematics.

For edge effects occurring in nonlinear problems see

K. O. Friedrichs and J. J. Stoker, Amer. J. Math. vol. 63 (1941) pp. 839–888,

E. Bromberg and J. J. Stoker, Quarterly of Applied Mathematics vol. 3 (1945) pp. 246–265

and a paper by E. Bromberg to appear in the Communications.

FLUID DYNAMICS

For the principles of *boundary layer theory* see

L. Prandtl, *The mechanics of viscous fluids*, Aerodynamic Theory, Ed. by W. F. Durand, vol. III G. (1935).

S. Goldstein, *Modern developments in fluid dynamics*, vol. I, Oxford, 1938.

For the *stability of viscous fluid flow* see the survey

W. Tollmien, *Laminare Grenzschichten*, Fiat Review of German Science, 1939–1946, vol. II, Hydro- and Aerodynamics, Wiesbaden, 1948, pp. 21–53.

Also see

C. C. Lin, *Hydrodynamic stability*, Proceedings of the Fifth Symposium in Applied Mathematics, New York, McGraw-Hill, 1954, pp. 1–18.

For the pertinent turning point analysis see the work by Wasow (1948–1952) quoted in the Cal. Tech. Review.

For the *shallow water theory* see the book

J. J. Stoker, *Water waves*, New York, Interscience

to appear.

For the principles of airfoil theory see the *Aerodynamic theory*, vol. I. For the approach described in the lecture see a paper by S. Ciolkowski and the author to appear in the Communications.

QUANTUM THEORY

For references to the work by Born and Fock (1926), Kato (1950) on the *adiabatic theorem* see a report by the author issued by the Institute of Mathematical Sciences, New York University.

The asymptotic expansion which leads from quantum mechanics to classical mechanics is clearly described by

G. D. Birkhoff, Bull. Amer. Math. Soc. vol. 39 (1933) pp. 681–700.

NEW YORK UNIVERSITY

THE SUMMER MEETING IN ANN ARBOR

The sixtieth Summer Meeting of the American Mathematical Society was held from Tuesday, August 30, through Friday, September 2, 1955, at the University of Michigan, Ann Arbor, Michigan. Several other organizations met in Ann Arbor at about the same time: the Mathematical Association of America, the Institute of Mathematical Statistics, the Econometric Society, the Association for Symbolic Logic, the Society for Industrial and Applied Mathematics, the Industrial Mathematics Society, and Pi Mu Epsilon.

Over 800 people registered for the meeting, including 600 members of the Society. This was a record attendance for a Society meeting.

Professor Nathan Jacobson of Yale University delivered the Colloquium lectures, entitled *Jordan algebras*, on Tuesday at 11:00 A.M. and Wednesday, Thursday, and Friday at 9:00 A.M. Presiding offiers were Professors R. L. Wilder, A. A. Albert, Marshall Hall, Jr., and T. H. Hildebrandt.

By invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings, Professors Raoul Bott, of the University of Michigan, and Edwin Hewitt, of the University of Washington, addressed the Society. Professor Bott spoke on *Some applications of the Morse theory to the topology of manifolds* at 1:30 P.M. Tuesday, and Professor Hewitt on *Convolution algebras* at 11:00 A.M. Wednesday. Presiding officers were Professors G. A. Hedlund and A. D. Wallace, respectively.

There was a joint session of the Society with the Econometric Society on *Extremum problems*, which was held in two parts on Thursday at 10:15 A.M. and 1:45 P.M. The lecturers were Professors Jacob Wolfowitz, Samuel Karlin, P. C. Rosenbloom, and Leonid Hurwicz. The presiding officers were Professors E. G. Begle and A. W. Tucker.

There was a joint session of the Society with the Industrial Mathematics Society on Friday at 10:15 A.M. There was an address by Dr. Saul Rosen of Wayne University entitled *The influence of digital computers on numerical analysis*, and four short papers. Professor A. W. Jacobson presided.

There were nineteen sessions for contributed papers. The presiding officers were Professors R. H. Bing, R. H. Bruck, A. H. Copeland, Dr. J. H. Curtiss, Dr. D. A. Flanders, Professors A. M. Gleason, J. W. Green, A. E. Heins, A. W. Jacobson, J. P. LaSalle, C. J. Nesbitt, G. de B. Robinson, H. J. Ryser, A. C. Schaeffer, R. D. Schafer, Seymour Sherman, P. A. Smith, R. M. Thrall, and C. B. Tompkins.

More papers were read at this meeting than at any other meeting of the Society.

The Committee on Arrangements provided a highly enjoyable program for the entertainment and recreation of the guests of the University. In addition, the running of the business of the meeting was carried on with commendable efficiency. The Associate Secretary takes this opportunity to pay tribute to Professor Wilfred Kaplan, the Chairman of the Committee; the meeting is a monument to his patience and energy.

The Council met on Tuesday evening, August 30, 1955.

The Secretary announced the election of the following forty-one persons to ordinary membership in the Society:

Mr. Abdul Kadir Aziz, University of Maryland;
Mr. Peter Greenough Bartlett, Askania Regulator Co., Chicago, Illinois;
Mr. Errett Albert Bishop, University of California, Berkeley;
Mr. Albert James Brinson, Seton Hall University;
Mr. Clarence Harlan Cook, The Glenn L. Martin Company, Baltimore, Maryland;
Mr. William Edwin Cummins, David Taylor Model Basin, Washington, D. C.;
Mr. Robert Edwin Daniels, Haller, Raymond and Brown, Inc., State College, Pennsylvania;
Mr. John Frank Detlef, Ballistic Research Labs.;
Reverend Edward Dennis Eichman, Epiphany Apostolic College;
Miss Constance Mary Foley, University of New Hampshire;
Dr. Gerald Henry Fraser Gardner, Cornell University;
Dr. Baruch Germansky, Universität Frankfurt, Mainz, Germany;
Mr. Hyman Housman, New York University;
Mr. Bernard Benedict James, Boeing Airplane Co., Seattle, Washington;
Pfc. Floyd Idwal John, United States Army, Dugway, Utah;
Miss June Rose Mary Jensen, Polytechnic Institute of Brooklyn;
Mr. James Kenyon Knowles, Massachusetts Institute of Technology;
Mr. Neophytos Kockinos, University of California, Berkeley;
Mr. Yehiel Lehrer, Weizmann Institute of Science, Israel;
Mr. Ronald Melvin Lewis, University of Southern California;
Mr. Ralph Aloysius Luebbe, University of Cincinnati;
Mr. Jon N. Mangnall, Cornell Aeronautical Laboratory, Buffalo, New York;
Miss Ann Muzyka, Polytechnic Institute of Brooklyn;
Mr. Anthony Joseph Nespole, College of the City of New York;
Mr. Edward Valentine Oborah, University of Chicago;
Mr. Julius Louis Pallone, Polytechnic Institute of Brooklyn;
Sister Julia Pflum, College of Mount St. Joseph;
Dr. Roger Stanton Pinkham, Duke University;
Mr. David Alexander Pope, University of California, Los Angeles;
Miss Susan Pyeatt, Catholic University of America;
Professor Fazlollah Reza, Massachusetts Institute of Technology;
Dr. Donald Eugene Richmond, Convair, Ft. Worth, Texas;
Mr. Robert Steele Roth, Aberdeen Proving Ground, Maryland;
Dr. Parfeny Pavlovich Saworotnow, Catholic University of America;
Mr. Thomas I. Seidman, AEC Computing Facility, New York;

Mr. Charles Robert Seliger, Rutgers University;
Mr. James Leroy Snyder, University of Cincinnati;
Mr. Bernard Albert Sobel, Ethyl Corp., Baton Rouge, Louisiana;
Mr. Vytas Francis Volertas, Aircraft Armaments, Inc., Cockeysville, Maryland;
Mr. Edward Joseph Voltaggio, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland;
Miss Martha Becket Walton, University of Nancy, Nancy, France.

It was reported that the following eight persons had been elected to membership on nomination of institutional members as indicated:

Brooklyn College: Mr. Abraham Karrass.
University of Cincinnati: Mr. Smbat Abian.
University of Michigan: Mr. Donald Warren Crowe.
Oregon State College: Mr. Robert Lewis Brock, Mr. Robert Wallace Brown, and Mr. Louis Baker Rall.
Santa Clara University: Mr. Samuel A. Francis.
Tulane University: Miss Ruth Evans Biggers.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematiker-Vereinigung: Dr. Heinz Bauer, Mathematical Institute, University of Erlangen, Germany; Indian Mathematical Society: Dr. N. Padma, Annamalai University, Annamalainagar, India; Svenska Matematikersamfundet: Professor Karl Gustav Natanael Hössjer, Chalmers University of Technology, Göteborg, Sweden; Unione Matematica Italiana: Mr. Flavio Marani, Banca d'Italia, Roma, Italy.

The following appointments by the President were reported: as Chairman of the Committee on Translations from Russian and Other Foreign Languages for a term of one year beginning July 1, 1955: J. L. Doob; and as members for terms of three years beginning July 1, 1955: R. P. Boas and Irving Kaplansky (Committee now consists of J. L. Doob, Chairman, R. E. Bellman, R. P. Boas, Irving Kaplansky, and Hans Samelson); as Chairman of the Organizing Committee for Summer Institutes for a period of one year beginning July 1, 1955: Edwin Hewitt; and as members for a period of three years beginning July 1, 1955: Nelson Dunford and E. R. Kolchin (Committee now consists of Edwin Hewitt, Chairman, S. S. Chern, Nelson Dunford, F. B. Jones, E. R. Kolchin, and J. J. Stoker; as an ad hoc committee on local arrangements for the 1955 Summer Institute: R. H. Bing, Chairman, R. H. Bruck, R. C. Buck, and A. C. Schaeffer; as auditors for 1955: C. R. Adams and C. H. W. Sedgewick; as a committee on Mathematicians Dismissed for Political Reasons: E. E. Moise, Chairman, W. L. Duren, H. W. Kuhn, R. D. James, and J. B. Rosser; as a committee to study privileges of membership insofar as publications are concerned: A. E. Meder, Jr., Chairman, E. G. Begle, and Einar Hille.

The following appointments to represent the Society were reported: at the 100th anniversary of St. Mary's College and the dedication of St. Mary's Science Hall at Notre Dame, Indiana, on April 23, 1955: Professor A. E. Ross; at the inauguration of President George W. Starcher of the University of North Dakota on April 20, 1955: Professor R. D. Staley; at the inauguration of Arthur B. Bronwell as President of Worcester Polytechnic Institute on April 30, 1955: Professor C. T. Bumer; at the inauguration of Willis McDonald Tate as President of Southern Methodist University on May 5, 1955: Professor C. R. Sherer; at the inauguration of Harold P. Rodes as President of Bradley University on May 6, 1955: Professor Evelyn K. Kinney; at the dedication of the Burlington Nuclear Laboratories on May 23, 1955: Professor J. H. Roberts; at the inauguration of Daryl Chase as President of Utah State Agricultural College on June 3, 1955: Professor C. R. Wylie, Jr.; at the 75th anniversary of the American Society of Mechanical Engineers in Boston, June 19–24, 1955: Professor W. T. Martin; at the dedication of the United States Air Force Academy, Lowry Air Force Base, Denver, Colorado, July 11, 1955: Professor W. J. Thron.

The Secretary reported that Professor Paul Olum had accepted an invitation to deliver an hour address at the meeting to be held at the University of Maryland on October 22, 1955; that Professor P. R. Garabedian will deliver an address at the meeting to be held at the University of Southern California on November 12, 1955; that Professor E. H. Spanier will deliver an address at the meeting to be held at the University of Wisconsin, Milwaukee, on November 25–26, 1955; that Professor Everett Pitcher will address the Society at the meeting to be held at Hunter College on February 25, 1956, and that Professor Fritz John will deliver an address at the Society meeting to be held at Columbia University on April 20–21, 1956.

The Secretary reported that the Agricultural and Mechanical College of Texas had been elected to institutional membership.

The Secretary reported that Professor A. W. Tucker had been elected to complete the term of Professor J. R. Kline on the Committee on Printing and Publishing.

The Council voted to set meetings November 12, 1955, at the University of Southern California; February 25, 1956, at Hunter College, New York City; and April 20–21, 1956, at Columbia University. The Council set the dates of the Annual Meeting in 1956 to be December 27–29.

The Council voted to grant an annual subvention of $1000 a year for three years to a new *Illinois Journal of Mathematics*.

The Council voted to approve a Summer Research Institute for the summer of 1956 on the topic "Differential Geometry in the Large."

The Council voted to approve a Symposium on Applied Mathematics on the topic "Calculus of Variations and Its Applications," to be held in conjunction with the spring meeting in Chicago in 1956 with the co-sponsorship of the Office of Ordnance Research.

The Council authorized and requested the President to appoint a committee to consider desirable extensions of the Society's activities.

The Council adopted the following resolution:

"It is with a sense of irreparable loss that the Council of the American Mathematical Society records the passing on May 2, 1955, of John Robert Kline, who was Secretary of the Society for ten years and who was a member of this body for all but one of the past twenty-seven years.

"Kline's association with the Society dates from 1915 when he presented as his first paper his Doctoral Dissertation "Double Elliptic Geometry in Terms of Point and Order." Since he did not become a member until the next year, it was necessary that he be "introduced" for this presentation by his teacher, Professor R. L. Moore. Over the next thirteen years the scientific programs of the Society were enriched by some twenty-three papers of his, chiefly on Analytic Topology.

"His official service to the Society seems to have begun in 1925 with membership on a Committee on Summer Meetings. Two years later he became an Associate Editor of the *Transactions* in which capacity he served for ten years. Soon he was elected to the Council and began twelve years as Associate Editor of the *Bulletin* and nine years in the same capacity for the *American Journal of Mathematics*. Later he served for four years on the Editorial Board for the *Colloquium Publications*. He was Associate Secretary from 1933 to 1936. Among the innumerable committees of which he was a member was a Preliminary Committee on the International Congress of which he became Secretary in September, 1938. This committee began planning to hold a Congress in 1940, but its labors were not to come to fruition until 1950.

"When Dean Richardson retired at the end of 1940, Kline was a natural choice for election as Secretary. He had experience, connections national and international, a love of work which gave him an almost unbelievable capacity for accomplishment; a powerful memory for facts, dates, and people, and thorough familiarity with the status of mathematical activity and personnel in this country. For the next decade he guided the Society's affairs wisely, firmly, gently. For he had those gifts of tact and diplomacy that come only from an innate gentleness of spirit, a spirit unyielding on matters of principle but never giving unnecessary offense.

"In the war years defense needs dominated mathematical activity, while the war's end saw the Society grow with almost alarming speed in numbers, in strength and in the scope of its activities; but Kline met both the variety and the number of demands upon him without wavering. He had a large share in the formation in 1945 of the Policy Committee for Mathematics which he served as Secretary and whose work continued the national emphasis that had arisen in wartime.

"Nor was his influence on the mathematical community due solely to his official acts. Many are the young mathematicians who have profited by his kindly guidance in launching their careers; and many are the seasoned department chairmen who have sought his advice on their administrative problems.

"Probably he himself regarded his work as Secretary of the Organizing Committee for the International Congress and then of Congress itself as his most satisfying achievement in mathematical organization. In a sense his entire career had prepared him for this—from his early studies abroad through his manifold administrative experience in university and scientific organizations. Patiently he guided and coordinated the work of the various subcommittees in conquering the difficulties of passport regulations, travel arrangements, finance, international politics, and the rest. And when the Congress had closed its most successful sessions, he was rightfully entitled to the true satisfaction of a great task well done, as he wound up its affairs and retired from the Secretaryship of the Society—from the Secretaryship but not from the service. In the last five years of his life he served as Trustee, was on several important committees, was a representative on the National Research Council, and was Chairman of the Policy Committee. But his health, which had never been robust though not too many of his associates knew of his physical sufferings, grew steadily worse until his final release from pain.

"To his widow, Mrs. Eunice Story Kline, the Council extends its sympathy and its appreciation of a life of truly devoted service to mathematics and to mathematicians."

(An In Memoriam article on Professor Kline, written by Professor B. P. Gill, will appear in an early issue of the *Bulletin*.)

A business meeting was held on Wednesday morning, August 31, 1955. Preceding the meeting Professor J. W. Green reported on the Survey of Research Potential and Training in Mathematics and requested the cooperation of all mathematicians in this Survey.

A resolution of thanks to the host institution was adopted.

At the business meeting, the following amendments to the by-laws were approved: To amend Article III, Sections 2, 3, and 4 to read:

Section 2. There shall be seven Editorial Committees as follows: committees for the *Bulletin*, for the *Proceedings*, for the *Colloquium Publications*, for *Mathematical Surveys*, for *Mathematical Reviews*, a joint committee for the *Transactions* and the *Memoirs* and a committee consisting of the representatives of the Society on the Board of Editors of the *American Journal of Mathematics*.

Section 3. There shall be a Committee on Printing and Publishing.

Section 4. The size of each Publication Committee shall be determined by the Council.

To amend Article VII, Section 1 to read:

Section 1. The term of office shall be one year in the case of the President Elect and the Ex-President, five years in the case of the Trustees, two years in the case of the President, Vice Presidents, the Secretary, the Associate Secretaries and the Treasurer. The term of office in case of members of the Publications Committees shall be three years, except that when the size of a Publications Committee is changed the Council may authorize the election of a member for a term of less than three years.

To add a new Section 9 to Article VII as follows:

Section 9. The following provisions shall govern the election of Trustees in 1955 and 1956. The Trustees in office on January 1, 1955, shall serve until their respective terms expire. At the annual meeting in 1955, one Trustee shall be elected for a term of five years and one for a term of two years. At the annual meeting of 1956, one Trustee shall be elected for a term of five years, one for a term of three years, and one for a term of two years. This section shall be repealed without further action on January 1, 1957.

The abstracts of the papers follow. Those having the letter "*t*" following the abstract number were read by title. Where a paper, presented in person, has more than one author, the symbol (p) follows the name of the author who presented it. Mr. Block was introduced by Professor A. A. Albert, Mr. Bratton by Professor T. C. Koopmans, Professors Fryer and Halperin by Professor H. W. Ellis, Professor Grothendieck by Professor G. B. Price, Mr. Gutierrez-Burzaco by Professor Ky Fan, Mr. Milnes by Professor G. G. Lorentz, Dr. Pinkham by Professor D. V. Widder, and Professor Wasel by Reverend T. J. Saunders.

ALGEBRA AND THEORY OF NUMBERS

632. Maurice Auslander (p) and D. A. Buchsbaum: *Global dimension of noetherian rings*. I.

The authors prove that if a local ring A has finite global dimension, then the Krull dimension of A equals global dimension of A. They show that for Krull dimension zero and one, finite global dimension implies A is regular. This leads to the conjecture that a local ring is regular if and only if it has finite global dimension. One consequence of this conjecture is that if P is any prime ideal of the regular local ring A, then A_P is also regular. A new proof of the Cohen-Macaulay theorem is obtained without the use of structure theorems. It is also shown that if B is a commutative, noetherian ring of finite global dimension, then it is the direct sum of a finite number of integral domains. Thus, if B is also hereditary, then it is the direct sum of a finite number of Dedekind rings. The techniques of homological algebra were employed extensively, in conjunction with standard ideal theory in noetherian rings. (Received July 21, 1955.)

633*t*. E. H. Batho: *On noncommutative semi-local and local rings*.

In this paper, the basic structure of a class of rings which reduce in the commutative case to the well-known semi-local and local rings are studied. A semi-local ring R is one having the following properties: (a) R satisfies the maximum condition for left ideals; (b) R modulo its Jacobson radical J satisfies the minimum condition for left ideals; (c) $\bigcap_{i=0}^{\infty} J^i = (0)$. R becomes a topological ring with respect to the topology induced by the powers of J and the basic topological and ideal-theoretic properties of R and its completion are studied. It is shown that a complete semi-local ring (or the completion of any semi-local ring) is isomorphic to a vector space direct sum of a finite number of full matrix rings over completely primary rings plus a subgroup of J and certain refinements of this decomposition are discussed. It is shown that a complete local ring (or the completion of any local ring) is isomorphic to a full ring of matrices over a completely primary ring and this structure is shown to characterize complete local rings. Certain concordancy conditions are discussed and a modified chain condition for a certain class of ideals is introduced. Some of the above results are applied to integral extensions of commutative local rings (Curtis, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 945–955) and some examples are discussed. (Received July 18, 1955.)

634. I. A. Barnett and Smbat Abian (p): *A note on skew-symmetric elements of a ring.*

Let $(')$ denote an involutionary anti-automorphism of a ring R^*. Let $A, C \in R^*$ where C is symmetric such that $A-C$ possesses an inverse B. Then A is skew-symmetric if and only if: $2B'CB+B'+B=0$. In particular if $(')$ denotes transposition in the ring of finite real n by n matrices and if $a_{ij}+a_{ji}=0$, the preceding relation will imply $|a_{ij}-c\delta_{ij}|(A_{ij}+A_{ji})+2c\sum_1^n A_{ik}A_{jk}=0$ where c is a real number and (A_{ij}) is the adjoint matrix of the matrix $(a_{ij}-c\delta_{ij})$. Again in the algebra of the operators (a, K) over real numbers, consider $(a, K)'f(x)=af(x)+\int_0^1 k(y, x)f(y)dy$, where $k(x, y)$ is the kernel of the operator (a, K). If $b(x, y)$ is the reciprocal of $k(x, y)$, in the sense of Fredholm, then a necessary and sufficient condition that $k(x, y)$ be skew-symmetric is that: $2\int_0^1 b(s, x)b(s, y)ds+b(y, x)+b(x, y)=0$. From this follows that the reciprocal of a skew-symmetric kernel is never skew-symmetric. (Received July 18, 1955.)

635. D. W. Blackett: *The near-ring of affine transformations.*

A few familiar facts about the affine transformations of a vector space over a field are: 1. The nonsingular affine transformations form the "affine group." 2. The translations are a normal subgroup of the affine group. 3. The quotient group of the affine group modulo the translation group is isomorphic to the group of nonsingular linear transformations. Unfamiliar analogues of these facts are: I. The affine transformations (singular ones included) form a near-ring with commutative addition. II. The constant transformations form the unique maximal two-sided ideal of the near-ring of affine transformations. III. The quotient near-ring of affine transformations modulo constant transformations is isomorphic to the simple ring of linear transformations. (Received June 27, 1955.)

636. Richard Block: *Some properties of new simple Lie algebras.* Preliminary report.

Several classes of new simple Lie algebras of characteristic p are examined for the properties of being restricted and having a nondegenerate trace form. The simple algebras $S_n(n>2)$ [M. S. Frank, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954)] of all truncated derivations of divergence zero of the polynomial algebra $B_n=F[x_1, \cdots, x_n]$ $(x_i^p=0)$ are restricted and any trace form for any of them is identically zero. The simple algebras V_m $(n=2m, p>2)$ of dimension $p^{2m}-2$ consisting of all derivations $D(\phi)=(a_1, \cdots, a_m; b_1, \cdots, b_m)$ where $a_i=\partial\phi/\partial x_{i+m}$, $b_i=-\partial\phi\partial x_i$ and ϕ has no term in $(x_1, \cdots, x_{2m})^{p-1}$ are restricted and have a nondegenerate trace form defined in terms of the form for the monomials $\phi=x_1^{i_1}\cdots x_{2m}^{i_{2m}}$, $\psi=x_1^{j_1}\cdots x_{2m}^{j_{2m}}$ by making $\tau(D(\phi), D(\psi))=1$ if $i_1+j_1=\cdots=i_{2m}+j_{2m}=p-1$ and zero otherwise. For $m=1$, these are the only trace forms, up to scalar multiples. The simple algebras T_n (n, $p>2$) of all derivations with divergence equal to coordinate sum are not restricted. (Received July 18, 1955.)

637*t*. J. L. Brenner: *Bounds for determinants. Conditions that a matrix of constants be regular.*

This paper concerns a condition for the nonsingularity of a matrix $A=(a_{ij})$ of complex numbers. The condition reads as follows. Define the pth power average $L_{i,p}$ for the ith row by $L_{i,p}=[\sum' |a_{ij}|^p]^{1/p}$, $p^{-1}+q^{-1}=1$. If $S_p=\sum(1+|a_{ii}|^q L_{i,p}^{-q})^{-1}$ is less than unity $(0\cdot 0^{-q}=0)$, then det A is not 0. (The condition is due to Ostrowski, Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) vol. 10 (1951) pp. 156–168; Mathematical Reviews vol. 14, p. 125.) Two new proofs of the validity of this condition are given, and upper and lower bounds for $|\det A|$ are found. From them, a nontrivial lower bound for the absolute value of the determinant of an arbitrary matrix is obtained, which is adapted for use with automatic computers. Credit is due the Office of Ordnance Research for sponsoring this work. (Received July 8, 1955.)

638*t*. Leonard Carlitz: *A degenerate Staudt-Clausen theorem.*

Put $x\{(1+\lambda x)^{\mu}-1\}^{-1}=\sum_0^\infty \beta_m(\lambda)x^m/m!$, where $\lambda\mu+1$, so that $\beta_m(\lambda)$ is a polynomial in λ with rational coefficients. The following results are proved. 1. Let $\lambda=a/b$, $(a, b)=1$. Then $\beta_{2m}(\lambda)=A_{2m}-\sum p^{-1}$, where A_m is a rational number whose denominator contains only primes occurring in b, and the summation is over primes p such that $p-1|2m$, $p|a$. 2. Let λ be integral (mod p), $p\nmid ab$, $p^r|m$. Then $\beta_m(a\lambda/b)\equiv(a/b)^m\beta_m(\lambda) \pmod{p^r}$. 3. Let $p\geqq 3$, $p^r|m$, $p|\lambda$. Then $\beta_m(\lambda)\equiv 0 \pmod{p^r}$ if $p-1\nmid m$, while $\beta_m(\lambda)+p^{-1}-1\equiv 0 \pmod{p^r}$ if $p-1|m$. (Received June 27, 1955.)

639*t*. Leonard Carlitz: *A further note on Dedekind sums.*

Let $\overline{B}_r(x)$ denote the Bernoulli function. Put $c_{r,s}(h, k)=\sum_\mu \overline{B}_r(\mu/k)\overline{B}_s(h\mu/k)$ and $f_m(h, k; \tau)=\sum_{s=0}^{m}\binom{m}{s}(h\tau-k)^{m-s-1}c_{m-s,s}(h, k)$. A transformation formula of Apostol (Duke Math. J. vol. 17 (1950) pp. 147–157 leads to the formula $f_m(H, K; \tau)=(-1)^m\cdot(k\tau-h)^{m-2}f_m(h_1, k_1, \tau_1)+f_m(h, k, \tau)$, where $\tau_1=(k'\tau+h')(k\tau-h)$, $hh'+kk'+1=0$, $h_1h_1'+k_1k_1'+1=0$ and $\begin{pmatrix}H' & K'\\ K & -H\end{pmatrix}=\begin{pmatrix}h_1 & h_1'\\ k_1 & -k_1\end{pmatrix}\begin{pmatrix}h & h'\\ k & -k\end{pmatrix}$. An elementary proof of this formula is given. It is then shown that for $m=2$ and $(k, k_1)=1$, the formula reduces to a three-term relation for Dedekind sums recently proved by Rademacher (Duke Math. J. vol. 21 (1954) pp. 391–397). (Received June 27, 1955.)

640*t*. Leonard Carlitz: *An extension of Bauer's congruence.*

Extending the well known congruence of Bauer, the author shows that $Q(s)=\prod_{r_i=0}^{p^m-1}(s+r_1u_1+\cdots+r_ku_k)\equiv\{\Delta(u_1, \cdots, u_k, s)/s\Delta(u_1, \cdots, u_k)\}^{p^{k(m-1)}} \pmod{p^m}$, where in the product $(r_1, \cdots, r_k, p)=1$ and $\Delta(x_1, \cdots, x_k)$ is the determinant $|x_i^{p^j}|$ $(1\leqq i\leqq k,\ 0\leqq j\leqq k-1)$. If $Q(s)=s^n+C_1s^{n-1}+\cdots+C_n$, where $n=p^{km}-p^{k(m-1)}$, it follows that $C_r\equiv 0 \pmod{p^m}$ for $p-1\nmid r$; moreover $C_{2r+1}\equiv 0 \pmod{p^{2m-1}}$ for all r and $C_{2r+1}\equiv 0 \pmod{p^{2m}}$ for $p-1\nmid 2r$. For small values of r these results can usually be improved. Put $R_t=\sum_0^{p^m-1}(r_1u_1+\cdots+r_ku_k)^t$. Then $R_t\equiv 0 \pmod{p^{mk-k'}}$ where $k'=[k/(p-1)]$. In particular it follows that $C_t\equiv 0 \pmod{p^{\min(mk,mk-k+t)}}$ for $1\leqq t<p-1$. We have also $R_t\equiv p^{mk}B_t^{(k)}[u_1, \cdots, u_k] \pmod{p^{mk+m-k'}}$, where $B_t^{(k)}[u_1, \cdots, u_k]$ is a Bernoulli number of order k in the notation of Nörlund. (Received June 27, 1955.)

641*t*. Leonard Carlitz: *A sum connected with quadratic residues.*

Let $p=2k+1$ be a prime, $m\geqq 1$, a arbitrary. Put $S_m(a)=\sum_{r=0}^{m}(-1)^{m-r}$

$\cdot((r+a)/p)\binom{m}{r}$. Define k and l by means of $(k-1)(p-1)<m-(p-1)/2\leqq k(p-1)$, $m+l=k(p-1)+(p-1)/2$. Then $S_m(a)$ is divisible by p^k at least; also the residue (mod p) of $p^{-k}S_m(a)$ is obtained. In particular $p^{-k}S_m(a)\equiv-(-1)^k/k!$ (mod p) for $l=0$, $p^{-k}S_m(0)\equiv-(-1)^k m/2\cdot k!$ (mod p) for $l=1$, etc. (Received June 27, 1955.)

642*t*. Leonard Carlitz: *The coefficients of* sinh x/sin x.

Put sinh x/sin $x=\sum_0^\infty \beta_{2m}x^{2m}/(2m)!$, so that the β's are rational numbers. In seeking a theorem of the Staudt-Clausen type for β_{2m} the following result was obtained. The denominator of β_{2m} contains only primes $p\equiv3$ (mod 4). Let $p^r\mid(2m+1)$, $p^{r+1}\nmid(2m+1)$ and define u, v by means of $2m+1\equiv u+vp$ (mod p^2-1), $0\leqq u\leqq p-1$; $0\leqq v\leqq p-1$. Then $(2m+1)\beta_{2m}\equiv1-\sum(-1)^s\binom{2m+1}{s(p-1)}(p^{-1}-1)$ (mod p^r), $0<s(p-1)<2m+1$. If $u+v\geqq p$ the denominator of β_{2m} is divisible by exactly p^r; if $u+v<p$ the denominator of β_{2m} is divisible by p^r at most. Like results hold for the coefficients in the expansion of $(e^{\omega x}-1)/(e^x-1)$, where ω is a primitive nth root of unity. (Received June 27, 1955.)

643*t*. Eckford Cohen: *A class of arithmetical functions*. II. *Applications*.

In Part I of this paper (Bull. Amer. Math. Soc. Abstract 61-4-446) it was shown that a function $f(n)=f(n, r)$ of period $r\geqq1$ is *even* (mod r) if and only if it has a Fourier expansion, $f(n, r)=\sum_{d\mid r}\alpha(d)c(n, d)$, $c(n, d)=\sum_{(x,d)=1}\exp(2\pi inx/d)$, or equivalently, an arithmetical representation, $f(n, r)=\sum_{d\mid(n,r)}g(d, r/d)$. In this paper the theory of Part I is applied to several functions. (1) The number of solutions $N(n, r, t)$ of the congruence $n\equiv x_1y_1+\cdots+x_ty_t$ (mod r) is deduced, furnishing a unified treatment, in a particular case, of a congruence problem considered previously (Bull. Amer. Math. Soc. Abstracts 61-3-388, 389). (2) Carmichael's sum, $c^{(s)}(n, r)=\sum_{d\mid(n,r)}d^s\mu(r/d)$, is represented in two different forms, from which the Ramanujan and Hölder evaluations of $c(n, r)$ are shown to result. (3) *Completely even* functions (mod r), defined by the property $f(n, r)=h((n, r))$, are introduced, and the cases $h(m)=\phi_s(m)/m^s$, $h(m)=\sigma_s(m)$, s arbitrary, are studied. (4) As corollaries, a number of arithmetical identities are derived, some of which are known while others appear to be new. (Received July 15, 1955.)

644*t*. C. W. Curtis: *On commuting rings of endomorphisms*.

A simple abstract method of proof is given for the results of Weyl (Duke Math. J. vol. 3 (1937) pp. 200–212) on the invariant subspaces relative to the centralizer C of a finite group G of semi-linear transformations in a finite dimensional vector space M over a field K. The method applies to the more general situation obtained by replacing K by an arbitrary division ring Δ, and by removing the hypotheses that M is finite dimensional over Δ, and that M is completely reducible relative to G. The result states that there exists a (1-1) correspondence, preserving direct sums, intersections, and isomorphism relations, between the C-direct summands of M and those right ideal direct components of the semi-linear group ring A of G which are contained in a certain two-sided ideal A_0 in A. If M is finite dimensional over Δ, then every indecomposable C-direct summand R of M possesses a unique maximal submodule S, and two indecomposable C-direct summands R_1 and R_2 are C-isomorphic if and only if R_1/S_1 and R_2/S_2 are C-isomorphic. By means of a natural duality in the theory, it is also shown how the structure of M as a G-module determines the ideal theoretic structure of the centralizer C. (Received July 14, 1955.)

645. C. W. Curtis: *On the structure and representation theory of Lie algebras of characteristic $p>0$.*

Let L be a semi-simple Lie algebra over the field of all algebraic numbers, and let H be a Cartan subalgebra of L. Then L has a basis (X_i), called an admissible basis of L relative to H, with certain special properties. With (X_i) is associated a number field K containing the constants of structure determined by the basis (X_i). A set of prime ideals in K is defined, which contains almost all prime ideals in K. For a fixed prime ideal P in the set, let O be the valuation ring, and $\overline{K}$ the residue field. Then $\Sigma=\Sigma OX_i$ is a Lie ring, and $\overline{L}=\Sigma/P\Sigma$ is a Lie algebra over $\overline{K}$ with nondegenerate Killing form. The natural mapping $T:\Sigma\rightarrow\overline{L}$ maps H onto a Cartan subalgebra of $\overline{L}$, and sets up a (1-1) correspondence between the root diagram of L relative to H and the root diagram of $\overline{L}$ relative to HT. In a natural way, $\overline{L}$ becomes a restricted Lie algebra in the sense of Jacobson. A theory of weights, corresponding to the Cartan-Weyl theory in the classical case, is developed for restricted representations of $\overline{L}$. Every irreducible representation U of L determines a restricted representation $\overline{U}$ of $\overline{L}$, and various properties of the correspondence $U\rightarrow\overline{U}$ are established. (Received July 14, 1955.)

646. R. A. Dean: *Completely free lattices generated by partially ordered sets.*

Let P be a partially ordered set with elements p_i and order relation $(\leqq)$, and let the set, $CF(P)$, of lattice polynomials (words) be ordered by: (i) $p_i\leqq p_j$ if and only if $p_i\leqq p_j$ in P, and, (ii) recursively, $A\leqq B$ if and only if one or more of the following hold: (1) $A=A_1\cup A_2$ and A_1 and $A_2\leqq B_1$, (2) $A=A_1\cap A_2$ and A_1 or $A_2\leqq B$, (3), (4), the dual conditions. Identifying words A and B when $A\leqq B$ and $B\leqq A$, $CF(P)$ becomes the completely free lattice generated by P. The word problem is solved in $CF(P)$ and $CF(P)$ can be mapped homomorphically into any lattice in which P may be embedded. Results analogous to those obtained for the free lattice on unordered elements by Whitman (Ann. of Math. vol. 42 (1941) pp. 325–330 and vol. 43 (1942) pp. 104–115) show the existence of canonical forms and give criteria for completely free sublattices. In general $CF(P)$ preserves neither least upper nor greatest lower bounds existing in P, but in special cases $CF(P)$ is isomorphic to the free lattice, $FL(P)$, generated by P and preserving those bounds of pairs of elements. In particular it is shown that when $P=[a, b, c, d; a>b, c>d]$ and $Q=[a, b, c, d, e; a>b>c>d, e]$, $CF(P)(\cong FL(P))$ contains infinite chains and is a sublattice of $CF(Q)(\cong FL(Q))$. (Received June 23, 1955.)

647t. G. D. Findlay and Joachim Lambek: *Calculus of bimodules.*

In addition to the ordinary tensor product of two bimodules A and B it is natural to define two tensor quotients: A over B and A into B. The former is written A/B and means Hom (B, A), where B and A are regarded as right modules, turned into a bimodule in the natural way. For "associative laws" involving the three tensor operations are then valid, for example $(A/B)/C\cong A/(C\cdot B)$. Four others hold conditionally, thus $A\cdot(B/C)\cong(A\cdot B)/C$ if A or C is direct summand of a right module with finite base. This condition on A can also be expressed by saying that there exists an identity matrix over A regarded as a right module. By the matrix ring over the right module A is understood the tensor product of A with its dual, turned into a ring in the natural way. The calculus of bimodules is applied to the extension of operator rings. In fact, the remaining eight associative laws depend on a generalization of

Frobenius extensions. Modules are not assumed to be unitary. (Received June 30, 1955.)

648. Malcolm Goldman: *Subalgebras of certain Banach algebras.*

In Annals of Mathematics (2) vol. 53 (1951) pp. 235–249 Kaplansky defines the concept of an AW*-algebra and the concept of type for such algebras. In American Journal of Mathematics vol. 75 (1953) pp. 839–858, he proves that an AW*-algebra B of type I is the algebra of all continuous endomorphisms of a certain kind of module. Using this spatial theory and methods of Kaplansky and von Neumann, the author proves: (1) an isomorphism between two maximal abelian (self-adjoint) subalgebras of B which leaves the center fixed can be extended to a unitary element of B; (2) if A is a self-adjoint subalgebra of B which is a double commutant and of type J (I, II, or III), then the commutant of A is of type J. (Received July 18, 1955.)

649. Franklin Haimo: *Component preserving automorphisms of semi-direct products.* Preliminary report.

Let G be a semi-direct product of a normal subgroup H by a subgroup K. It is proved that those automorphisms of G which do not alter the components in H of each element of G form a subgroup of the group of automorphisms of G, and that this subgroup is itself a semi-direct product of an abelian group by a group of automorphisms of K. Let A be an automorphism of G which carries each element of G onto an image with the product of the components in H of element and image equal to the identity. Then A carries the centralizer of H in G onto a (direct) product HL where L is that subgroup of K such that the inner automorphisms of G generated by the elements of L are precisely those inner automorphisms of G generated by elements of K which reduce to the identity on H. Likewise A carries HL onto the centralizer of H in G. See W. H Mills, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 428–443, where, in the case of the holomorph of H, $A(H)$, the conjoint of H, is this centralizer. This work has been supported in part by the OSR, USAF. (Received July 18, 1955.)

650*t*. Bruno Harris: *Centralizers of simple Jordan algebras.* Preliminary report.

Let A be an involutorial ring $L(M, M)$ (all linear transformations on the vector space M over a division ring D having adjoints relative to a nondegenerate scalar product in M). Let J be the Jordan ring of self-adjoint elements of A, K a Jordan subring of J containing $F_1 \cdot 1$, F_1 being the self-adjoint elements of the center F of D, and $J(K)$ the linear transformations in J commuting with every linear transformation in K. Theorem: Let char. $D \neq 2$. Let K be simple finite-dimensional over $F_1 \cdot 1$, B its enveloping associative ring. Assume (a) $B \supseteq F \cdot 1$, (b) dimension of M over $D > 2 \cdot$ dimension over F of a minimal right ideal of B. Then: 1. If B is simple, $J(K)$ is the set of self-adjoint elements of an involutorial ring $L(N, N)$. 2. If B is not simple, there exist a pair of dual spaces N_1, N_2 such that $J(K)$ is the set of self-adjoint elements of $L(N_1, N_2) \oplus L(N_2, N_1)$ relative to the involution mapping elements of $L(N_i, N_j)$ on their adjoints in $L(N_j, N_i)$. Also, unless K is of type D (i.e. space of elements of degree ≤ 1 of a Clifford algebra), $J(J(K)) = K$. (Received July 18, 1955.)

651. Melvin Henriksen: *Concerning adequate rings and elementary divisor rings.* III.

A commutative ring S with unit is called an *elementary divisor* ring if, for every matrix M over S, there exist nonsingular matrices K, L such that KML is diagonal. S is *adequate* if (i) all finitely generated ideals are principal and (ii) for every $a, b \in S$, $a \neq 0$, there exist $r, s \in S$ such that $a = rs$, $(b, r) = S$, and for every non-unit divisor t of s, $(t, b) \neq S$. The following improvements of results announced in Bull. Amer. Math. Soc. Abstract 61-2-182 have been obtained: 1. If S is adequate, every proper prime ideal of S is contained in a unique maximal ideal. 2. There exist *integral domains* that are elementary divisor rings, but are not adequate. 3. If S satisfies (i) and each element of S outside its Perlis-Jacobson radical is in at most a finite number of maximal ideals, then S is an elementary divisor ring (but need not be adequate). (The author was supported by the National Science Foundation, contract no. NSF-G1129.) (Received July 18, 1955.)

652t. J. H. Hodges: *An exponential sum for symmetric matrices in a finite field*. Preliminary report.

For $\alpha \in GF(q)$, q odd, let $e(\alpha) = e^{2\pi i t(\alpha)/p}$, where $t(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{n-1}}$. If $A = (\alpha_{ij})$ is a square matrix with elements in $GF(q)$, let $\sigma(A) = \sum \alpha_{ii}$. Consider the sum $S = S(B, \Lambda, A) = \sum e\{\sigma(\Lambda' X + X' \Lambda)\}$, where A is a nonsingular symmetric matrix of order m, B is a symmetric matrix of order t, Λ is an arbitrary $m \times t$ matrix, and the sum is over all $m \times t$ matrices X satisfying the equation $X'AX = B$, all matrices having elements in $GF(q)$. If $\Lambda = 0$, S is the number of solutions X of $X'AX = B$, which is given in (L. Carlitz, *Representations by quadratic forms in a finite field*, Duke Math. J. vol. 21 (1954) pp. 123–128). In *Weighted quadratic partitions over a finite field* (Canadian J. of Mathematics vol. 5 (1953) pp. 317–323) L. Carlitz has considered S for $t = 1$. Explicit values for this S are given in all except one case and in all cases S is given in terms of certain Kloosterman sums over $GF(q)$. In the present paper it is shown that the general sum S can be expressed in terms of Kloosterman sums defined for symmetric matrices over $GF(q)$. A number of the properties of these Kloosterman sums are also given. (Received July 15, 1955.)

653. R. E. Johnson: *Structure theory of faithful rings*. II. *Restricted rings*.

A ring R is (left) faithful if $aR \neq 0$ for every nonzero $a \in R$. Designate by $\mathcal{L}(R)$ the lattice of right ideals of R and by $\mathcal{J}(R)$ the set of all ideals A of R for which $A \cap A^l = 0$ and $A = A^{ll}$, where A^l is the left annihilator of A. The set $\mathcal{J}(R)$ is shown to be a Boolean algebra if the union operation $\cup$ is defined by: $A \cup B = (A + B)^{ll}$. If $\mathcal{J}(R) = \{0, R\}$, R is called irreducible. Associated with R is a unique maximal extension ring $\overline{R}$ that contains R as an ideal and is such that $\overline{R}^l = 0$ in $\overline{R}$. The Boolean algebras $\mathcal{J}(R)$ and $\mathcal{J}(\overline{R})$ are shown to be isomorphic. The ring R is called restricted if $\mathcal{J}(R)$ is a complete Boolean algebra. A closure operation f is induced on $\mathcal{L}(R)$ by $\mathcal{J}(R)$ if R is restricted. If f is atomic and if the union B of the atoms of $\mathcal{J}(R)$ is such that $B^l = 0$, then there exists a set $\{A_i\}$ of irreducible rings such that $\sum_i A_i \subseteq R \subseteq \sum_i^* \overline{A}_i$, where $\sum (\sum^*)$ designates the discrete (full) direct sum. An analysis is made of the structure of a right R-module M induced by an atomic closure operation on $\mathcal{L}(R)$. The extended centralizer of R over M is a full direct sum of primitive rings. (Received June 8, 1955.)

654. Irving Kaplansky: *Rings resembling operator algebras*.

An AW^*-algebra can be defined as a C^*-algebra A in which every annihilator is of the form eA with e an idempotent. The latter condition is meaningful in any ring, and in this paper its consequences are studied. In due course it becomes advisable to postulate an involution and add two axioms concerning projections and square roots. The theory can be pushed to the point just before one introduces a trace. Some of the proofs are shorter than the old ones. (Received July 29, 1955.)

655. J. H. B. Kemperman: *Complexes in arbitrary groups.*

Consider two finite nonempty subsets A and B of a multiplicative group G and let $C=AB$ be the set of all the elements $c=ab$ with $a\subset A$, $b\subset B$. Let $k=[A]+[B]-[AB]$, where $[D]$ denotes the number of elements in D. Let $c_0=a_0b_0$ ($a_0\subset A$, $b_0\subset B$) be a fixed element of C. Conjecture: (*) c_0 admits at least k different representations of the form $c_0=ab$ with $a\subset A$, $b\subset B$. This is trivially true when $k\leqq 1$. It is shown that: (1) Assertion (*) holds when $[B]\leqq k+2$, hence, when $[B]\leqq 4$. (2) Let $b_0, b_1, \cdots, b_p$ ($p\geqq k-1$) be the set of all those elements b in B for which $a_0bb_0^{-1}\subset A$. Then (*) holds when $b_ib_0^{-1}b_j=b_jb_0^{-1}b_i$ ($i, j=1, \cdots, p$). Thus $p=1$ implies (*) (also $p=2$ and $[B]\leqq k+3\leqq 6$). (3) From (2), (*) holds for each element c_0 in C when the elements of B commute with each other, hence, when G is abelian. An equivalent form of conjecture (*) is: (**) When some element c_0 of AB admits at most k_0 representations $c_0=ab$, then $[AB]\geqq[A]+[B]-k_0$. From (3), (**) holds when G is abelian. For $k_0=1$, this was shown by L. Moser and P. Scherk, Amer. Math. Monthly vol. 62 (1955) p. 46. (Received July 18, 1955.)

656. L. A. Kokoris: *On a class of almost alternative algebras.*

A finite dimensional algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ is called an algebra of (γ,δ)-type if it satisfies the identities $z(xy)=(zx)y+\gamma(xz)y-\gamma x(zy)+\delta(yz)x-\delta y(zx)$ and $(xy)z=x(yz)+\gamma(xz)y-\gamma x(zy)+(\delta-1)(yz)x-(\delta-1)y(zx)$ where γ and δ are elements of $\mathfrak{F}$ satisfying $\gamma^2-\delta^2+\delta=1$. These algebras are a type of almost alternative algebras which arise in the study of almost alternative algebras relative to quasi-equivalence. When $\mathfrak{F}$ has characteristic $\neq 2, 3, 5$, $\mathfrak{A}$ is power-associative. If also $\delta\neq 0, 1$ and $\mathfrak{A}$ has an idempotent e, $\mathfrak{A}$ may be decomposed relative to e in the same way as an associative algebra. The radical of $\mathfrak{A}$ is defined to be the maximal nilideal of $\mathfrak{A}$ and it is then proved that a simple algebra is either associative or has a unity quantity which is absolutely primitive. A semisimple algebra is shown to be a direct sum of simple algebras. (Received July 13, 1955.)

657t. Walter Ledermann and B. H. Neumann: *On the order of the automorphism group of a finite group.* I.

A function $f(n)$ is found with the property that every finite group whose order is greater than or equal to $f(n)$ has at least n automorphisms. The least such bound satisfies $f(1)=1$, $f(2)=3$, $f(3)=f(4)=7$, and $f(n)\leqq(n-1)^{n+(n-2)[\log_2(n-1)]}$ when $n>4$. (Received July 7, 1955.)

658t. C. S. Lin and H. B. Mann: *A generalisation of a theorem of Cauchy.* Preliminary report.

Let G be a cyclic group of prime power order, A, B sets of elements in G. Put $C=A\oplus B=\{a\oplus b;\ a\in A,\ b\in B\}$, where $\oplus$ denotes the addition in G. Let H be the largest subgroup of G for which $A\oplus B\oplus H=A\oplus B$ and let (A) denote the number of elements in A. Then $(C)+(H)\geqq(A)+(B)$. The corresponding theorem for sets of nonnegative integers is not true. The methods are those introduced by H. B. Mann in several previous papers. (Received May 27, 1955.)

659*t*. R. C. Lyndon: *A nonspecial operator Lie ring.*

A Lie ring L, over a commutative ring R of operators, is constructed, which is not R-isomorphic to a Lie subring of any associative R-ring A. (The present example seems to be of a quite different nature from that given by Shchirshchov, Usp. Mat. Nauk vol. 8 (1953) pp. 173–175.) R is the ring of polynomials over the integers in indeterminates ξ_1, ξ_2, ξ_3. L has generators x_1, x_2, x_3, and defining relations $\xi_i x_j = \xi_j x_i$. Let $u = \sum \xi_i [x_j, x_k]$, summed cyclically. Then $u \neq 0$ in L (although $2u = 0$), while u vanishes under any R-homomorphism of L into associative A. (Received July 19, 1955.)

660*t*. P. J. McCarthy: *Sufficient conditions for a genus of indefinite ternary quadratic forms to contain only one class.* II.

Let f be a ternary quadratic form with integral matrix A. Let Ω be the g.c.d. of the two-rowed minor determinants of A, and let Δ be the integer determined by $|A| = \Omega^2\Delta$. Let F be the reciprocal form of f, i.e., the form with matrix $\Omega^{-1}|A|A^{I}$. The genus of f contains only one class if the following conditions are satisfied: $\Omega \equiv 4 \pmod 8$ and Δ is odd, $(\Omega, \Delta) = 1$, $(-1)^{(f-1)/2} = (-1)^{(\Delta-1)/2}$, and the congruence $F \equiv -\Omega/4 \pmod 4$ is solvable. This extends a theorem of A. Meyer (see L. E. Dickson, *Studies in the theory of numbers*, Chapter 4). (Received July 13, 1955.)

661. P. J. McCarthy: *Witt's cancellation theorem in valuation rings.*

Let R be a valuation ring in which 2 is a unit. It is not assumed that R is complete. Two n-ary quadratic forms over R are said to be congruent in R if there is a unimodular transformation (i.e., a transformation whose determinant is a unit in R) taking one of the forms into the other. A short and quite simple proof is given of the following theorem: if $f(x_1, \cdots, x_r)+g(x_{r+1}, \cdots, x_n)$ and $f(x_1, \cdots, x_r)+h(x_{r+1}, \cdots, x_n)$ are congruent in R, then g and h are congruent in R. This generalizes the cancellation theorem of Witt (J. für Math. vol. 176). It has been proved by Durfee with the added assumption that R is complete (Duke Math. J. vol. 11). (Received July 13, 1955.)

662. T. S. Motzkin and E. G. Straus (p): *Some combinatorial extremum problems.*

For a class of functions $F(x_1, \cdots, x_n)$ including Gauss brackets (denominators of continued fractions) of positive integers and lengths of routes through certain point sets in metric spaces the arrangements of the given n numbers which maximize and minimize F are $\cdots a_5 a_3 a_1 a_2 a_4 \cdots$ and $a_1 a_n a_3 a_{n-2} \cdots a_4 a_{n-1} a_2$ where $a_1 \geqq a_2 \geqq \cdots \geqq a_n$. Related results are obtained for cyclic and infinite problems. (Received June 10, 1955.)

663. L. J. Paige: *A class of simple Moufang loops.*

Let R be a simple alternative, not-associative ring possessing an idempotent not its unit element. Let L be the loop of all regular elements of R and let Z be the center of L. Then either L/Z is a simple, not-associative Moufang loop or L/Z contains a simple, not-associative Moufang subloop. This result yields finite simple Moufang loops whose possible orders are $(2^{7n}-2^{3n})$ or $2^{-1}(p^{7n}-p^{3n})$ if p is an odd prime. The corresponding result for the Cayley-Dickson division algebra over the real field R^* is valid but is not true for all fields. (Received July 5, 1955.)

664. Irving Reiner: *Integral representations of cyclic groups of prime order.*

Let G be a cyclic group of prime order p. It is shown here that G has exactly $1+2h_p$ inequivalent indecomposable rational integral representations, where h_p is the class number of the cyclotomic field of pth roots of unity over the rationals. In matrix terminology, this shows that every $n\times n$ integral matrix solution of $X^p=I$ is integrally similar to a direct sum of matrices chosen from a certain set of $1+2h_p$ such solutions. This generalizes results of Latimer and MacDuffee (Ann. of Math. vol. 34 (1933) pp. 313–316) for the case $n=p-1$, and of Hua and Reiner (Trans. Amer. Math. Soc. vol. 71 (1951) pp. 331–348) for the case $p=2$. The proof depends upon (i) the work of Latimer and MacDuffee (op. cit.) on classes of nonderogatory matrices with given characteristic polynomial, (ii) Zassenhaus' results (Abh. Math. Sem. Hansischen Univ. vol. 12 (1938) pp. 276–288) connecting rational and integral equivalence of representations, and (iii) a generalization of Maschke's theorem due to Gaschütz (Math. Zeit. vol. 56 (1952) pp. 376–387). (Received April 18, 1955.)

665t. Irving Reiner and J. D. Swift: *Congruence subgroups of matrix groups.*

Let $\mathcal{D}$ be the ring of integers in an algebraic number field and G the group of 2×2 matrices over $\mathcal{D}$ with determinant $+1$. Let German letters denote ideals in $\mathcal{D}$ and define $G(\mathfrak{N})$ to be the subgroup of all matrices $(a_{ij})\in G$ for which $a_{21}\in\mathfrak{N}$. It is shown that if H is a group satisfying $G(\mathfrak{M}\mathfrak{N})\subset H\subset G(\mathfrak{N})$ where $(\mathfrak{M}, (6))=(1)$, then $H=G(\mathfrak{D}\mathfrak{N})$ for some $\mathfrak{D}\supset\mathfrak{M}$. This generalizes a theorem of M. Newman (Duke Math. J. vol. 22 (1955) pp. 25–32) for the case where $\mathcal{D}$ is the ring of rational integers. Examples are given to show that the restriction $(\mathfrak{M}, (6))=(1)$ is not superfluous. A second generalization of Newman's theorem is obtained by considering matrices of higher order. Let M_r^+ denote the modular group consisting of all integral $r\times r$ matrices with determinant $+1$. Define groups $C_m=\{(a_{ij})\in M_r^+ ; m\,|\,a_{i1} \text{ for } i>1\}$, and $R_n=\{(a_{ij})\in M_r^+ ; n\,|\,a_{rj} \text{ for } j<r\}$. It is proved that if H is a group satisfying $(C_{mn}\cap R_{mn})\subset H\subset (C_m\cap R_n)$, where $(am, bn)=1$, then $H=C_{\alpha m}\cap R_{\beta n}$ with $\alpha\,|\,a$, $\beta\,|\,b$. (Received May 25, 1955.)

666. D. W. Robinson: *Continuity of a matric function.*

Let T be an element of the space of finite square matrices over the field of complex numbers. Let f be a complex valued function of a complex variable, defined in a neighborhood of each of the eigenvalues of T and analytic at the eigenvalues of index greater than one. Then using the classical definition of a matrix function and the ordinary notion of a limit of a matrix, the following theorem is proved: The matric function f is continuous at T if and only if f is continuous at the eigenvalues of T and analytic at each repeated eigenvalue of T. (Received July 15, 1955.)

667t. Alex Rosenberg and Daniel Zelinsky: *Cohomology into the ring of all linear transformations.* Preliminary report.

Let C be the algebra of all linear transformations on a vector space V over a division algebra K. Then (*) $H^n(C, C)=H^n(K, K)$. When $n=1$, this is a theorem on derivations which will appear in a forthcoming book by N. Jacobson. In the course of proving (*) we also show that if B is any subalgebra of C then (**) $H^n(B, C)$

$=\mathrm{Ext}^n_{B\otimes K}(V, V)$. In case $n=1$, (**) yields two more theorems in Jacobson's book. First, if the base field is the center of K and B is simple of finite order, then $H^1(B, C)=0$. Secondly, if B is any primitive ring with nonzero socle S and V is a left B-module with $SV=V$, then every derivation of B into the ring of endomorphisms of the group V is inner [(**) is used after extending B to be an algebra over a prime field K]. Both (*) and (**) are direct consequences of identities to be found in *Homological algebra* by H. Cartan and S. Eilenberg. (Received July 18, 1955.)

668*t*. Alex Rosenberg and Daniel Zelinsky: *Cohomology of infinite algebras*. II.

If A is an algebra over a field F, F-dim A is the supremum of the integers n for which there is a two-sided A-module N with $H^n(A, N)\neq 0$. If the field K is a subalgebra of the center of A, then $F\text{-dim } K \leqq F\text{-dim } A \leqq F\text{-dim } K + K\text{-dim } A$. If A is a field of transcendence degree t over F and K is generated by a transcendence basis of A, the inequality yields $t \leqq F\text{-dim } A$. When $t<\infty$, $t=F\text{-dim } A$ if and only if A is finitely, separably generated over F. The case $t=1$ was already reported in (Bull. Amer. Math. Soc. Abstract 61-2-207). If A has no separable generation, then $F\text{-dim } A=\infty$. In proving these results, it is also shown that if A is a union of a countable tower of simple algebras of finite orders, then $H^1(A, A\otimes A)\neq 0$. (Received July 18, 1955.)

669. Alex Rosenberg (p) and Daniel Zelinsky: *Dimension of matrix rings*.

H. Cartan and S. Eilenberg have defined the left global dimension of a ring R as the supremum of the projective dimensions of left R-modules. We show that if M is the ring of n by n matrices over R and T is the ring of n by n triangular matrices over R, then l. gl. dim. $M=$ l. gl. dim. R and l. gl. dim. $T=1+$l. gl. dim. R. Furthermore, if R is an algebra over a commutative ring F, then $F\text{-dim } M=F\text{-dim } R$ and $F\text{-dim } T=1+F\text{-dim } R$. When l. gl. dim. $R=0$ or $F\text{-dim } R=0$ the results for T have already been proved by Eilenberg, Nagao, and Nakayama. (Received July 18, 1955.)

670*t*. M. F. Ruchte: *Rings of quotients*.

A ring S is said to be a (right) quotient extension of a ring R if $S\supseteq R$ and if, for nonzero s in S, $sR\cap R\neq 0$. R. E. Johnson [Bull. Amer. Math. Soc. vol. 60 (1954) p. 96] conjectures that the maximal elements of the set of all quotient extensions of R, ordered by inclusion, are isomorphic and that quotient extensions of S, where S is a quotient extension of R, are quotient extensions of R. A proof is given for both of these statements based upon Eckmann and Schopf [Arch. Math. vol. 4 (1953) pp. 75–78]. (Received July 18, 1955.)

671*t*. H. J. Ryser: *Maximal determinants in combinatorial investigations*.

Let Q be a 0, 1 matrix of order v, containing exactly t 1's. Let k denote a positive real, and set $\lambda=k(k-1)/(v-1)$. If $t\leqq kv$ and $0<\lambda\leqq k-\lambda$ or if $t\geqq kv$ and $0<k-\lambda\leqq\lambda$, then $|\det Q|\leqq k(k-\lambda)^{(v-1)/2}$. In the preceding theorem, if $|\det Q|=k(k-\lambda)^{(v-1)/2}$, then Q is the incidence matrix of a v, k, λ configuration (see S. Chowla and H. J. Ryser, *Combinatorial problems*, Canadian Journal of Mathematics vol. 2 (1950) pp. 93–99). Thus if Q is a 0, 1 matrix of order v containing exactly t 1's, and if $k=t/v$, with $\lambda=k(k-1)/(v-1)$, $0<\lambda<k<v$, then $|\det Q|\leqq k(k-\lambda)^{(v-1)/2}$, equality holding if and

only if Q is the incidence matrix of a v, k, λ configuration. (Received May 31, 1955.)

672. R. L. San Soucie: *Flexible rings.*

A. A. Albert (Trans. Amer. Math. Soc. vol. 64 (1948) pp. 552–593) has called a (not necessarily associative) ring *flexible* in case $(x, y, x) = 0$, all $x, y \in R$. For a flexible ring R, define $S = [s \in R \mid (s, x, y) = (y, x, s) = 0]$, and $N = [n \in R \mid (n, x) = (x, n) = 0]$. The flexible identity is studied, using the sets S and N. The theorems proved include: (1) A flexible division ring has a unit element if and only if $S \neq 0$. (2) If R is not associative and has no proper divisors of zero, then R is commutative if and only if $(R, R) \subset S$. (3) If $x^3 - x \in N$, all x in R, then R is commutative. Call a flexible ring R *strongly flexible* if and only if $xy \cdot zx = x(yz \cdot x)$ and observe that all alternative rings are strongly flexible (R. H. Bruck and Erwin Kleinfeld, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 878–890). Representative theorems include: (1) Let p be zero or any prime greater than one. There exists a finite-dimensional strongly flexible algebra of characteristic p which is not alternative. (2) Every strongly flexible division ring is alternative, and hence has a unit element. (3) Let R be an arbitrary ring having a unit element and characteristic not two. Then R is alternative if and only if R is strongly flexible. (4) Let R be a strongly flexible ring of characteristic zero and embed R in the ring S with unit element in the usual way. Then S is strongly flexible if and only if R is alternative. (5) Let R be a ring of characteristic not two and all nilpotent elements in N. Suppose further that R has a left unit e. Then R is alternative if and only if R is strongly flexible. (Received July 18, 1955.)

673*t*. E. V. Schenkman: *A direct proof of a theorem of Gruenberg.*

If G is an arbitrary group, then a sequence of commutators and an ordering in the sequence is defined so that every element of G is an ordered sum of terms of the sequence. This leads to a direct proof of the fact (cf. Gruenberg, Proc. Cambridge Philos. Soc. vol. 49 (1953) pp. 377–380) that a finitely generated solvable Engel group is nilpotent and enables one also to give a bound for the class of nilpotency in terms of the number of generators, the length of the derived series, and the order of the Engel condition. (Received July 5, 1955.)

674*t*. E. V. Schenkman: *The principal ideal theorem for some infinitely generated groups.*

G will stand for a group, G' for its commutator subgroup, and G^* for the intersection of the members G^m of the descending central series of G. We then prove the following three facts. LEMMA 1. Let G be a p-group with no elements of infinite height; let G/G^* be finite and let G^* be Abelian. Then G^* consists only of the identity and G is finite. LEMMA 2. Let G be a group with no elements of infinite order; let G/G^* be finite and let G^* be Abelian. If the elements of infinite height of G have order prime to that of G/G^*, then there is a finite nilpotent subgroup H of G such that H intersects G^* only in the identity and such that H and G^* generate G. THEOREM. If G/G' is finite then the transfer of G into G' is contained in the second derived group G'' of G provided that G has no elements of infinite order and that the elements of infinite height have orders prime to the order of G/G'. (Received June 2, 1955.)

675. D. S. Scott: *A new characterization of α-representable Boolean algebras.*

C.-C. Chang characterized those α-complete Boolean algebras (α-BA's) that are

α-representable (α-RBA's), i.e., α-homomorphic images of α-complete fields of sets (Bull. Amer. Math. Soc. Abstract 61-4-579). The author has proved: (*) *an α-BA is an α-RBA iff every proper principal ideal is included in a prime ideal preserving any α-system of α-termed sums given in advance.* This follows directly from four lemmas: (I_α) α-RBA's are characterizable by algebraic equations, which may involve α-termed sums. (II_α) Every two-element BA is an α-RBA. (III_α) Every algebraic equation holding in two-element BA's holds in all α-RBA's. (IV_α) In an α-field of sets every proper α-complete ideal is included in a prime ideal preserving any α-system of α-termed sums given in advance. (I_α) is proved either by Birkhoff's theorem or by a simple transformation of Chang's condition to an explicit equational characterization. Contrast (I_α)–(III_α) with (I_σ)–(III_σ) of A. Tarski (Bull. Amer. Math. Soc. Abstract 51-6-677) noticing, however, that not every α-BA is an α-RBA for $\alpha \geqq 2^{\aleph_0}$. Compare (*) with the known fact that an α-BA is isomorphic to an α-field of sets iff every proper principal ideal is included in a prime ideal preserving *all* α-termed sums. (Received July 11, 1955.)

676. Ernst Snapper: *Higher-dimensional field theory.* II. *Linear systems.*

Let E/F be a finitely generated field extension, i.e., E and F are commutative fields and $E=F(a_1, \cdots, a_n)$, where $a_1, \cdots, a_n$ belong to E. The present paper is the second of a series of three articles which have as purpose to develop an intrinsic theory for E/F, by giving a purely field-theoretic treatment of the theory of linear systems of algebraic varieties. Let L denote a finitely generated module of E/F. See the author's paper *Higher-dimensional field theory*, I, Bull. Amer. Math. Soc. Abstract 61-4-458, for the notion of the integral closure $|L|_i$ of L and its properties. In the present paper, the author introduces the notion of a *projective class of modules of E/F*, which is the field-theoretic substitute for a projective model of E/F. Precisely, when $L\neq 0$, its class C consists of all the modules $(1/a)L$, where $a\in L$ and $a\neq 0$. In terms of C, he defines the notions of divisors of the first kind, zeros and poles of elements of E, linear equivalence, linear systems, complete linear systems, etc. The theory of linear systems which arises in this way leans heavily on the properties of $|L|_i$. (Received July 18, 1955.)

677. Alfred Tarski: *Metamathematical proofs of some representation theorems for Boolean algebras.*

Consider the following weaker form of Stone's representation theorem for Boolean algebras (BA's): *every BA is a homomorphic image of a field of sets.* This statement is directly derivable from three elementary lemmas: (I) The class K of homomorphic images of fields of sets is closed under the formation of homomorphic images, subalgebras, and direct products (and hence, by Birkhoff's theorem, is characterizable by algebraic equations). (II) K contains all two-element BA's. (III) Every algebraic equation holding identically in two-element BA's holds in all BA's. Consider now Loomis' theorem: *every σ-complete BA is a σ-homomorphic image of a σ-field of sets.* This follows from lemmas (I_σ)–(III_σ) exactly analogous to (I)–(III); *σ-complete* BA's are regarded here as BA's with an additional infinite operation, σ-addition, hence homomorphisms and subalgebras in (I) are replaced by σ-homomorphisms and σ-subalgebras, and equations in (I_σ) and (III_σ) may involve σ-sums. The proofs of (I_σ) and (II_σ) are elementary. (III_σ) was obtained by D. Scott as a direct consequence of Rasiowa-Sikorski's lemma by which every proper principal ideal is included in a prime ideal preserving any denumerable system of infinite sums given in advance.

As opposed to (III) the proof of (III_c) involves the axiom of choice. (Received July 11, 1955.)

678t. Bernard Vinograde: *A decomposition of the product of inseparable fields.*

From the uniqueness of the composite LM of pure inseparable algebraic extensions L, M of finite degree over K (see, for instance, G. Pickert, *Inseparable Korpererweiterungen*, Math. Zeit. vol. 52 (1949) p. 96) it follows that L and M possess a unique maximal pair of isomorphic subfields. Denote each by D. Then $L\times_K M \cong L\times_D M + N_0$, where N_0 is an ideal in the radical. In terms of the invariants of the composite LM (Pickert, loc. cit. p. 97) the summand $L\times_D M$ is a field if and only if $m' = m$, while $K = D$ if and only if $q_1 = \cdots = q_{m'} = 1$. (Received July 21, 1955.)

679. E. A. Walker: *The solution of Kaplansky's test problem* III. Preliminary report.

In Kaplansky's book *Infinite Abelian groups*, p. 13, the following question for Abelian groups, called test problem III, is asked: if F is finitely generated and $F\oplus G$ is isomorphic to $F\oplus H$, are G and H isomorphic? This paper answers this question in the affirmative and in fact shows that if F is torsion, commutativity of G and H is not needed. It suffices to answer the question for the cases F infinite cyclic and F cyclic of prime power order. These cases are treated separately and only elementary group theory methods are used. If F, G, and H are modules over a principal ideal ring the answer is still in the affirmative and the proof is almost identical to the one for groups. (Received June 24, 1955.)

680t. A. L. Whiteman: *On the cyclotomic numbers of order sixteen.*

Let g be a primitive root of a prime p of the form $16f+1$. Let (h, k) denote the number of values of y, $1\leqq y\leqq p-2$, for which $y\equiv g^{16s+h}$, $1+y\equiv g^{16t+k}$ (mod p), where the values of s and t are each selected from the integers $0, 1, \cdots, f-1$. Emma Lehmer [Canadian Journal of Mathematics vol. 6 (1954) pp. 449–454] has raised the question whether or not $256\ (h, k)$ is expressible (at least for some h, k) as a linear combination with integer coefficients of p, x, y, a and b, where $p=x^2+4y^2=a^2+2b^2$, and $a\equiv x\equiv 1$ (mod 4). In this note the following four formulas are obtained. If the integer m is defined by means of the congruence $g^m\equiv 2$ (mod p), then (1) $256\ (00) = p-47-18x$ (f even, $m\equiv 4$ (mod 8)), (2) $256\ (80) = p-15-18x-32a$ (f even, $m\equiv 4$ (mod 8)), (3) $256\ (00) = p-31-18x-16a$ (f odd, $m\equiv 0$ (mod 8)), (4) $256\ (40) = p-15-2x+16a$ (f odd, $m\equiv 4$ (mod 8)). The following theorem is also proved. If 2 is an octic residue of p, then the set of 16th power residues modulo p cannot form a difference set. (Received July 12, 1955.)

681. L. R. Wilcox: *Modular extensions of semi-modular lattices.*

Let $(M, \cup, \cap)$ be a complemented modular lattice. A subset L of M is a *join-regular subsystem* of M if (1) $0, 1\in L$; (2) if $a, b\in L$, then $a\cup b\in L$; (3) every $a, b\in L$ possess a g.l.b. with respect to elements of L; (4) if $a, b\in M$, $c\in L$ and $a<b<c$, there exists $x\in L$ such that $x\cup b=c$, $x\cap b=a$. Necessary and sufficient conditions on a lattice L of length $\geqq 6$ that it be isomorphic to a join-regular subsystem of some M are found. These include conditions of semi-modularity, complementation, *fullness* (i.e., if $b<a\cup b$ and b is not covered by $a\cup b$, then x exists with $b<x<a\cup b$ such that (a, x) is a modular pair), and certain conditions governing the behavior of non-

modular pairs of hyperplanes if such are present. When M exists, it is unique up to isomorphisms. M is constructed by means of *quasi-dual-ideals* (subsets S of L satisfying (a) $x \in S$, $y \geqq x$ implies $y \in S$ and (b) if $z = x \cap y$ with $x, y \in S$, (x, y) a modular pair, then $z \in S$) which are generated in a certain sense by one or two elements. The present theory generalizes the well known addition of ideal elements in geometry. (Received July 15, 1955.)

682. Y. K. Wong: *On the solution of linear inequalities.*

A simple proof of Minkowski's theorem on the finiteness of a base for all solutions of a finite system of linear inequalities is given. The method is based on the concept of E. H. Moore's general reciprocals of rectangular matrices (which is related to von Neumann's theory of regular rings). A fundamental property of homogeneous linear inequalities is stated as follows: Let A be a matrix of m rows and n columns with elements in an ordered field, and R its general reciprocal. If the columns of A are linearly independent, then $Ax = b$ has a nonnegative solution if and only if $Rb \geqq 0$, and $x = Rb$ is the unique nonnegative solution. Theorems on the extreme supports of a finite system of points in terms of Moore's general reciprocals are obtained. The results are then applied to the theory of games. (Received July 18, 1955.)

683. F. B. Wright: *Structure of topological abelian groups.*

Let G be a topological abelian group, written additively, M an open semigroup in G not containing 0 and maximal with respect to these properties. Let $b(M)$ denote the complement of the union of M and $-M$. (1) Then $b(M)$ is a closed subgroup of G. Define T to be the intersection of all such subgroups $b(M)$ (with $T = G$ if there are none). The closed subgroup T is called the torsion subgroup of G. (2) If G is discrete, T consists precisely of the elements of finite order. (3) If G is compact, then $G = T$. (4) If G is connected, there is a continuous homomorphism of G into a real linear topological space having T as kernel. (5) For the additive group of a real linear topological space, the torsion subgroup consists of precisely those elements annihilated by every continuous linear functional. (6) Elementary topological and algebraic proofs are given for most of the structure theory of locally compact abelian groups. (Received July 19, 1955.)

ANALYSIS

684. M. I. Aissen: *Summable sets and L^*-spaces.*

The Weierstrass development of the real number system is imitated to develop the idea of absolute convergence of series, independently of the concept of sequential limit. Then the concepts of limit of a sequence and convergence of a series are introduced, using the notion of absolute convergence. (Received July 18, 1955.)

685. E. J. Akutowicz: *A qualitative characterization of Blaschke products in a half-plane.*

Any holomorphic function $f(w)$ in $|w| < 1$ satisfying the conditions (i) $|f(w)| < 1$, (ii) $\lim_{r \to 1-} (1/2\pi)\int_0^{2\pi} \ln |f(re^{i\theta})| d\theta = 0$ is necessarily of the form $f(w) = e^{ic}b(w)$, c a real number and $b(w)$ the Blaschke product in the unit circle formed with the zeros of $f(w)$. Consider the upper half-plane $\mathrm{Im}\, z = \mathrm{Im}\,(x+iy) = y > 0$ and prove: Theorem I. *Let $B(z)$ be a Blaschke product in the upper half-plane. Then $\int_{-\infty}^{\infty} \ln |B(z)|/(1+x^2)dx \to 0$ as $y \to 0+$.* Theorem II. *Let $F(z)$ be holomorphic and bounded for $y > 0$, and such that (1) $\int_{-\infty}^{\infty} \ln |F(z)|/(1+x^2)dx \to 0$ as $y \to 0+$. Then $F(z) = M \cdot (\exp (ikz + ic))B(z)$, where*

c and k are real, $k \geq 0$, and $B(z)$ is the Blaschke product with the zeros of $F(z)$ as its set of zeros. This uniqueness result is less sharp than the one stated above for the unit circle in that the factor $\exp(ikz)$ can occur. This difference originates from the fact that (1) does not correspond to (ii). (Received April 21, 1955.)

686. R. A. Askey (p) and I. I. Hirschman, Jr.: *Weighted quadratic norms and ultraspherical polynomials.*

Let $p_n^\lambda(x)$ $(0<\lambda<\infty)$ be the normalized ultraspherical polynomials; let $N_{\alpha,\beta}[f]$ be defined as $[\int_{-1}^{1}|f(x)|^2(1-x^2)^{\lambda-1/2}(1-x)^\alpha(1+x)^\beta dx]^{1/2}$; and let $a_n=\int_{-1}^{1}f(x)(1-x^2)^{\lambda-1/2}$ $\cdot p_n^\lambda(x)dx$ so that $f(x)\sim\sum_0^\infty a_np_n^\lambda(x)$. Consider the multiplier transformation $Tf(x)$ $\sim\sum_0^\infty t_na_np_n^\lambda(x)$. It is shown that if $|t_n|\leq C$ $(n=0, 1, \cdots)$ and $\sum_{2^n}^{2^{n+1}}|t_{n+1}-t_n|\leq C$ $(n=0, 1, \cdots)$, then $N_{\alpha,\beta}[Tf]\leq A(\alpha, \beta)CN_{\alpha,\beta}[f](-1/2<\alpha, \beta<1/2)$. (Received June 24, 1955.)

687. Frederick Bagemihl and Leonard Gillman (p): *Some cofinality theorems on ordered sets.*

Theorem 1. Let $\phi>0$, $\rho<\omega_\alpha$, $\beta=\omega_\alpha\phi+\rho$; then there exists $M\subset W(\beta)$, with $|M|$ $=\aleph_\alpha$ but $|M\cap W(\gamma)|<\aleph_\alpha$ for all $\gamma<\beta$, *iff* $\rho=0$ and either ϕ is isolated or $cf(\phi)$ $=cf(\alpha)$. Let $T(\alpha, \beta)$ denote the lexicographically ordered set of all ω_β-sequences $t=(\tau_\xi)$, where each $\tau_\xi<\omega_\alpha$, all but finitely many are 0, but not all are 0; denote by $R(t)$ the set of all elements $>t$. *Theorem* 2. If $\beta\leq cf(\alpha)$, then $\beta=cf(\alpha)$ *iff* there exists $M\subset T(\alpha, \beta)$, with $|M|=\aleph_\alpha$, but $|M\cap R(t)|<\aleph_\alpha$ for all $t\in T(\alpha, \beta)$. *Theorem* 3. If $\beta<cf(\alpha)$ and $M\subset T(\alpha, \beta)$, with $|M|=\aleph_\alpha$, then M contains ω_α, and $T(\alpha, \beta)$ has $\exp\aleph_\alpha$ subsets similar to M. Call M *$\aleph_\alpha$-homogeneous* if every interval has power $\aleph_\alpha$. *Theorem* 4. For $cf(\alpha)<\alpha$, if M is $\aleph_\alpha$-homogeneous and every interval contains both $\omega_{cf(\alpha)}$ and $\omega_{cf(\alpha)}^*$, then there exists $E\subset M$, with $|E|=\aleph_\alpha$, that contains neither ω_α nor ω_α^*. *Corollary.* If $\alpha>0$, then $cf(\alpha)>0$ *iff* there exists $\aleph_\alpha$-homogeneous M such that every $E\subset M$, with $|E|=\aleph_\alpha$, contains ω_α or ω_α^*. (Received August 29, 1955.)

688. R. B. Barrar: *Some estimates for solutions of linear parabolic equations.*

This paper treats the Dirichlet type boundary value problem for linear parabolic equations in n dimensions. The method used is patterned after that developed for the study of elliptic equations by J. Schauder [*Über lineare elliptische Differentialgleichungen zweiter Ordnung*, Math. Zeit. vol. 38 (1934) pp. 257–282]. Estimates for solutions and extensions of present existence theorems are given. (Received July 7, 1955.)

689. J. H. Barrett: *Boundedness of solutions of systems of second order differential equations.*

Let $\alpha(x)$ be a nontrivial (column) vector solution of the vector-matrix differential equation (I) $(P(x)\alpha')'+Q(x)\alpha=0$, where $P(x)$ and $Q(x)$ are square matrices of real continuous scalar functions on $x\geq a$ and for each $x\geq a$, $P(x)$ is nonsingular. If $W(x)$ is a nonsingular matrix of differentiable scalar functions on $x\geq a$ such that W^*W' is negative semi-definite (for example, a constant matrix) then $\|\alpha(x)\|\leq k\exp\{(1/2)$ $\cdot\int_a^x\|G(t)\|dt\}$ where $G=(WP)^{*-1}-WQ$ and the (*) denotes the transpose of the matrix. Furthermore, if QP^* is symmetric, positive definite and differentiable, and its derivative $(QP^*)'$ is positive semi-definite; then there exists a nonsingular differentiable matrix $W(x)$ which satisfies $G=0$ and such that W^*W' is negative semi-definite.

Under the above conditions on Q and P every solution of equation (I) is bounded as $x\rightarrow\infty$. These results are matrix analogs of scalar results of the author (Proc. Amer. Math. Soc. vol. 6 (1955) pp. 247–251). (Received July 15, 1955.)

690. J. W. Brace: *Almost uniform convergence.*

The paper defines almost uniform convergence on an abstract set for a net of real or complex valued functions. The concept is used to investigate topologies on linear spaces. The weak topology (*not* the weak* topology) on the adjoint space of a locally convex space is determined to be the topology of almost uniform convergence on the bounded sets of the original space. Along with the corresponding dual proposition for the weak topology on the original space, it is shown that the weak* topology on an adjoint space is the topology of almost uniform convergence on the weakly compact sets of the original space. The results are used to investigate the continuity of linear transformations. The notion of almost uniform convergence for sequences of functions can be found in the works of Sirvent and Kantorovitch. (Received July 18, 1955.)

691*t*. J. W. Brace: *The structure of the second adjoint space.*

The paper presents a short proof of the fact that the second adjoint space of a locally convex linear topological space is the union of the closures of the bounded subsets of the original space when the closure operation is performed in the second adjoint space under the topology of uniform convergence on the finite point sets of the first adjoint space. The first known approach to such a theorem was made by Gelfand [see Gantmacher and Smulian, C. R. Acad. Sci. URSS. vol. 17 (1937) pp. 91–94]. For the special case of a Banach space proofs have been given by M. M. Day, Duke Math. J. vol. 8 (1941) pp. 763–770, and H. H. Goldstine, Duke Math. J. vol. 4 (1938) pp. 125–131. The proof of the theorem makes extensive use of polars in moving from one space to its adjoint and conversely. (Received July 18, 1955.)

692. L. R. Bragg: *The solutions of a linear ordinary differential equation of the third order about a turning point of the second order.*

A study is made of the solutions, for large values of the complex parameter λ, of the differential equation $y'''(z)+\lambda k_1(z, \lambda)y''(z)+\lambda^2 k_2(z, \lambda)y'(z)+\lambda^3 k_3(z, \lambda)y(z)=0$, where the $k_i(z, \lambda)$ are series of the form $\sum_{j=0}^{\infty} k_{ij}(z)/\lambda^j$ with the $k_{ij}(z)$ analytic functions of z, in a neighborhood Θ_0 of a turning point of the second order—that is, a point at which the discriminant of $\gamma^3(z)+k_{10}(z)\gamma^2(z)+k_{20}(z)\gamma(z)+k_{30}(z)=0$ has a zero of the second order. The author develops a method for obtaining expansions formally satisfying this equation and shows that these expansions are true solutions in a certain asymptotic sense. The method developed is similar to that developed by R. E. Langer in the case of a turning point of the first order, but differs markedly in a number of places since the role taken by the Bessel function is now taken by the confluent hypergeometric function. The author uses several results of R. W. McKelvey's paper on the solutions of a second order differential equation with a second order turning point. (Received July 15, 1955.)

693. Donald Bratton: *The duality theorem in linear programming.*

We suppose given two ordered locally convex spaces E, F and a continuous linear transformation T between them, and two points $a\in E'$, $b\in F$. One can then define a convex set K in E (resp. K' in F') and a corresponding "program" I (resp. II) of maximizing (resp. minimizing) the functional a (resp. b) in K (resp. K'), and (III) a third program of minimax for the function $\langle x, a\rangle+\langle b, y\rangle-\langle Tx, y\rangle$ in $E_+\times F_+$. The duality theorem states an intimate connection between these three programs (a result similar to Lagrange multipliers), and is known to hold under finite-dimensionality. In the general case, by consideration of the graph of T, the present writer reverts the theorem to the situation of just one space E and replaces T by a closed subspace S of E. The corresponding results are stated and proved under the hypothesis that S is of type $\mathcal{E}$, i.e., that each continuous positive linear functional in S can be extended as such to all of E. Finally, a study of the subspaces of type $\mathcal{E}$ is made. A dual notion, "type Q," is introduced, and it is shown that each closed subspace of B is of type $\mathcal{E}$ if and only if each closed subspace is of type Q. (Received June 22, 1955.)

694. C. C. Braunschweiger: *A geometric construction of L-spaces.*

An L-space (cf. S. Kakutani, Ann. of Math. vol. 42 (1941) pp. 523–537) is a Banach lattice whose norm and ordering satisfy the property: $x\geqq\theta$, $y\geqq\theta$ imply $\|x+y\|=\|x\|+\|y\|$. Denote the convex hull of a set A in a linear space X by the symbol $\langle A\rangle$. A subspace Y of the linear space X is constructed as follows: For a subset B of X (called the "base set" of the construction) let $F=\langle B\rangle$, $S=\langle F\cup(-F)\rangle$, $Y=\{sS: s\geqq 0\}$, and let $p(x)$ be the Minkowski functional for S on Y. Conditions on B and X sufficient to insure that Y', the Cantor completion of Y, is an L-space are considered. For example, if B is a (finitely) linearly independent subset of the linear space X then Y' is an L-space. In particular the construction is used to prove that a Banach lattice X is homeomorphic to an L-space under a linear order-preserving homeomorphism if and only if there exists in X a hyperplane whose intersection with the positive cone is a nonvoid bounded set not containing θ. (Recieved July 8, 1955.)

695*t*. F. E. Browder: *Regularity properties of solutions of elliptic differential equations.* I.

Let $A=\sum_{|\alpha|,|\beta|\leqq m}(-1)^{|\beta|}D^{\beta}(a_{\alpha\beta}(x)D^{\alpha})$ be an elliptic differential operator of order $2m$ on the smoothly bounded domain G of E^n. ($\alpha=(\alpha_1,\cdots,\alpha_r)$, $|\alpha|=r$, $D^{\alpha}=\partial^r/\partial x_{\alpha_1}\cdots\partial x_{\alpha_r}$.) By a weak solution of the equation $Au=f$ is meant a function u having m-strong derivatives in L^1 on every compact subdomain of G and satisfying $\sum_{\alpha,\beta}(a_{\alpha\beta}(x)D^{\alpha}u, D^{\beta}\phi)=(f,\phi)$ for $\phi\in C_0^{\infty}(G)$. It is shown that if u is a weak solution of an elliptic boundary-value problem in the variational sense on G, and if $a\in C^{m+t}(\bar{G})$, $t\geqq 0$, then u has $2m+t$ strong derivatives in $L^2(G)$ and there exist constants $c_t\geqq 0$ independent of u such that $\|D^{2m+t}u\|_{L^2(G)}\leqq c_t\{\|D^tf\|_{L^2(G)}+\|u\|_{L^2(G)}\}$. It follows from a theorem of Sobolev that for $t>n/2-2m$, u will have $2m+t-n/2$ continuous derivatives on the closure of G if the tth derivatives of f lie in $L^2(G)$. In particular, the derivatives of u involved in the boundary-value problem will exist and satisfy the boundary conditions in the classical sense. The proof utilizes a special case of an inequality due to Aronszajn on coercive differential forms. (Received July 19, 1955.)

696*t*. F. E. Browder: *Regularity properties of solutions of elliptic differential equations.* II.

Let u be a weak solution of the equation $Au=f$ of I on the arbitrary domain G of E^n. Suppose $a\in C^t(G)$, $t\geqq 0$. Then there exist constants $c_t(G_1)$ independent of u for compact subdomains G_1 of G such that u has $m+t$ strong derivatives in $L^p(G_1)$ satisfying $\|D^{m+t}u\|_{L^p(G_1)}\leqq c_t(G_1)\{\|f\|_{L^p(G)}+\|u\|_{L^1(G)}\}$. ($s=np/(n+p)(m-t)$) if $t\leqq m$, p if $t\geqq m$.) For $a\in C^{\infty}(G)$, the existence of a fundamental solution in the small and Green's functions for variational boundary value problems follow, while for $a\in C^{m+1}(G)$ it is shown that every weak solution must be a differentiable solution in the classical sense for $f\in L^p(G)$, $p>n/2m+1$, and the $2m$th derivatives will satisfy a

Holder condition. Finally for regular solutions of $Au=\sum_{|\gamma|\leq 2m} c_\gamma D^\gamma u=f$, the $2m-1$st derivatives will satisfy a Holder condition if $g\in C^p(G)$, $p>n/2m$, while the $2m$th derivatives will satisfy a Holder condition if c_γ and f do. The proofs rest upon the trick of Korn which has been applied to second order equations by E. Hopf, Schauder, and Morrey as well as the idea due to Nirenberg of injecting into Korn's trick the singular integral theorems of Calderon-Zygmund. (Received July 19, 1955.)

697*t*. F. H. Brownell: *A Gaussian averaged error estimate for a plane lattice point problem.*

It has long been known that the number $F(x)$ of lattice points (m_1, m_2), m_i being integers, in the plane which satisfy $m_1^2+m_2^2\leq x$ for $x>0$ is given by the equation $F(x)=\pi x+R(x)$. For the remainder $R(x)$ it is known as $x\to+\infty$ that $R(x)=O(x^\beta)$ for some $\beta<1/3$ (van der Corput), that $R(x)$ is not $o(x^{1/4})$ (Hardy), and that $(1/x)\cdot\int_0^x|R(t)|dt=O(x^{1/4})$ (Cramér). As a consequence of an investigation of the extended asymptotic distribution of eigenvalues of the membrane equation, we have found that $R(x)=\bar{O}(x^{-r})$ for every $r>0$. The precise definition of this Gaussian averaged error estimate denoted by $\bar{O}(x^{-r})$ is that $\left|\int_{1/2}^\infty \exp\ (-\rho^2(\ln\ x/y)^2)dR(y)\right|\leq M_\rho x^{-r}$ over $x\geq 1/2$ for some finite M_ρ existing for every $\rho>0$, and that $\psi_2(S)=\int_{1/2}^\infty Y^{-s}dR(y)$ possesses an analytic continuation in s from the region $\Re[s]>2$ throughout $\Re[s]\geq 0$ for which $\psi_2(0)=-R(1/2)$. (Received July 5, 1955.)

698. Lamberto Cesari: *Retraction of mappings from finitely connected Jordan regions.*

Notations: J, $J_0\subset E_2$ finitely connected closed Jordan regions; $S=(T, J)$: $p=T(w)$, continuous mapping from J into E_3 (surface); $L(S)$ Lebesgue area of S; Γ the decomposition of J into maximal continua of constancy g for (T, J); $K\subset J$ any continuum; $\{\gamma\}_{KTJ}$ the collection of the components γ of $J-K$. Properties P′, P$_1$, P$_2$ of K with respect to (T, J): (P′) $K=\sum g$, $g\in\Gamma$; (P$_1$) $k\subset g$, $g\in\Gamma$, for every component k of $F(\gamma)=\bar{\gamma}-\gamma$ and $\gamma\in\{\gamma\}_{KTJ}$; (P$_2$) $F(\gamma)$ is a continuum, for every $\gamma\in\{\gamma\}_{KTJ}$. I. If $K\subset J$ has properties P′, P$_1$ with respect to (T, J), then there is some J_0 with $K\subset J_0\subset J$ such that K has properties P′, P$_1$, P$_2$ with respect to (T, J_0). Then a mapping (T_0, J_0) is the retraction of (T, J) with respect to K in J_0 if $T_0=T$ on K, $T_0(w)=T[F(\gamma)]$ for every $w\in\gamma$ and $\gamma\in\{\gamma\}_{KTJ_0}$. Now $K\subset J$ is said to be a fine-cyclic element (set) (fce) for (T, J) if (i) K is a continuum; (ii) P′ and P$_1$ hold, and $K\notin\Gamma$; (iii) K is minimal with respect to (i), (ii). Then $\sigma=(T_0, J_0)$ [retraction of (T, J) with respect to K in a region J_0 as above] is said to be a fine-cyclic element (surface) (fce) of $S=(T, J)$. II. The collection of all (fce) of (T, J) is countable; III. For every fce K there is a cyclic element K_0 with $K\subset K_0$[$K=K_0$ if J is simply connected]. IV. $L(S)=\sum L(\sigma)$ where $\sum$ ranges over all fce of S (fine-cyclic additivity theorem). (Received July 15, 1955.)

699*t*. Paul Civin and Bertram Yood: *Boundaries for families of continuous functions.*

Let H be a multiplicative semi-group of real or complex bounded continuous functions on a topological space E. If there exists a unique minimal closed subset E_1 of E such that for all $f\in H$, sup $|f(t)|$, $t\in E_1$, is the same as $|f(t)|$, $t\in E$, then E_1 is called the Šilov boundary of E relative to H. A characterization and an analytic representation are obtained for the Šilov boundary of E relative to H, whenever the latter exists. (Received July 5, 1955.)

700*t*. Paul Civin and Bertram Yood: *Ideals in semi-groups of continuous functions.*

Let C be an abstract multiplicative semi-group which is known to be the semi-group of the continuous functions vanishing at infinity on some locally compact space X. Sub-semi-groups B of C are characterized in which suitable classes of multiplicative ideals are in 1-1 correspondence with the closed sets of X. The ideals used are similar to those used by Milgram (Duke Math. J. vol. 16 (1949) pp. 377–383) and Yood (to appear in the Duke Math. J.) in the compact case. Applications are made to regular Banach algebras. (Received July 5, 1955.)

701*t*. Paul Civin and Bertram Yood: *Invariant functionals.*

Let E be a normal linear space and G a solvable group of bounded linear operators on E. Conditions for the existence of a nontrivial bounded linear functional x^* invariant under G were given by Yood (Proc. Amer. Math. Soc. vol. 2 (1951) pp. 225–233). Related problems for E a real linear space were treated by Klee (Pacific Journal of Mathematics vol. 4 (1954) pp. 37–46). In this paper E is taken to be a Banach space and category arguments are used to improve the sufficient conditions for the existence of the functional x^*. The conditions involve hypotheses on the orbits of $x \in E$ under G rather than the metric properties of G as heretofore. (Received July 5, 1955.)

702*t*. Harvey Cohn: *Perturbation modular functions.* II.

The author continues the study of perturbation modular functions defined in an earlier abstract (Bull. Amer. Math. Soc. Abstract 61-2-231). The following application is made: Consider a Fuchsian group of transformations $Sz = (az+b)/(cz+d)$ with real unimodular transformation matrices S and a fundamental domain intersecting the real axis at one or more cusps. Consider further the regular modular functions $f(z)$ of degree -4, i.e., $f(Sz)(cz+d)^{-4} = f(z)$. Obviously every sum $E(z) = \sum (cz+d)^{-4}$ is such an $f(z)$ (where the summation is taken over cusps $-d/c$ equivalent under S to a fixed cusp). Conversely, however, not every $f(z)$ is a linear combination of these sums. The sums (or rather linear combination with real coefficients) can be characterized by the fact that their values possess a third primitive $P(z)$ (i.e., $P'''(z) = E(z)$), such that $P(z) - (cz+d)^2 P(Sz)$ is a quadratic polynomial in z with *real* coefficients. The proof of this result on the dimensionality of differentials is accomplished by a geometrical construction, i.e., it is shown that any $P(z)$ of this type can be arrived at only by the perturbation process defined in the earlier abstract. (Work supported by ONR contract.) (Received April 14, 1955.)

703. V. F. Cowling: *On Borel summability.*

Denote by $R(k,\psi_1, \psi_2)$ a region in the complex w-plane which contains the origin as an interior point and whose boundary consists of two rays $I_1 = I(k, \psi_1)$ and $I_2 = I(k, \psi_2)$ emanating from the point k, $-1 < k < 0$, making angles ψ_1 and ψ_2 respectively with the positive direction of the axis of reals. The following theorem is proved. Let $a(w)$, $w = k + Re^{i\psi}$, be analytic in $R(k, \psi_1, \psi_2)$ and there satisfy the condition $|a(w)/\Gamma(w+1)| = O(|w|^{\pi}e^{\delta|w|})$ for some δ $(0 < \delta < \pi)$ and some real k, ψ_1 and ψ_2 satisfying the inequalities $\pi/2 < \psi_1 < \pi$, $-\pi < \psi_2 < -\pi/2$, $\sin \psi_1 > 2\delta\pi/(\pi^2+\delta^2)$, and $\sin \psi_2 < -2\delta\pi/(\pi^2+\delta^2)$. Suppose also that $\operatorname{Lim\,Sup}_n |a(n)/\Gamma(n+1)|^{1/n} = 1$. Then $f(z) = \sum a(n)z^n$ is Borel integral summable for each z for which $\delta \operatorname{Csc} \psi_1 < \operatorname{Arg} z < 2\pi - \delta|\operatorname{Csc} \psi_2|$ and the Borel integral defines a function analytic for each z for which $\delta \operatorname{Csc} \psi_1 - \pi/2 < \operatorname{Arg} z < 5\pi/2 - \delta|\operatorname{Csc} \psi_2|$. This theorem is a generalization of a particular case of a theorem due to S. Bernstein (Bull. Sci. Math. vol. 63 (1928)). A similar theorem for Dirichlet is obtained. (Received July 13, 1955.)

704. Philip Davis and H. O. Pollak (p): *On the analytic continuation of mapping functions.*

The problem of this paper is to study the relationships between the continuability of various mapping functions of a domain B with analytic boundary and the moments $\iint_B z^n\overline{g(z)}\,dA$, where $g(z)$ is an arbitrary function analytic in B. Let $\bar{t}(z)$ denote $\overline{t(\bar{z})}$ for any analytic function $t(z)$, and let $h(z)=\int_0^z g(u)\,du$. Then a typical theorem reads as follows: Let B be a simply connected domain with analytic boundary and containing the origin, and let $m(z)$ map the unit circle onto B with $m(0)=0$. Then $\bar{h}(\bar{m}(z))$ is entire if and only if $\lim_{n\to\infty}\left|\iint_B z^n\overline{g(z)}\,dA\right|^{1/n}=0$. In case the lim sup of the preceding expression does not vanish, it may be identified by introducing a certain sequence of "Radienbilder" of B and finding the largest to which $\bar{h}(\bar{m}(z))$ continues analytically. (Received July 13, 1955.)

705. Ruth M. Davis: *On The Euler-Poisson-Darboux equation.*

Let $u(x)$ denote $u(x_0, x_1, \cdots, x_{n-1})$ and let k be a real parameter, $-\infty<k<\infty$. The explicit solution of a regular Cauchy problem for the n-dimensional Euler-Poisson-Darboux equation (E.P.D. equation) written as (1) $L[u]=u_{x_0x_0}+kx_0^{-1}u_{x_0}-\sum_{i=1}^{n-1}u_{x_ix_i}=0$, (2) $u=f(x_1, x_2, \cdots, x_{n-1})$, $u_{x_0}=0$ for $x_0=t_0>0$, is found by an extension of the method of Marcel Riesz. The essential point of this extended method is to find a function $V^\alpha(x; y)$ depending on a parameter α for which $M[V^{\alpha+2}]=V^\alpha$ and such that V^α and its first derivatives vanish on the surface of the characteristic cone with vertex x. Here M is the adjoint operator to L. The method of solution which relies on analytic continuation of divergent improper integrals leads to final formulas of entirely different character for odd and even dimensions. It is seen that there is no Huyghen's principle for n odd but for n even Huyghen's principle holds when k is even and $0\leq k\leq n-2$. For certain values of k and n the solution of the singular Cauchy problem for (1) is obtained immediately by simple limiting processes. Finally it should be noted that once the function V^α is found the solution to any regular correctly set Cauchy problem for either the homogeneous or the inhomogeneous E.P.D. equation should be readily obtainable. (Received June 13, 1955.)

706. M. M. Day: *Every L-space is isomorphic to a strictly convex space.*

A normed space B is called sc (sm) if it is isomorphic to a space with strictly convex (smooth) unit sphere. A recent paper (Day, Trans. Amer. Math. Soc. vol. 78 (1955) pp. 516–528) showed that $L(\mu)$, the space of μ-summable functions on a measure space X, is sm if and only if $\mu(X)$ is σ-finite. In this note other results from that paper are used with a theorem of Maharam on the structure of measure algebras to prove that every $L(\mu)$ is sc. (Received June 15, 1955.)

707. Allen Devinatz (p) and A. E. Nussbaum: *On the permutability of normal operators.*

It is proved that if N_1, N_2, and N are (in general unbounded) normal operators acting in a Hilbert space such that $N_1N_2=N_2N_1=N$, then N_1 and N_2 commute; i.e. the canonical resolutions of the identity of N_1 and N_2 commute. The method of proof is analogous to the one used by the authors in collaboration with J. von Neumann in a similar theorem for self-adjoint operators (cf. *On the permutability of self-adjoint operators* by A. Devinatz, A. E. Nussbaum and J. von Neumann, to appear in Ann. of Math.). Further, it is shown how this theorem yields a spectral representation of an n-parameter semi-group of normal operators: Let $\{N_x\}$ be a weakly continuous semi-group of normal operators (in general unbounded), where the index $x=(x_1, x_2, \cdots, x_n)$ ranges over the semigroup which is the Cartesian product $I_m^+ \times E_{n-m}^+$ where I_m^+ is the set of vectors in Euclidean m-space with non-negative integer components and E_n^+ is the set of vectors in Euclidean $(n-m)$-space with non-negative components such that $x \neq (0, 0, \cdots, 0)$. Then there exists a unique $2n$-parameter resolution of the identity $\{K(t, s)\}=\{K(t_1, t_2, \cdots, s_1, s_2, \cdots, s_n)\}$ such that $K(t, s)=0$ if any $t_k<0$, $k=1, 2, \cdots, n$, and $N_x=\int \rho e^{ix\cdot s} dK(t, s)$, where $\rho=\prod_1^n t_k^{x_k}$, $x\cdot s=\sum_1^n x_k s_k$. (Received June 24, 1955.)

708. L. E. Dubins: *A Radon-Nikodym type theorem for vector valued measures.*

Let M be the measure ring of a probability space and let S be the dual of a metrizable topological vector space B. A notion of generalized random variable is defined. The author determines necessary and sufficient conditions for a function defined on M with values in S to be the indefinite integral of a suitably measurable integrable generalized random variable. (Received July 19, 1955.)

709*t*. R. J. Duffin: *Infinite programs.*

Most of the development work on linear programming theory has been confined to finite programs. A program is termed finite if it involves only a finite number of variables and a finite number of constraint inequalities on these variables; otherwise, a program is termed infinite. In this paper the interesting duality theory which has been developed for finite linear programs is extended in such a manner that it applies to infinite linear programs. The theory, so extended, finds application to integral inequalities of the following type. Let $A(s, t)$ and $a(s)$ be continuous functions and let $A(s, t)=-A(t, s)$. The problem is to minimize $f=\int_0^1 a(t)x(t)dt$ for positive functions $x(t)$ such that $\int_0^1 A(s, t)x(t)dt+a(s)>0$ for s in the interval $(0, 1)$. It is shown that the greatest lower bound of f is zero. (Received July 14, 1955.)

710. Albert Edrei: *Analogue of a theorem of Boas and Pólya.*

Let $f(z)$ be holomorphic for $|z|<1$ and $f(z)f'(z)f''(z) \neq 0$. Assume $|(f(0)f''(0)/\{f'(0)\}^2)-1| \geqq \kappa (0<\kappa \leqq e^{-1})$. Then $|f(z_1)/f(z_2)| \leqq \exp(A\kappa^{-3/2}|z_1-z_2|)$ where A is an absolute constant and z_1, z_2 any two points of the disk $|z| \leqq 1/4$. The index m is said to be critical if $f^{(m)}(z)f^{(m+1)}(z)f^{(m+2)}(z) \neq 0$ in $|z|<1$, and if $|(f^{(m)}(0)f^{(m+2)}(0)/\{f^{(m+1)}(0)\}^2)-1| \geqq \kappa$. The above result and definition lead to the following analogue of a theorem of Boas and Pólya [Duke Math. J. vol. 9 (1942) p. 407]. Let $f(z)$ be holomorphic in the unit disk and let $\{n_k\}$ be an infinite sequence of critical indices (corresponding to a fixed value of $\kappa(>0)$). Then (i) if $n_k-n_{k-1}=O(1)$, $f(z)$ is an entire function of order one and finite type; (ii) if $n_k-n_{k-1}=O(n_k^\delta)$ where δ is fixed, $0<\delta<1$, then $f(z)$ is an entire function of finite order not exceeding $1/(1-\delta)$; (iii) if $n_k-n_{k-1}=O(n_k)$, then $f(z)$ is an entire function. (Received July 18, 1955.)

711. H. W. Ellis: *On the basis problem for function spaces determined by length functions.*

In a paper entitled *Haar functions and the basis problem for Banach spaces**, to appear in the Journal of the London Mathematical Society, Israel Halperin and the author defined generalized Haar functions on a general (not necessarily metric) measure space S and showed that there is a countable basis of generalized Haar functions in separable L^{λ} spaces of real valued functions on S for suitable λ, including λ a levelling length function. (For the definition of length functions and L^{λ} spaces see H. W. Ellis and Israel Halperin, Canadian Journal of Mathematics vol. 5 (1953) pp. 576–592.) The existence of a countable basis in vector valued Banach spaces $L^{\lambda'}(L^{\lambda})$ is considered. It is shown that if $L^{\lambda'}$ and L^{λ} are spaces of real valued functions on the general measure spaces S and S' and if $L^{\lambda'}(L^{\lambda})$ is separable then $L^{\lambda'}(L^{\lambda})$ is equivalent to a Banach space $L^{\lambda'\lambda}$ of real valued functions on the Cartesian product space $S\times S'$. The existence of a countable basis of generalized Haar functions in $L^{\lambda'\lambda}$ when both L^{λ} and $L^{\lambda'}$ satisfy the conditions of * is then shown. (Received July 6, 1955.)

712. N. J. Fine: *Cesàro summability of Walsh-Fourier series.*

The Walsh-Fourier series of an integrable function $f(x)$ is $(C, 1)$ summable to $f(x)$ almost everywhere. For $f\in L^p$, $p>1$, a proof was indicated by Paley [Proc. London Math. Soc. (2) vol. 34 (1932) pp. 241–279]. (Received May 16, 1955.)

713. Tomlinson Fort: *Limits of the characteristic values for certain boundary-value problems associated with difference equations.*

This paper is concerned with the n-dimensional generalization of $\Delta\{k(i-1)\cdot\Delta y(i-1)\}+[\lambda/m^2-s(i)]y(i)=0$ subject to boundary conditions; $y(1)=c_1y(0)$, $y(m+1)=c_2y(m)$. Several theorems are proved with reference to the characteristic equation. Then if $k(i)=k(x_i)$ where $x_{i+1}-x_i=1/m$ and $\text{limit}_{m\to\infty}\, k(x_i)=K(x)$ and if the characteristic equation is transformed so that its roots are multiplied by $1/m$ and if m^2/λ is denoted by ρ then explicit forms for the limits of the coefficients of the equation in ρ are obtained. Several theorems relative to the characteristic values for the difference problem and the approximated differential problem are then proved. The n-dimensional case involves only expected variations from the one-dimensional case. (Received July 18, 1955.)

714t. Evelyn Frank: *A new class of continued fraction expansions for the ratios of hypergeometric functions.*

A new class of continued fraction expansions for the ratios of two Gauss hypergeometric functions is described in detail. A number of such ratios are expanded, and the convergence of the corresponding continued fractions is determined as well as the functions to which the continued fractions converge in various regions. Many special relations are thus obtained between functions, finite continued fractions, and infinite continued fractions. Specialized continued fraction expansions and specialized functions are discussed, and integral representations for such continued fractions are found. (Received July 6, 1955.)

715t. Casper Goffman: *Convergence in area of integral means.*

The theorem of Radó that the Lebesgue areas of the surfaces defined by the integral means of a continuous $f(x, y)$ converge to the Lebesgue area of the surface given by $f(x, y)$ is extended to summable functions. This allows a definition of area of surfaces associated with summable functions to be made which agrees with a definition given previously by L. Cesari (Annali Scuola Normale Superiore Pisa (2) vol. 5 (1936) pp. 299–313) and the author (Rend. Circ. Mat. Palermo (2) vol. 2 (1953) pp. 203–235). (Received June 6, 1955.)

716. Michael Golomb: *Spectral properties of commutators.*

Let A, B be linear operators on a Banach space $\mathfrak{X}$, $C=AB-BA$ their cummutator, and assume $AC=CA$. The conjecture that the spectrum $\sigma(C)$ consists of 0 alone is confirmed in this paper by the following results. (1) $C=0$ if $\mathfrak{X}$ is Hilbert and one of the operators A, B, C is normal. (2) $C=0$ if $\mathfrak{X}$ is Hilbert and A or B is maximal hermitian (in this case there are certain additional hypotheses on the domains of the nonbounded A, B, C). (3) $C=0$ if $C=\sum_i \lambda_i E_i$ where $E_iE_j=\delta_{ij}E_i$. (4) $\sigma(C)$ has no component not connected with 0. (5) If C is a "spectral operator of finite type" defined to possess a certain spectral decomposition then $\sigma(C)=[0]$. To prove that certain operators, not normal nor completely continuous, are spectral operators of finite type on the Hilbert space $\mathfrak{X}$ the following theorem is proved. If $N=\int_\sigma s\,dE_s$ is normal, $\sigma=\sigma_1+\sigma_2+\cdots$ where the σ_i are both open and closed in σ, and R is such that $R\int_{\sigma_i}dE_s=\int_{\sigma_i}dE_sR$ is nilpotent for every i then $A(N+R)=(N+R)A$ implies $AE_s=E_sA$ for every s. (Received July 18, 1955.)

717. P. E. Guenther: *Canonical solutions of a q-difference system.* Preliminary report.

It is known that the q-difference system $Y(qx)=A(x)Y(x)$ where the elements of the coefficient matrix $A(x)$ are functions analytic except possibly for poles at $x=\infty$ may be reduced to canonical form $Y(qx)=C(x)Y(x)$ where the coefficient matrix $C(x)$ involves only polynomial or zero elements (Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 218–222). Starting with this latter form of the q-difference system analytic solutions are determined by a variety of methods employed by Birkhoff, Carmichael, Nörlund, and others in the case of additive difference equations. The relations between these solutions are studied and their asymptotic representation by the complement of formal series solutions is established. In particular, the convergent part of the formal series complement is found to be predictable from the array of vanishing elements in $C(x)$. (Received July 15, 1955.)

718. Meyer Jerison: *Martingale formulation of ergodic theorems.*

Let f be an integrable function on a σ-finite measure space X, T a 1-1 measure-preserving transformation of X, and N the set of non-negative integers with measure 1 assigned to each integer. On the product space $X\times N$ with the product measure, a sequence of functions integrable on sets of finite measure is defined: $h_k(x, 0)=k^{-1}\cdot\sum_{i=0}^{k-1} f(T^ix)$, $h_k(x, n)=h_k(T^{-n}x, 0)$ if $n<k$, and $h_k(x, n)=f(x)$ if $n\geqq k$. The ergodic theorem asserts the existence of $\lim_{k\to\infty} h_k(x, n)$ almost everywhere on $X\times N$. The sequence $\{h_k\}$ is a martingale [Doob, *Stochastic processes*, New York, 1953] with the index going backward, but the existence of $\lim_{k\to\infty} h_k(x, n)$ cannot be inferred from the standard martingale convergence theorem because $X\times N$ has infinite measure. A similar formulation of the generalized ergodic theorem [Hurewicz, Ann. of Math. vol. 45 (1944) pp. 192–206] is given by replacing the product measure on $X\times N$ with one that reflects the effect of a transformation which does not preserve measure. This formulation sheds light upon the relation between the ergodic theorems and the martingale convergence theorem. (Received July 14, 1955.)

719. R. V. Kadison: *Operator algebras with a faithful weakly-closed representation.*

A W^*-algebra is defined to be a C^*-algebra in which each bounded, monotone-increasing, directed sequence $\{A_\gamma\}$ of operators has a least upper bound A and which has a family of states ω such that $\{\omega(A_\gamma)\}$ has the directed limit $\omega(A)$ and such that if $\omega(B)=0$ for some positive operator B in the algebra and each state of the family then $B=0$. Main theorem: A C^*-algebra has a faithful representation as a ring of operators if and only if it is a W^*-algebra. (Received June 6, 1955.)

720. M. S. Klamkin: *On a graphical solution of the general second order linear differential equation.*

The general second order linear differential equation can be transformed into the canonical form $[D^2+\phi(x)]y=F(x)$. This 2d order equation is equivalent to the following set of 1st order equations: $Du=u^2+\phi(x)$, $[D+u]y=r$, and $[D-u]r=f(x)$; which are then solved graphically. (Received July 18, 1955.)

721*t*. V. L. Klee, Jr.: *Solution of a problem of E. M. Wright on convex functions.*

With R denoting the real number field and f a function on R to R, consider the following two statements about f: (A) $f(x+d)-f(x)\geq f(y+d)-f(y)$ for all $d>0$ and $x>y$. (B) $f(x/2+y/2)\leq f(x)/2+f(y)/2$ for all x, y. It is well known that (A) implies (B), and that for continuous f the two are equivalent. In a recent paper [*An inequality for convex functions*, Amer. Math. Monthly vol. 61 (1954) pp. 620–622], Wright asked whether there is a function f for which (B) is true and (A) is false. It is proved here that if τ is an arbitrary additive, rationally homogeneous isomorphism of R onto R^2, and if $f(x)=|\tau(x)|^2$ for each $x\in R$, then f is such a function. (This note will appear in the American Mathematical Monthly.) (Received June 15, 1955.)

722. R. E. Langer: *On the construction of related differential equations.*

Linear differential equations that depend upon a parameter λ are commonly of the form (1) $d^nu/dz^n+\lambda p_1(z, \lambda)d^{n-1}u/dz^{n-1}+\cdots+\lambda^np_n(z, \lambda)u=0$, with each coefficient $p_j(z, \lambda)$ a polynomial or power series in $1/\lambda$. For many purposes the forms of the solutions of such an equation, when $|\lambda|$ is large, must be determined. That can be done by classical methods, if the z-region in which the equation is being considered is one in which the roots of the auxiliary algebraic equation (2) $\chi^n+p_1(z, \infty)\chi^{n-1}+\cdots+p_n(z, \infty)=0$ maintain an invariant configuration, i.e. roots that are single at any z are so for all z, ones that are multiple, are identically so in z. Other methods are required if the z-region contains a *turning point* (transition point) at which the auxiliary roots have a different configuration than elsewhere. A method that is then applicable is based upon a so-called "related equation." This is a differential equation that fulfills the specifications (a), its coefficients are identical with those of the given equation to the terms in some specified power in $1/\lambda$, and (b), its solution forms are known. This paper considers the differential equation (1) in a z-region in which $(n-m)$ roots of the equation (2) remain simple, whereas the other m roots, $\chi_1, \chi_2, \cdots, \chi_m$, are involved in multiplicities at a turning point. It shows how a related equation is to be constructed if such an equation is known, or constructible, for the differential equation of the lower order m whose auxiliary equation has the roots $\chi_1, \chi_2, \cdots, \chi_m$. (Received July 11, 1955.)

723*t*. J. J. Levin: *On singular perturbations related to conditional stability.*

The behavior, as $\epsilon\to0+$, of the solutions of the system of real differential equations (1) $dx/dt=f(x, y, t, \epsilon)$, $\epsilon dy/dt=Ay+g(x, y, t, \epsilon)$ is investigated. It is assumed that x and y are vectors of m and n components respectively, that A is a constant n by n matrix k of whose characteristic roots have negative real parts, while the remaining $n-k$ characteristic roots have positive real parts, and that f and g satisfy several mild smoothness hypotheses. It is shown that there exists a k-dimensional initial manifold in y space, which depends on ϵ and the initial x vector, such that if the initial y vector lies on the manifold then the solution of (1) exists, for all sufficiently small $\epsilon>0$, over a fixed t interval which does not depend on ϵ. Further properties of this initial manifold and of solutions starting near it are established. (Received July 13, 1955.)

724. A. J. Lohwater and George Piranian (p): *On a conjecture of Lusin.*

A point $e^{i\theta}$ is called an L-point of a function $f(z)$, regular in $|z|<1$, if for every t in $0<t<1$ the (possibly many-sheeted) image of the disc $|z-te^{i\theta}|<1-t$ under the mapping $w=f(z)$ has infinite area. It was conjectured by N. Lusin [Doklady Akad. Nauk 56 (1947)] that there exists a bounded analytic function $f(z)$ for which the set of L-points on $|z|=1$ is of measure 2π. It is shown, by means of a Taylor series, that there exists a function $f(z)$, regular in $|z|<1$ and continuous in $|z|\leq 1$ for which every point of $|z|=1$ is an L-point. It is also shown that there exists a function $f(z)$, regular in $|z|<1$ and continuous in $|z|\leq 1$, which maps $|z|<1$ into a region of infinite area, but which has no L-points on $|z|=1$. (Received May 23, 1955.)

725. R. M. McLeod: *On the zeros of the derivatives of some entire functions.*

Let $\mathcal{L}(f)$ be the set of limit points of the zeros of the derivatives of the entire function f. Let $\mathcal{J}_1$ be the class of entire functions $f(z)=z^mP(z)\exp(Q(z))$ where $Q(z)$ is a polynomial of degree q and $P(z)$ is a canonical product of genus p with $0\leq p\leq q-2$. Pólya (Math. Zeit. vol. 12 (1922) pp. 36–60) found $\mathcal{L}(f)$ for those $f\in\mathcal{J}_1$ for which $P(z)$ is a polynomial. Let $Q(z)=\sum_{k=0}^{q} b_kz^k$ with $b_0>0$. Set $w=|w|e^{i\theta}=z+b_1/(qb_0)$ and $S_k(t)=\{z\,|\,|w|>0, |\theta-2\pi k/q|<t\}$. Let $\mathcal{R}$ be the complement of $\bigcup_{k=0}^{q-1} S_k(\pi/q)$. If $f\in\mathcal{J}_1$ and if f has at most a finite set of zeros in $\bigcup_{k=0}^{q-1} S_k(t)$ for some positive t, then by methods similar to those of Pólya it is proved that $\mathcal{L}(f)=\mathcal{R}$. Similar results are obtained for the class $\mathcal{J}_2$ of functions $f(z)=z^mP(z)\exp(b_0z^2+b_1z+b_2)$, $b_0>0$, where $P(z)$ is a canonical product of genus one with zeros a_k satisfying $\sum_{k=1}^{\infty}|\mathrm{Re}\,(1/a_k)|<\infty$. Moreover, it is shown that there are functions $f\in\mathcal{J}_1$ for which $\mathcal{R}$ is a proper subset of $\mathcal{L}(f)$. (Received July 15, 1955.)

726t. G. W. Mackey: *Unitary representations of group extensions.*

In an earlier paper (Proc. Nat. Acad. Sci. U.S.A. (1949) pp. 537–545) the author used the theory of induced representations to study (unitary) representations of semi-direct products of separable locally compact groups in the special case in which the normal subgroup is required to be abelian. In the present paper this work is extended to a study of the connection between the representations of K, G, and G/K where K is a (not necessarily abelian) closed normal subgroup of G and G is an arbitrary extension of K by G/K. The representations of G may depend upon the projective representations of G/K and its subgroups. However the whole theory may be developed ab initio for projective representations so that the results may be used inductively to study complicated groups. In particular a theory of induced projective representations is presented which extends previous work of the author on induced representations. K is not allowed to be completely arbitrary but must have a suitably regular decomposition theory for its representations. Investigation of this point leads to a refinement of existing work on decomposition theory and to a division of groups into those with "smooth" and "rough" duals according to the behavior of a certain natural Borel structure in the set of equivalence classes of irreducible representations. (Received July 18, 1955.)

727t. G. R. MacLane: *On the zeros of the derivatives of an entire function.*

Pólya (Bull. Amer. Math. Soc. vol. 49 (1943) p. 179) has defined the *final set* L_f of an entire function $f(z)$ as follows: $z_0 \in L_f$ if and only if for each neighborhood U of z_0 there are infinitely many of the functions $\{f^{(n)}(z)\}$ which have zeros in U. It is a simple consequence of known facts that either $f(z) = a \exp(bz)$, in which case L_f is void, or $\infty \in L_f$. The present paper proves the following theorem. Let K be any compact subset of the sphere such that $\infty \in K$. Then there exists an entire function f such that $L_f = K$. Furthermore, f may be chosen to satisfy any one of the following conditions: (1) if $p(r)$, $r>0$, is such that $\lim_{r\to\infty} r^{-n}p(r) = \infty$, $n>0$, then $|f(z)| < p(|z|)$; (2) f is of order μ, $0<\mu<\infty$; (3) f is of arbitrarily rapid growth. (Received July 11, 1955.)

728t. J. S. MacNerney: *Hellinger integrals in inner product spaces. I.*

An *inner product space* is an ordered pair $\{S, Q\}$ such that S is a (complex) linear space and Q is an inner product for S; $B\{S, Q\}$ is the space of continuous linear transformations in $\{S, Q\}$, normed in the usual way. It is assumed *throughout this sequence* that $\{S_0, Q_0\}$ is a complete inner product space. The idea of a certain Hellinger integral is extended to obtain an integral denoted by $\int_0^1 Q_0(df, [d\phi]^{-1}dg)$, where each of f and g is a function from $[0, 1]$ to $\{S_0, Q_0\}$ and the function ϕ, from $[0, 1]$ to $B\{S_0, Q_0\}$, is nondecreasing in the sense that $\phi(b)-\phi(a)$ is Hermitian and non-negative for each subinterval $[a, b]$ of $[0, 1]$. LEMMA: For H and K non-negative Hermitian in $B\{S_0, Q_0\}$, $Q_0(x, H^{-1}x)+Q_0(y, K^{-1}y) \geq Q_0([x+y], [H+K]^{-1} \cdot [x+y])$ where $H^{-1}x$ is any point u in S_0 such that $Hu = x$, *etc.* Let H be non-negative Hermitian in $B\{S_0, Q_0\}$ and S be the linear space of all functions f from $[0, 1]$ to S_0 such that $f(0)$ is in $H(S_0)$ and $\int_0^1 Q_0(df, [d\phi]^{-1}df)$ exists. Let $Q(f, g) = Q_0(f[0], H^{-1}g[0]) + \int_0^1 Q_0(df, [d\phi]^{-1}dg)$ for f and g in S. The inner product space $\{S, Q\}$ is complete only in case $H(S_0)$ and all $[\phi(b)-\phi(a)](S_0)$ are closed linear manifolds in $\{S_0, Q_0\}$. Let $K(s, t) = K(t, s) = H+\phi(s)$ for $0 \leq s \leq t \leq 1$. For "fixed" t in $[0, 1]$ and x in S_0, $K(\cdot, t)x$ is in S and $Q(f[\cdot], K[\cdot, t]x) = Q_0(f[t], x)$ for f in S. (Received July 29, 1955.)

729t. J. S. MacNerney: *Hellinger integrals in inner product spaces. II.*

Let E be a set. A ***kernel*** in the inner product space $\{S, Q\}$ of functions from E to $\{S_0, Q_0\}$ is a function K from $E \times E$ such that each $K(s, t)$ is a linear transformation from S_0 to S_0 and, for each t, x in $E \times S_0$, the function $K(\cdot, t)x$ belongs to S and $Q(f[\cdot], K[\cdot, t]x) = Q_0(f[t], x)$ for all f in S. There is a kernel in $\{S, Q\}$ only in case convergence in $\{S, Q\}$ (assumed complete) implies point-wise convergence over E: K is unique. A kernel K must satisfy $\sum_{i,j} Q_0(x_i, K[t_i, t_j]x_j) \geq 0$ for all finite sets $\{t_i\}$ in E and $\{x_i\}$ in S_0, whence the values of K are in $B\{S_0, Q_0\}$. If K is a function from $E \times E$ to $B\{S_0, Q_0\}$ satisfying this condition, there is only one complete inner product space $\{S, Q\}$ of functions from E to $\{S_0, Q_0\}$ in which K is the kernel, and the set of f of the form $f(\cdot)=\sum K(\cdot, t_j)x_j$ is dense in $\{S, Q\}$; a function f from E to $\{S_0, Q_0\}$ belongs to S only if there is a number N such that $|\sum Q_0(f[t_i], x_i)|^2 \leq N\sum Q_0(x_i, K[t_i, t_j]x_j)$ for finite sets $\{t_i\}$ and $\{x_i\}$ as above, in which case $Q(f, f)$ is the least such N; if $\{S_0, Q_0\}$ is separable and E is a separable topological space and the functions in S are weakly continuous, then $\{S, Q\}$ is separable. This extends earlier results [Bull. Amer. Math. Soc. Abstract 61-2-266] of the author. (Received July 29, 1955.)

730*t*. J. S. MacNerney: *Hellinger integrals in inner product spaces.* III.

A ***kernel system*** is a sequence $\{K, E, S, Q\}$ such that K is the kernel in the complete inner product space $\{S, Q\}$ of functions from the set E to $\{S_0, Q_0\}$. Let each of $\{K_1, E_1, S_1, Q_1\}$ and $\{K_2, E_2, S_2, Q_2\}$ be a kernel system. If $E_1=E_2$ then S_1 is a subset of S_2 only in case there is a positive number N such that NK_2-K_1 is the kernel in some space. Suppose $E_1=E_2$ and S_1 is a subset of S_2: the equation $Q_1(f, Tg)=Q_2(f, g)$ for f, g in $S_1\times S_2$ defines a continuous linear transformation T from $\{S_2, Q_2\}$ to $\{S_1, Q_1\}$; $S_1=T(S_2)$ only in case either S_1 or $T(S_2)$ is a closed linear manifold in $\{S_2, Q_2\}$. Suppose E_1 is a proper subset of E_2 and K_1 is the contraction of K_2 to $E_1\times E_1$: S_1 is the set of contractions to E_1 of functions in S_2; $\{S_1, Q_1\}$ is isomorphic to $\{S_3, Q_3\}$, where S_3 is the orthogonal complement in $\{S_2, Q_2\}$ of the set of f in S_2 such that $f(t)=0$ for t in E_1, and Q_3 is the contraction of Q_2 to $S_3\times S_3$. Suppose S is a linear subspace of S_1: there exists a kernel system $\{K, E, S, Q\}$ only in case there exist a positive number N and an inner product Q' for S such that $\{S, Q'\}$ is complete and $Q_1(f, f)\leq NQ'(f, f)$ for all f in S. For details in the one-dimensional case (where $B\{S_0, Q_0\}=\{S_0, Q_0\}$), see Aronszajn [Trans. Amer. Math. Soc. vol. 68 (1950) pp. 337–404]. (Received July 29, 1955.)

731*t*. J. S. MacNerney: *Hellinger integrals in inner product spaces.* IV.

The kernel system $\{K, E, S, Q\}$ is ***ideal*** provided, for each finite sequence $t=\{t_i\}_1^n$ in E, the set M_t of all f of the form $f(\cdot)=\sum_1^n K(\ , t_j)x_j$ is a closed linear manifold in $\{S, Q\}$. If S_0^n is the n-fold direct product of S_0 with itself and if $Q_0^n(x, y)=\sum_1^n Q_0(x_j, y_j)$, each of the following is a necessary and sufficient condition that a set M_t (as above) be closed in $\{S, Q\}$: (1) the transformation H, $Hx=\{\sum K(t_i, t_j)x_j\}$ for $x=\{x_i\}$, maps S_0^n onto a closed linear manifold in $\{S_0^n, Q_0^n\}$; (2) the set V_t, to which x belongs only in case there is an f in S such that $x=\{f(t_i)\}$, is a closed linear manifold in $\{S_0^n, Q_0^n\}$; (3) for each f in S there is an $\{x_i\}$ in S_0^n such that $\{f(t_i)\}=\{\sum K(t_i, t_j)x_j\}$ in which case $\sum K(\cdot, t_j)x_j$ is the projection $P_t f$ of f on M_t. In the latter notation, if $\{K, E, S, Q\}$ is an ideal kernel system then Q is a ***Hellinger integral*** over E in the sense that if f, g is in $S\times S$ and $c>0$ there is a finite sequence s in E such that if t is a finite sequence in E of which s is a subsequence then $|Q(f, g)-Q(f, P_t g)|<c$. NOTE: in the ideal system $\{K, [0, 1], S, Q\}$ of part I (above), if t is a subdivision of $[0, 1]$ then $Q(f, P_t g)$ is the corresponding ***approximating sum*** for the integral $Q(f, g)=Q_0(f[0], H^{-1}g[0])+\int_0^1 Q_0(df, [d\phi]^{-1}dg)$. (Received July 28, 1955.)

732*t*. J. S. MacNerney: *Hellinger integrals in inner product spaces.* V.

This is an application of parts I–IV to integral equations. Let S be the linear space of all functions f from $[0, 1]$ to $\{S_0, Q_0\}$ such that $\int_0^1 Q_0(df, [d\phi]^{-1}df)$ exists. A study is made of connections between S and a system of Stieltjes (mean) integral equations: (*) $f(b)-f(a)+\int_a^b d\phi\cdot g=0$ and $\int_a^b d\theta\ f+g(b)-g(a)=h(b)-h(a)$ for each subinterval $[a, b]$ of $[0, 1]$, where h is a function of bounded variation from $[0, 1]$ to $\{S_0, Q_0\}$. With suitable restrictions on ϕ and θ (which are nondecreasing functions from $[0, 1]$ to $B\{S_0, Q_0\}$), the equation $Q(f, g)=\int_0^1 Q_0(df, [d\phi]^{-1}dg)+\int_0^1 Q_0(f, d\theta\cdot g)$ defines an inner product Q for S and there is a kernel in the space $\{S, Q\}$. Various boundary value problems connected with the system (*) determine certain subspaces of $\{S, Q\}$, and lend themselves to classification in terms of the relationships among the associated subspaces; the boundary condition $g(0)=g(1)=0$ determines the whole space. These boundary value problems lead to integral equations, of the form $f(s)-z\cdot\int_0^1 K(s, u)\cdot d\Psi(u)\cdot f(u)=g(s)$ with Ψ of bounded variation from $[0, 1]$ to $B\{S_0, Q_0\}$, which resemble those arising in the classical Sturm-Liouville theory of second order differential equations with two-point boundary conditions. (Received July 28, 1955.)

733*t*. J. S. MacNerney: *Hellinger integrals in inner product spaces.* VI.

This is an application of parts II–IV to function theory. Let E be a connected domain in the number plane, and P be a weakly continuous function from E to $B\{S_0, Q_0\}$ such that, for each closed bounded set D lying in E, $Q_0(x, P[z]x)$ is real and has positive upper and lower bounds for z in D and $Q_0(x, x)=1$. The function f from E to $\{S_0, Q_0\}$ is *regular* provided that for each x in S_0 the function g, $g(z)=Q_0(f[z], x)$ for z in E, is a regular (analytic) function in E. Let S be the set to which f belongs only in case f is a regular function from E to $\{S_0, Q_0\}$ and the real function $Q_0(f, Pf)$ is integrable over E: (1) S is a linear space; (2) $Q(f, g)=\iint_E Q_0(f, Pf)$ defines an inner product Q for S; (3) the inner product space $\{S, Q\}$ is complete and there is a kernel in $\{S, Q\}$; (4) if the condition that P be weakly continuous is replaced by any condition which ensures (1), then (2) and (3) still hold. An example is given, with E the interior of the unit circle and $\{S_0, Q_0\}$ the 2-dimensional complex Euclidean space, wherein the only 2×2 constant matrices which belong to $B\{S, Q\}$ as "constant multipliers" are those matrices A such that $A_{11}-A_{22}=A_{12}-A_{21}$: $P(z)_{11}=[1-|z|]$, $P(z)_{12}=P(z)_{21}=[1-|z|^2]/2$, and $P(z)_{22}=[1-|z|^3]/3$. (Received July 29, 1955.)

734. H. W. Milnes: *Convexity of Orlicz spaces.*

Let L_Φ be the Orlicz space over $\Delta=[a, b]$ generated by an increasing function $0\leqq\phi(u)\leqq+\infty$ with inverse $\psi(v)$ and respective integrals $\Phi(u)$, $\Psi(v)$, and the norm defined by $\|f\|=\sup\int_\Delta|f(x)|g(x)dx$ (supremum taken for all $g(x)\geqq 0$ with $\int_\Delta\Psi(g)dx\leqq 1$). Conditions for strict and uniform convexity of L_Φ are obtained. L_Φ is strictly convex (i.e. boundary of the unit sphere in L_Φ does not contain segments) if and only if $\psi(v)$ and $\Psi(v)$ are continuous. L_Φ is uniformly convex if and only if in addition (i) $\liminf[\phi(u+\epsilon u)/\phi(u)]>1$ for all $\epsilon>0$, (ii) $\limsup[\Phi(2u)/\Phi(u)]<+\infty$. Essential for the proofs is the fact that when $\lim_{v\to\infty}\psi(v)\to\infty$ for each $f\in L_\Phi$, there is a $g(x)$ with the properties $\int_\Delta\Psi(g)dx=1$, $\psi(g(x))=\text{Const.}\ |f(x)|$ such that $\|f\|=\int_\Delta|f(x)|g(x)dx$. (Received September 1, 1955.)

735*t*. C. J. Neugebauer: *A strong cyclic additivity theorem of a surface integral.*

Let X be a compact subset of E_3 (Euclidean 3-space), and let T be a continuous mapping from the unit square Q of E_2 (Euclidean 2-space) into E_3 such that $T(Q)\subset X$ and $L(T)<\infty$, where $L(T)$ is the Lebesgue area of the surface S represented by T.

For such a continuous mapping, L. Cesari has introduced a surface integral $J(T) = (S)\int F d\sigma$ as a Weierstrass integral (Ann. Scuola Norm. Sup. Pisa (2) vol. 13 (1944) pp. 77–117). If $T = lm$, $m: Q \Longrightarrow M$, $l: M \to E_3$ is a monotone-light factorization of T, then J. Cecconi has shown that $J(T)$ is *weakly cyclicly additive* in the following sense. If r_C denotes the monotone retraction from M onto a proper cyclic element C of M, then $J(T) = \sum J(lr_C m)$, $C \subset M$. This result can be extended by using the concept of an unrestricted factorization of T due to E. J. Mickle and T. Radó (Trans. Amer. Math. Soc. vol. 66 (1949) pp. 347–365). Let $T = sf$, $f: Q \to M$, $s: M \to E_3$ be an unrestricted factorization of T. If r_C has the same meaning as above and if $\sum'$ denotes the summation over all proper cyclic elements C of M for which $f(Q) \cap C \neq \phi$, then $J(T)$ is *strongly cyclicly additive*, i.e., $J(T) = \sum' J(sr_C f)$. This formula also extends a strong cyclic additivity theorem of $L(T)$ due to T. Radó. Extensions are available. (Received July 14, 1955.)

736. Robert Osserman: *On Ahlfors' counting function $n(t)$.*

The class of Riemann surfaces considered consists of simply-connected covering surfaces of the plane having only algebraic singularities over finite points. Fixing a point on the surface, we define $n(t)$ to be the total order of branch points whose distance along the surface to the fixed point is less than t. Ahlfors has proved (Comment. Math. Helv. vol. 3 (1931) pp. 173–177) that a sufficient condition for parabolic type is $\int^\infty dt/tn(t) = \infty$. In the present paper further results are obtained relating the growth of $n(t)$ to the type of the surface. In particular, if all branch points of the surface project onto a bounded set in the plane, then the divergence of $\int^\infty dt/n(t)$ is already sufficient. On the other hand, if no further restrictions are made on the class of surfaces considered, then Ahlfors' condition is the best possible. In fact, a subclass of surfaces is exhibited for which Ahlfors' condition is both necessary and sufficient for parabolic type. This class includes examples of hyperbolic surfaces with $n(t) < t^\epsilon$ for arbitrarily small $\epsilon > 0$. Finally, it is pointed out that no criterion for hyperbolic type can depend on $n(t)$ alone, since there exist parabolic surfaces with $n(t)$ arbitrarily large. (Received July 18, 1955.)

737. R. S. Pinkham: *Inversion of the Laplace and Stieltjes transforms utilising difference operators.*

The paper is divided into four parts. Part I contains a new real inversion formula for the Stieltjes transform which utilizes differences of the generating function rather than derivatives. A generalized form of the asymptotic method of Laplace is developed and used to evaluate the singular integrals which appear. Part II deals with a new inversion formula for the Laplace transform which requires only knowledge of the generating function along the axis of reals. Part III presents the representation theory resulting from the operators of Parts I and II, while in Part IV an operator is constructed for solving the moment problem for the Stieltjes transform. (Received June 30, 1955.)

738. Pasquale Porcelli: *Inversion of a Lebesgue-Stieltjes integral equation.*

The integrals in this paper are in the Lebesgue-Stieltjes sense. The following theorem is proved; Theorem: If each of $s(t)$ and $g(t)$ is a nondecreasing function on $[a, b]$ and $f(t)$ is a nonnegative g-summable on $[a, b]$ such that $s(a) = g(a) = 0$ and (i) $s(u) = \int_a^u f(t)dg(t)$ for $a < u \leq b$, then in order for $1/f(t)$ to be s-summable and (ii)

$g(u)=\int_a^u\{1/f(t)\}ds(t)$ it is necessary and sufficient that $l_gR(f)=0$, where $R(f)$ is a subset of $[a, b]$ and t is in $R(f)$ only if $f(t)=0$, and $l_gR(f)$ denotes the outer g-length of $R(f)$. (Received July 15, 1955.)

739t. R. T. Prosser: *On the representations of W^* algebras.*

Let H be a hilbert space, and A, $A'W^*$ algebras of operators acting on H, where A' is the commutant of A. For each pair of cardinals m, n, zero or infinite, there exist a hilbert space $H_{m,n}$ and W^* algebras A_m and A'_n acting on $H_{m,n}$ such that A'_n is commutant of A_m, and: A_0 is finite, A_m is either zero or cardinally homogeneous of degree m, for $m>0$; likewise for A'_n; H is the direct sum of the $H_{m,n}$, A is the direct sum of the A_m, and A' is the direct sum of the A'_n. Moreover, there exist hilbert spaces M_m, N_n, and $P_{m,n}$, and W^* algebras B_m, C_m and B'_n, C'_n, such that: M_m has dimension m, N_n has dimension n; B_m consists of all operators acting on M_m, B'_n consists of all operators acting on N_n, C_m and C'_n are strongly semi-simple algebras acting on $P_{m,n}$ with C'_n the commutant of C_m; $H_{m,n}$ is the tensor product of M_m, N_n and $P_{m,n}$, A_m is the tensor product of B_m and C_m, and A' is the tensor product of B'_n and C'_n. The $H_{m,n}$, A_m, A'_n, M_m, N_n, B_m and B'_n are unique (up to unitary equivalence), and the possible choices for $P_{m,n}$, C_m and C'_n may be described in terms of a suitable double dimension function. These results summarize the multiplicity theory for representations of general W^* algebras. (Received July 18, 1955.)

740. R. T. Prosser: *On the structure of W^* algebras.*

Let A be any W^* algebra. Then for each cardinal n, zero or infinite, there exists a W^* algebra A_n such that: A_0 is finite, A_n is either zero or cardinally homogeneous of degree n, for $n>0$, and A is the direct sum of the A_n. Moreover, there exist W^* algebras B_n and C_n such that: B_n is the full $n\times n$ matrix algebra, C_n is strongly semi-simple, and A_n is the W^* tensor product of B_n and C_n. The A_n and B_n are unique (up to equivalence), and the possible candidates for C_n may be described in terms of a suitable dimension function. In particular, if A is of type I, each C_n may be chosen abelian; if A is of type II, each C_n may be chosen finite; if A is of type III, each C_n is unique. These results summarize the known structure theorems for general W^* algebras. (Received July 18, 1955.)

741t. M. O. Reade: *A class of univalent functions.* Preliminary report.

In a recent paper, W. Kaplan introduced the class of near-to-convex functions [Michigan Mathematical Journal vol. 1 (1952) pp. 169–185]. In this article, the present author uses Kaplan's new class to prove the following result. Let $f(z)=z+\sum_2^\infty a_kz^k$ be analytic and let $f'(z)\neq 0$ for $|z|<1$. If $\int_{\theta_1}^{\theta_2}\mathrm{Re}\,[1+re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta})]\,d\theta>-\pi/2$ for all $\theta_1<\theta_2$ and for all $0\leqq r<1$, then $f(z)$ is univalent, and $|a_n|\leqq(n+1)/2$ for $n=1, 2, 3, \cdots$. This includes results due to Robertson [Amer. J. Math. vol. 58 (1936) pp. 465–472] and Renyi [Publ. Mathem. Debrecen. vol. 1 (1949) pp. 18–23]. (Received May 11, 1955.)

742. W. T. Reid: *A comparison theorem for self-adjoint differential equations of second order.*

For $L_\alpha[y]\equiv(R_\alpha(t)y')'+P_\alpha(t)y=0$ $(\alpha=1, 2)$, with $P_\alpha(t)$ real-valued continuous and $R_\alpha(t)$ positive functions of class C' on an interval J, and $T_\alpha^+(t_0)\{T_\alpha^-(t_0)\}$ the least $\{$greatest$\}$ value of t succeeding $\{$preceding$\}$ t_0 and defining a focal point of $t=t_0$

relative to $L_2(y)=0$ on J, the following comparison theorem is established: if $(R_2/R_1)' \geqq 0$, $P_1 \geqq 0$ and $P_2/R_2 \leqq P_1/R_1$ on J, then the existence of $T_2^+(t_0)$ implies that $T_1^+(t_0)$ exists and $T_1^+(t_0) \leqq T_2^+(t_0)$; similarly, if $(R_2/R_1)' \geqq 0$, $P_2 \geqq 0$, $P_1/R_1 \leqq P_2/R_2$ then the existence of $T_1^-(t_0)$ implies that $T_2^-(t_0)$ exists and $T_1^-(t_0) \leqq T_2^+(t_0)$. Various consequences of this result are given. In particular, it is shown that this result implies a comparison theorem on conjugate points for equations $(r(x)u'(x))'+r(x)u(x)=0$, $x\in I_x$, and $(s(y)v'(y))'+s(y)v(y)=0$, $y\in I_y$, that has been obtained by H. F. Weinberger using the maximum property of Cauchy's problem for hyperbolic partial differential equations. In addition for a differential equation $(R(t)y')'+P(t)y=0$, with $R(t)$, $P(t)$ positive functions of class C' on $[a, \infty)$, there are obtained certain criteria for oscillation and nonoscillation. (Received July 18, 1955.)

743*t*. I. F. Ritter: *Computation of quadratic factors of a polynomial.*

The first part of this article describes a new algorithm for the approximate computation of the coefficients a, b in a quadratic factor $(s^2-as-b)^m$, of multiplicity m, of a polynomial. It is shown that the procedure is equivalent to the Bernoulli-Euler method, but avoids the explicit computation of the recurrent sequence of that method. This has the advantage that the difficulty of scaling the numbers in the recurrent sequence does not arise. The process also explains the convergence behavior of the iteration of S-N. Lin by bringing out that the Lin process anticipates the monotonic convergence of the Bernoulli-Euler method. In the second part the Lin process is generalized to quadratic factors of any multiplicity m and supplied with practical tests for finding m and for convergence. For dealing with cases in which the Lin process diverges, the Newton iteration given by Bairstow is extended to any m. Although this process is generally applicable, a similar and equally practical higher order iteration is finally described that converges faster than the Bairstow method whenever the Lin iteration would converge. (Received July 18, 1955.)

744*t*. D. W. Robinson: *A note on the definition of a matric function.*

Let T be a finite square matrix over the field of complex numbers, with distinct eigenvalues $\lambda^1, \cdots, \lambda^k$, and corresponding projections (Frobenius covariants) $E^1, \cdots, E^k$. Let $F[T]$ be the class of all complex valued functions of a complex variable that are analytic on an open set N containing the eigenvalues of T. A definition (proposed by E. Cartan in a letter to G. Giorgi) of a matrix value corresponding to T under $f\in F[T]$ is (1) $(2\pi i)^{-1}\int_C (sI-T)^{-1}f(s)ds$, where $C=\sum_{i=1}^k C^i$, C^i being a simple closed curve in the neighborhood $N^i\subset N$ of λ^i and enclosing λ^i. The following theorem is proved: For every T and every $f\in F[T]$ let there correspond a matrix $f(T)$. Then $f(T)$ is given by (1) above if and only if this correspondence satisfies either of the following sets of conditions: (1.1) The combinatorial requirements of Fantappiè, and, (1.2) If $f, f_n\in F[T]$, $\{f_n\}$ analytic in and converging to f on N^i, uniformly on every compact subset of N^i, then $E^if_n(T)$ converges and $E^if(T)=\lim E^if_n(T)$; or, (2.1) If each eigenvalue of T is of index one then $f(T)=\sum_{i=1}^k E^if(\lambda^i)$, and, (2.2) If $\{T_n\}$ is a sequence of matrices such that $f\in F[T_n]$ for all n and $\lim T_n=T$, then $\{f_n(T)\}$ converges and $f(T)=\lim f(T_n)$. (Received July 15, 1955.)

745*t*. L. B. Robinson: *On tensors and covariants of differential systems.*

The author has computed a complete system of tensors associated with differential systems. They are given by the formula $I_{ij}=(1/M)\sum_{k=1}^{s}\sum_{l=1}^{s} M^{kl}_{ij}\Psi_{kl}$ where the Ψ are arbitrary functions of the covariants. Now reduce the covariants to canonical form. If the Ψ are rational functions it is then easily seen that they are rational functions of the *fundamental* covariants. If the Ψ are polynomials they are polynomial functions of the *fundamental* covariants. So the system of tensors is functionally, rationally, and algebraically complete. This is what Gordan proved for the covariants of binary forms. (Received May 6, 1955.)

746. F. Virginia Rohde: *A note on partial fractions.*

In cases where the study of partial fractions is left until the calculus it would seem worth while to use methods of the calulus in determining the coefficients. For all cases of linear factors in the denominator these coefficients can be expressed in terms of the derivatives of the product of the function and the highest power of the given factor evaluated at the zero of the factor in much the same manner as is done in the determination of residues in the theory of complex variables. (Received July 18, 1955.)

747t. L. A. Rubel: *Sets of positive integers of upper density one.*

This paper is concerned with several of the most useful upper density functions. The main result is that the class of sets of positive integers of upper density one is the same for each of these upper density functions. For sets A of positive integers, the author defines (1) the upper asymptotic density, $\overline{D}(A) = \lim\sup_{(t\to\infty)} A(t)/t$ where $A(t)$ denotes the number of integers n in A such that $n \leq t$; (2) the upper mean density, $\overline{D}_M(A) = \lim\sup_{(t\to\infty)} t^{-1}\int_0^t A(x)/x\,dx$; (3) the upper Abel density, $\overline{D}_a(A) = \lim\sup_{(t\uparrow 1)}(1-t)\sum_{n\in A} t^n$; (4) the upper Poisson density, $\overline{D}_P(A) = 2/\pi \lim\sup_{(t\to\infty)}\int_0^\infty (A(y)/y)t(t^2+y^2)^{-1}dy$; (5) the modified upper logarithmic density, $\overline{D}_L(A) = \inf_{(\lambda>1)} \lim\sup_{(x\to\infty)} \{(\log\lambda)^{-1}\phi(x,\lambda)\}$ where $\phi(x,\lambda) = \sum 1/n$, the sum being taken over all n in A for which $x \leq n \leq \lambda x$. The main theorem states: *If, for a given set A of positive integers, any one of the numbers* $\overline{D}(A), \overline{D}_M(A), \overline{D}_a(A), \overline{D}_P(A), \overline{D}_L(A)$ *is equal to* 1, *then all of these numbers are equal to* 1. The proof depends principally upon the author's optimal extension of Carlson's theorem on entire functions (Bull. Amer. Math. Soc. Abstract 61-3-417). This extension states that for each set A of positive integers, each entire function $f(z)$ satisfying the conditions (i) $f(n) = 0$ for each n in A, (ii) $f(z) = O(1) \exp \tau|z|$ for some $\tau < \infty$, (iii) $f(iy) = O(1) \exp c|y|$ for some $c < \pi$ must vanish identically if and only if $\overline{D}(A) = 1$. In view of the present paper, we may replace the condition $\overline{D}(A) = 1$ by any one of the conditions $\overline{D}_L(A) = 1, \overline{D}_P(A) = 1, \overline{D}_a(A) = 1, \overline{D}_M(A) = 1$. (Received June 27, 1955.)

748t. Walter Rudin: *An algebraic characterization of conformal equivalence.*

Let $A(R)$ be the algebra whose members are the single-valued regular analytic functions on the open Riemann surface R. Theorem: If $A(R_1)$ and $A(R_2)$ are algebraically isomorphic, then R_1 and R_2 are conformally equivalent (the converse is obvious). The proof depends on the fact that for every $p \in R$ there is an $f \in A(R)$ which has a simple zero at p and vanishes nowhere else (see Florack, Schr. Math. Inst. Münster, no. 1, 1948). It follows that every principal maximal ideal of $A(R)$ consists of the functions vanishing at some point of R. If T is an isomorphism of $A(R_2)$ onto $A(R_1)$, there is therefore a one-to-one mapping t of R_1 onto R_2 such that $(Tf)(p) = f(t(p))$ for all $p \in R_1$ and all $f \in A(R_2)$; this implies that t is conformal. (Received July 25, 1955.)

749. Walter Rudin: *The automorphisms and endomorphisms of the group algebra of the unit circle.*

Theorem I. For any mapping t of the set J of all integers into itself, the statements

(A) and (B) are equivalent: (A) If $\sum_{-\infty}^{\infty} c(n)e^{in\theta}$ is the Fourier series of a function $f \in L_1$, then $\sum_{-\infty}^{\infty} c(t(n))e^{in\theta}$ is the Fourier series of a function $Tf \in L_1$. (B) There exists a sequence $\{s(n)\}$ and a positive integer p such that (i) $t(n) = s(n)$ except possibly for a finite set of values of n; (ii) $s(n+p)+s(n-p) = 2s(n)$ for every n; (iii) $s(n+p) \neq s(n)$ for every n. The fact that (B) implies (A) is a matter of simple computation; the proof of the converse depends on the observation that, if (A) holds, then the sequences $\{e^{ixt(n)}\}$ are, for each real x, sequences of Fourier-Stieltjes coefficients of measures μ_x such that $\mu_x * \mu_y = \mu_{x+y}$. Theorem II. If t is a one-to-one mapping of J onto itself which satisfies (B), then T is an automorphism of the convolution algebra L_1 on the unit circle. Every automorphism of L_1 is obtained in this manner. Theorem III. If t satisfies (B) and if N is a set of integers which differs from a periodic set by only a finite number of elements, then the mapping which assigns to the series $\sum_{-\infty}^{\infty} c(n)e^{in\theta}$ the series $\sum_{n \in N} c(t(n))e^{in\theta}$ defines an endomorphism of L_1. Every endomorphism of L_1 is obtained in this manner. (Received June 10, 1955.)

750. E. C. Schlesinger: *Conformal invariants and prime ends.*

A sequence of points $\{a_n\}$ of points of a simply connected plane region G "converges to a prime end of G" if $\lambda(\Gamma) = 0$. Here Γ is the family of cross-chains (= union of a finite number of connected crosscuts of G) which separate (in G) all but a finite number of the a_n from E; E is a closed subset of G which contains at most a finite number of the points a_n; finally, $\lambda(\Gamma)$ is the extremal length (L. Ahlfors and A. Beurling, Acta Math. vol. 83 (1950) pp. 101–129) of the family Γ. Two sequences of points of G "converge to the same prime end of G," if the mixed sequence formed from them converges to a prime end of G. These formulations do not depend on the set E. The "prime ends of G" are the equivalence classes (of sequences of points) obtained in this way. *Theorem*: The above definition is equivalent to that of Carathéodory (Math. Ann. vol. 73 (1913) pp. 323–370). *Theorem*: If $\{a_n\}$ is a sequence of points of G converging to the prime end π of G then all limit points of the sequence are "principal" on π (Carathéodory, op. cit. pp. 353–354), provided the sequence $\{\lambda_n+\lambda_n^{-1}\}$ remains bounded, where λ_n is the extremal length of a certain family of crosscuts determined by the region G, the prime end π, and the point a_n. The second of these theorems is a conformally invariant and intrinsic form of a result due to Lindelöf (Acta Soc. Sci. Fenn. vol. 46 (1915) No. 4, p. 28) on "angular" approach. (Received July 8, 1955.)

751*t*. I. E. Segal: *Note on induced representations.*

The need for quasi-invariant measures and certain measurability considerations in the development of the notion of induced representation of a locally compact group are eliminated in a new approach, which also makes possible the extension of the basic theory to not necessarily separable groups. The method uses abelian operator rings. (Received July 19, 1955.)

752*t*. George Seifert: *A note on a result concerning limit cycles for an autonomous system.*

Consider the system $\dot{x} = y$, $\dot{y} = -g(x) - f(x, y)y$, where g and f are continuous and satisfy Lipschitz conditions in every bounded domain in the (x, y) plane, $xg(x) > 0$ for $x \neq 0$, and $\int_0^{\pm\infty} g(x)dx = +\infty$. It has been shown by A. DeCastro (Boll. Un. Mat. Ital. (3) vol. 9 (1954) pp. 280–281) that if this system has a periodic solution whose trajectory, a cycle, will be denoted by C, then the system $\dot{x} = y$, $\dot{y} = -g(x) - f_1(x, y)y$ will have a periodic solution whose cycle C_1 is interior to C provided (1) $f_1(0, 0) < 0$ and

(ii) $f(x, y) < f_1(x, y)$ for all x and y. It is observed here that this same result may be obtained without the hypothesis $\int_0^{\pm\infty} g(x)dx = +\infty$, and under a condition weaker than (ii); namely, $f(x, y) \leq f_1(x, y)$ on C, the equality holding there for at most a finite number of points. (This research was supported by the United States Air Force through the Office of Scientific Research of the Air Research and Development Command). (Received July 19, 1955.)

753*t*. V. L. Shapiro: *Localisation of Laplace series.*

Let $T = \sum_{n=1}^{\infty} Y_n(x)$ be a series of surface spherical harmonics where $Y_n(x) = O(n^w)$, $w \geq 0$, uniformly for x on S, the surface of the unit sphere in three-dimensional Euclidean space. To T can then be associated a function $F(x) = \sum_{n=1}^{\infty} (-1)^k \cdot Y_n(x)/[n(n+1)]^k$ where k is the smallest integer greater than $(w+1)/2$. It is then shown in this paper that if $F(x)$ is of class $C^{(2k)}$ on a domain D of S, then T is Abel summable to $\Delta^k F(x)$ uniformly in every closed domain contained in the interior of D, where $\Delta^k = \Delta(\Delta^{k-1})$ and Δ is the Laplacian on the surface of the sphere. Another localization theorem of a slightly different nature also obtained in this paper is the following: Let $F(x)$ be a function integrable on S and let $T[F]$ designate the Laplace series of F. Let $\Delta T[F]$ be the series of surface spherical harmonics obtained by formally applying the Laplacian operator to T. Suppose $F(x)$ has a generalized Laplacian at the point x_0 equal to p. Then $\Delta T[F]$ is Abel summable to p at the point x_0. This latter result can also be stated in the following manner: Given a series of surface spherical harmonics $T = \sum_{n=1}^{\infty} Y_n(x)$ such that $\sum_{n=1}^{\infty} Y_n(x)/n(n+1)$ is the Laplace series of an integrable function. Then the Riemann summability of T implies the Abel summability of T. (Received May 17, 1955.)

754*t*. Paul Slepian: *Theory of Lebesgue area of continuous maps of 2-manifolds into n-space.*

Let X be a finitely triangulable topological space and let f be a continuous function on X into Euclidean n-space, E_n, with $n \geq 2$. For any integers i and j with $1 \leq i < j \leq n$, let $P_n^{i,j}$ be the projection of E_n onto E_2 defined by $P_n^{i,j}(x) = (x_i, x_j) \in E_2$ for each $x = (x_1, x_2, \cdots, x_n) \in E_n$. Let L_2 be 2-dimensional Lebesgue area. Federer has shown (Ann. of Math. vol. 61 (1955) p. 289) that if X is a subset of the plane, then $L_2(f) \leq \sum_{1 \leq i < j \leq n} L_2(P_n^{i,j} \circ f)$. The principal result of this paper states that the above inequality remains true if X is a compact 2-manifold (with or without boundary). This inequality is of fundamental importance for the extension of the whole theory of Lebesgue area to continuous maps of 2-manifolds into n-space. (Received June 3, 1955.)

755*t*. D. B. Sumner: *A generalization of the averaging operator.*

The averaging operator $\nabla f(s) = [f(s+h) + f(s)]/2$ has been studied by Norlund and certain aspects of its integer powers by Milne-Thomson. The operator ∇^λ is defined for any real λ, and is shown to be applicable to a wide class of functions. Generalized Euler polynomials associated with the operator are defined, and some of their more important properties established; and the inversion theorem for the functional equation $\nabla f(s) = \phi(s)$ is proved. (Received July 5, 1955.)

756*t*. D. B. Sumner: *Generalized powers of the difference operator.* Preliminary report.

A generalized power Δ^λ of the difference operator $\Delta f(s)$ is defined and its more important properties established, including those of the generalized Bernoulli polynomials associated with it. An inversion theorem for the functional equation $\Delta f(z) = \phi(z)$ is proved; and eigenvalues and eigenfunctions of the operator are discussed. (Received July 5, 1955.)

757. W. C. Swift: *An iterated matrix method of summability.*

Given a matrix $A = (a_{nk})$, a sequence $\{x_i\}$ is said to be summable-A to x if one has $x = \lim_k \lim_j \sum_{i=1}^{j} a_{ki}x_i$. If, in this last expression, one or both of the limiting operations are themselves replaced by matrix summability methods, then the resulting transform is referred to as an iterated matrix method of summability. In general, the set of iterated matrix operators is generated recursively by replacing the natural limiting operations as they appear in conventional matrix summability by previously determined iterated matrix operators. If, from a fixed regular matrix A, it is possible to generate in this fashion an operator which sums a given sequence, then the value obtained is uniquely determined; and the sequence is said to be ultimately summable-A to that limit. For the usual method of summability, Okada (Math. Zeit. vol. 23, pp. 62–71) has proved a theorem concerning the effectiveness for summing a Taylor's series to its analytic continuation. A generalization of this theorem for the iterated methods is found to be valid. As a result, it can be shown that for a very simply derived class of regular matrices, the sequence of partial sums of a Taylor's series is ultimately summable to its analytic continuation throughout the principal star domain of the function. (Received July 18, 1955.)

758. T. T. Tanimoto: *Minimal paths through n points.*

Given n points $P_1, P_2, \cdots, P_n$ in Euclidean space E_r of dimension r, the following method is applied to determine minimal paths passing through each of the n points: Let A be the matrix consisting of elements $a_{ij} = a_{ji} = d(P_i, P_j) > 0$ for $i \neq j$, and $a_{ii} = 0$, where $d(P_i, P_j)$ is the undirected distance between P_i and P_j; and let S be the associated doubly stochastic matrix consisting of elements $s_{ij} = s_{ji} = 1/(n-1)$, $i \neq j$, $s_{ii} = 0$. If X is the set of row vectors and Y the column vectors of S, it is shown that A considered as a two-person zero sum game has the solution X_{i_0}, Y_{j_0}, $(2/(n-1)^2) \cdot [T - W_{i_0 j_0}]$ where X_{i_0} is the vector with $1/(n-1)$ as every component except for a zero in the i_0th component, and similarly for Y_{j_0}, $T = (1/2)\sum_{ij} a_{ij}$, and $W_{ij} = (1/2) \cdot (\sum_\alpha a_{\alpha i} + \sum_\beta a_{j\beta} - a_{ij})$. The first moves of the game and the subgames determined by deleting the points of order two are played and the weight of a segment (P_i, P_j) viz., W_{ij} is used to determine optimal segments. The restrictions of the linear graphs desired are imposed and minimal paths are determined. (Received July 18, 1955.)

759. W. J. Thron: *Analytic solutions of the functional equation $f(f(z)) = g(z)$.*

Using a theorem of Pólya (J. London Math. Soc. vol. 1 (1926) pp. 12–15) and Hadamard's factorization theorem it is proved that the functional equation $f(f(z)) = g(z)$, where $g(z)$ is an entire function of finite order, which is not a polynomial, and which takes on a certain value p only a finite number of times, does not have a solution which is an entire function. (Received July 15, 1955.)

760t. W. R. Utz: *A third order differential equation.*

The following theorem is proved. Let $f(x)$ be a real differentiable function such that $\lim_{x\to\infty} f(x) = c$, $0 < c < \infty$, and such that $f'(x)$ is bounded as $x \to \infty$. Let $p(x)$ be a real polynomial and let $a > 0$ be a real number such that $p(a) = 0$, $p'(a) > 0$. If $y = y(x)$

is a solution of $f(x)y'''-yy''+p(y')=0$ valid for large values of x and for which $\lim_{x\to\infty} y'(x)=a$, then $y=ax+b$, where b is a constant. This theorem generalizes a theorem of G. H. Hardy (Proc. Cambridge Philos. Soc. vol. 35 (1939) pp. 652, 653) in which $f(x)=1$, $p(y')=2(y'^2-1)$, $a=1$. Hardy's theorem was suggested by a paper of S. Goldstein (Proc. Cambridge Philos. Soc. vol. 35 (1939) pp. 338–340) who uses the equation in his study of the boundary layer for a steady two-dimensional flow of a fluid of constant density and constant kinematic viscosity. (Received July 19, 1955.)

761*t*. S. E. Warschawski: *Distortion in conformal mapping of variable domains*. I. *Estimates for the L_p mean.*

Let C_i ($i=1, 2$) be closed Jordan curves possessing continuous tangents, let $\beta_i(\sigma)$ be moduli of continuity of the tangent angles, $\tau_i(s)$, as functions of the arclength s. Suppose: (i) C_i are contained in the ring $0<d\leq|w|\leq D$; (ii) if Δs is the (shorter) arc of C_i between w' and w'' then $(\Delta s/|w'-w''|)\leq c$; (iii) for some fixed $\epsilon>0$, C_2 is in the ϵ-neighborhood of C_1, i.e. every point of C_2 is contained in a circle of radius ϵ about some point of C_1; (iv) let $\tau_i^*(w)$ denote the tangent angle at w on C_i; there exists an η such that, for any $w_2\in C_2$, there is a point $w_1\in C_1$, with $|w_2-w_1|\leq\epsilon$, such that $|\tau_1^*(w_1)-\tau_2^*(w_2)|\leq\eta$, $0<\eta\leq\pi/2-\alpha<\pi/2$. Suppose $w=f_i(z)$ maps $|z|<1$, conformally onto the interior of C_i such that $f_i(0)=0$, $f_i'(0)>0$. Then it is proved: *Under the assumptions* (i), (ii), (iii), (iv) *there exist, for every* $p>1$, (1) *an* M_p *such that* $\mathfrak{M}_p\{f_1-f_2\}=\{(1/2\pi)\int_0^{2\pi}|f_1(e^{i\theta})-f_2(e^{i\theta})|^p d\theta\}^{1/p}\leq M_p\epsilon$; (2) *a* K_p *such that* $\mathfrak{M}_p\{f_1'-f_2'\}\leq K_p[\eta+\beta_1^*(2cM_p\epsilon)]$, *where* $\beta_1^*(\sigma)$ *is a convex majorant of* $\beta_1(\sigma)$. The constants M_p, K_p depend on p, d, D, c, α and the β_i only. This contains an earlier result for $\beta_1(\sigma)=H\sigma^\gamma$, $0<\gamma\leq 1$, $\eta=q\epsilon^\gamma$ (Archiv fur Mathematik vol. 4 (1955) pp. 102–114). (Received July 18, 1955.)

762*t*. S. E. Warschawski: *Distortion in conformal mapping of variable domains*. II. *Uniform estimates.*

Using the notations of part I (preceding abstract) define: $\omega(\delta)=\mathrm{Max}_{i=1,2}\int_0^\delta\sigma^{-1}\cdot\beta_i(\sigma)d\sigma$, $\mu=\mathrm{Sup}_{|z|\leq 1}|f_1(z)-f_2(z)|$, $\mu'=\mathrm{Sup}_{|z|\leq 1}|f_1'(z)-f_2'(z)|$. It is proved: (1) If $\omega(\delta)<\infty$ and (i), (ii), (iii) of part I are satisfied, then $\mu\leq M\epsilon\log(1/\epsilon)$. (2) If $\omega(\delta)<\infty$ and (i)–(iv) hold, then $\mu'\leq N\{\omega(\lambda\delta)+[\eta+\beta_1(2c\mu)]\log(\pi/\delta)\}$ for every δ, $0<\delta<\pi/e$. The constants M, N, λ depend on d, D, c, and the β_i *only*. In particular, for $\beta_1=H_1\sigma$, $\beta_2=H_2\sigma^\gamma$, $0<\gamma\leq 1$, $\eta=q\epsilon$, $\delta=\epsilon^{1/\gamma}$ one has $\mu'\leq N_1\epsilon\log(1/\epsilon)$, where $N_1=N_1(d, D, c, H_1, H_2, \gamma, q)$. (3) Suppose that C_i are represented by $w=w_i(t)$, $0\leq t\leq a$, such that $0<b_1\leq|w_i'(t)|\leq b_2$, $|w_i''(t)|\leq k$ (b_1, b_2, k are constants) and that, for some $\epsilon>0$ and some $p>1$, $\mathfrak{M}_p\{w_1''(t)-w_2''(t)\}\leq\epsilon$ and $|w_1(0)-w_2(0)|\leq\epsilon$. Suppose, in addition, (i) and (ii) hold. If $\phi_i(w)$ denotes the inverse function of $f_i(z)$, then $|\phi_1'(w_1(t))-\phi_2'(w_2(t))|\leq\Lambda\epsilon$, $0\leq t\leq a$, where $\Lambda=\Lambda(d, D, c, b_1, b_2, k, p)$. This provides an estimate for the variation of the normal derivative of Green's function of first kind on the boundary under quite general deformations of the domain. (Received July 15, 1955.)

763*t*. A. D. Wasel: *An initial value problem for mixed compressible fluid flows.*

In order to study the stream function ψ for compressible fluid flows, Bergman (Proc. Nat. Acad. Sci. U.S.A. vol. 29 (1943) p. 276) and Bers and Gelbart (Quarterly of Applied Mathematics vol. 1 (1943) p. 168) introduced solutions ψ_{nj}, $j=1, 2$; $n=1, 2, \cdots$, of $m(\psi)=\psi_{HH}+l\psi_{\theta\theta}=0$, generated by an integral operator. Here $l=(1-M^2)/\rho^2$ (M the Mach number, ρ the density), H is a known function of the speed, and θ is the velocity direction. By modifying these solutions, the author con-

structs solutions $\psi(H, \theta) = \{ \sum_{n=1}^{\infty} n^{-1}a_{n-1}^{(2)}[\binom{n}{1}\theta^{n-1}\int_{H_0}^{H} dH_1 - 3!\binom{n}{3}\theta^{n-3}\int_{H_0}^{H} dH_3 \int_{H_0}^{H_3} dH_2 \cdot \int_{H_0}^{H_2} dH_1 + \cdots] + \sum_{n=0}^{\infty} a_n^{(1)}[\theta^n - 2!\binom{n}{2}\theta^{n-2}\int_{H_0}^{H} dH_2 \int_{H_0}^{H_2} dH_1 + \cdots]\}$ satisfying $\psi(H_0, \theta) = \chi_1(\theta)$ and $\psi_H(H_0, \theta) = \chi_2(\theta)$ where $\chi_k(\theta)$, $k = 1, 2$, are arbitrary analytic functions of θ possessing expansions $\sum_{n=0}^{\infty} a_n^{(k)}\theta^n$, $k = 1, 2$, respectively. (Received May 27, 1955.)

764*t*. A. D. Wasel: *An initial value problem for subsonic compressible fluid flows.*

The author considers $s(\psi) = \psi_{z'\bar{z}'} + N(\psi_{z'} + \psi_{\bar{z}'}) = 0$ (a form of $m(\psi) = 0$, see preceding abstract, valid for subsonic flows) with $z' = \lambda' + i\theta'$, $\bar{z}' = \lambda' - i\theta'$, $\lambda' = \lambda - \lambda_0$, λ a conveniently chosen function of M, and $\lambda_0 < 0$. He defines ψ to have a pole-like singularity at $z' = \alpha$ with respect to the Bergman operator (Amer. J. Math. vol. 74 (1952) p. 444) generating solutions of $s(\psi) = 0$, if the associate function $g(z') = \sum_{n=0}^{\infty} \mu_n z'^n$ of that operator has a simple pole at $z' = \alpha$. He sets $\Psi^*(0, \theta') = \sum_{n=0}^{\infty} \beta_n \theta'^n$ and $\Psi^*_{\lambda'}(0, \theta') = \sum_{n=0}^{\infty} \gamma_n \theta'^n$, where $\Psi(\lambda', \theta') = \psi(z', \bar{z}')$ and $\Psi^* = H^{-1}\Psi$, H being a known function of λ', and using Theorem 5.2 of the above reference he expresses μ_n in terms of β_n, γ_n, and certain constants depending on the coefficient N of $s(\psi) = 0$. Finally he shows that if α lies in the domain of validity of the operator and the coefficients β_n and γ_n are chosen in such a way that $|(\mu_{n+1}/\mu_n) - (1/\alpha)| < k^n$ for all n sufficiently large and $k < 1$, then ψ has a pole-like singularity at $z' = \alpha$. (Received May 27, 1955.)

765. R. E. Zindler: *End points of trajectories.*

Given the system of equations $dx_i/dt = X_i(x_1, \cdots, x_n)$ where the X_i are real continuous functions of the x_i (real) in an open region R except possibly at a closed set of singular points therein. A singular point is a point at which (a) all X_i vanish, (b) one or more of them becomes infinite or, (c) the limit of any one of them is not independent of the direction of approach. It is shown that all trajectories (extensions of local solutions through nonsingular points by repeated application of the existence theorem) must end in singular points if they have a finite end point. (Received July 13, 1955.)

APPLIED MATHEMATICS

766. R. J. Arms: *Numerical calculation of the eigenvalues of matrix.*

With A a square matrix and $x^{(0)}$ a fixed vector, we consider the sequence $\{x^{(n)}\}$ where $x^{(n)} = A^n x^{(0)}$. If $x_i^{(n)}$ is the ith component of $x^{(n)}$ and $f_i(z) = \sum_{n=0}^{\infty} x_i^{(n)} z^n$, it is noted that $f_i(z)$ is a rational function whose denominator has the reciprocals of the eigenvalues of A as roots. Hence, by theorems of Hadamard, Kronecker, Konig and others, there are algorithms for using the $x_i^{(n)}$'s to compute the poles of $f_i(z)$. (Received July 7, 1955.)

767. R. C. F. Bartels (p) and A. C. Downing, Jr.: *A note on the stability and convergence criteria for hyperbolic systems of equations.*

The paper follows a method by Keller and Lax to establish criteria for the stability and convergence of the numerical solution of an initial value problem for hyperbolic systems. If the classical condition of Courant, Friedrichs and Lewy is satisfied, then the solution of the difference equations converges to the solution of the differential equations as $\overline{\Delta x}^2/\Delta t$ tends to zero. This result is obtained from an inequality giving a bound for a norm of the error in the solution in terms of bounds on the second derivatives of the solution. From this inequality, it is also possible to say something about an optimum choice of grid size. (Received July 18, 1955.)

768. J. S. Bendat: *Optimum time-variable filters for finite operating periods.*

This paper presents a general solution to an integral equation characterizing optimum time-variable linear filters which operate from a finite starting point in time. Input signals are permitted to be arbitrary nonstationary random processes expressible in Fourier series ensemble form. The particular noise perturbation treated is represented by a nonstationary damped exponential-cosine autocorrelation function, which is of major importance in such applied fields as meteorology, radar and automatic control. A new technique is established for solving integral equations, involving Laplace Transform theory and sets of simultaneous time-varying equations. An asymptotic system design method is outlined for making concrete applications of the mathematical solutions. This practical procedure is illustrated with details of actual engineering systems. (Received July 5, 1955.)

769. Garrett Birkhoff and J. B. Diaz (p): *Non-linear networks.*

The main concern is with "connected networks," i.e. finite *connected* graphs consisting of a finite set of *nodes* (vertices), certain nodes being joined in pairs by a finite set of oriented *links* (branches), with the *boundary* (a certain subset of the set of nodes) supposed specified. In network flow problems, Kirchhoff's node law is supposed to hold at each interior node, while certain "boundary conditions" (of Dirichlet, Neumann or of mixed type) hold at boundary nodes. For physical equilibrium, the currents satisfy certain equilibrium relations: $i(a_\gamma)=c_\gamma(\Delta u_\gamma)$ on each link a_γ, where Δu_γ is the potential difference across the directed link a_γ, and the "conductivity" functions c_γ are strictly increasing continuous functions of Δu. Uniqueness and existence theorems are proved for the various boundary value problems mentioned above. Under certain additional hypotheses regarding the conductivity functions, the existence of the solution of the mixed problem is shown by means of a constructive process of "exact point by point relaxation." (Received July 18, 1955.)

770. J. W. Carr, III: *Row-operation numerical inversion of high-order matrices.* Preliminary report.

An analysis is made of a so-called "row-operation procedure" for the determination of the inverse of a (nonsingular) nonsymmetric real matrix. This method is a logical extension of the procedure for nonsymmetric matrices discussed by J. von Neumann and H. Goldstine, *Numerical inverting of matrices of high order*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1021–1099. This algorithm is the equivalent of decomposition of an arbitrary nonsingular nonsymmetric matrix A into the product $A=OT$, where T is an upper-triangular matrix and O an orthogonal matrix. The inverse A^{-1} equals $T^{-1}O^{-1}$, and since the transpose of an orthogonal matrix is also its inverse, is obtained directly by row-operations alone, combined with determination of the maximum norms of the rows being operated on at any of the iterative stages of the procedure. Rearrangement of the rows in order of maximum norm similar to the "pivotal element technique," combined with successive row orthogonalizations, provides an inversion procedure stable under "round-off." The "row-operation procedure" requires the same number of multiplications ($2n^3$, where n is the order of the matrix) as the von Neumann-Goldstine technique of pre-multiplication by the transpose, but is much easier to "code" for a high-speed electronic digital computer, because it deals only with additions of rows, multiplication of rows by constants, and row inner products. The final result obtained is the transpose of the inverse desired. (Received July 20, 1955.)

771. R. V. Churchill: *Generalized finite Fourier cosine transforms.*

Let k_n $(n=1, 2, \cdots)$ denote the positive roots of the equation $h \cos k_n = k_n \sin k_n$ where h is a constant other than zero. The integral transformation $T\{F\} = \int_0^1 F(x) \cdot \cos k_n x dx$ transforms $F''(x)$ into $-k_n^2 f(n) - F'(0) + [hF(1) + F'(1)] \cos k_n$, where $f(n) = T\{F\}$. Further operational properties of the transformation are found, some in terms of certain dual transformations. The generalized Fourier series corresponding to the orthogonal functions $\cos k_n x$ furnishes a formula for the inverse transformation. The following convolution property of the transformation is established. If $f(n)$ and $g(n)$ are the transforms of two bounded integrable functions $F(x)$ and $G(x)$ on the interval (0, 1), if $F_e(x)$ denotes the even periodic extension of $F(x)$ with period 2, and if the asterisk denotes the convolution operation for the Laplace transformation, then under the transformation T the image of the product $f(n)g(n)$ is the operation $2^{-1}\int_0^1 [F_e(x+t) + F_e(x-t)]G(t)dt - he^{-hx} * F(1-x) * G(1-x)$. Methods used here suggest procedures that may apply to other finite integral transformations whose kernels have no common period for all n. The results are applied to obtain a closed form of the solution of a boundary value problem. (Received July 18, 1955.)

772. C. L. Dolph and R. K. Ritt (p): *On the Schwinger variational principle.* Preliminary report.

The Schwinger variational principle for harmonic time dependent quantum mechanical scattering at arbitrary angles of reflection is formulated vigorously in a Hilbert space and its relation to more classical variational principles discussed. The completeness and validity of the approximation process is demonstrated in some special cases by use of the spectral operator theory of Dunford and Schwartz (Pacific Journal of Mathematics, September 1954). (Received July 19, 1955.)

773*t*. Jim Douglas, Jr.: *On the errors in analogue solutions of heat conduction problems.*

The approximate solution of heat conduction problems on analogue machines such as differential analyzers and resistance-capacitance circuits are most commonly obtained by replacing the space derivatives by finite differences, retaining the time derivative, and integrating the resulting system of ordinary differential equations on the machine. It has been conjectured by Hartree that the truncation error (as distinguished from various mechanical or electrical errors) is $O((\Delta x)^2)$, where Δx is the largest space increment. This conjecture is affirmed for the quasi-linear parabolic equation $\Delta u = F(x_1, \cdots, x_m, t, u)u_t + G(x_1, \cdots, x_m, t, u)$. (Received July 18, 1955.)

774. Jim Douglas, Jr.: *The relationship between stability and convergence in numerical solution of partial differential equations.*

It is the purpose of this paper to prove that stability in the sense of von Neumann is sufficient to insure convergence in the mean of the solution of a difference analogue of a linear parabolic or hyperbolic differential equation with time independent coefficients to that of the differential equation. No restriction on the number of independent variables nor on the order of the differential equation is involved. The difference equation may be either implicit or explicit in form and may involve the values of its solution on an arbitrary number of time levels. (Received July 18, 1955.)

775. Jim Douglas, Jr. and T. M. Gallie, Jr. (p): *Numerical integra-*

tion of a parabolic differential equation subject to a moving boundary condition.

Physical problems such as melting of ice, recrystallization of metals, evaporation of droplets and invasion processes of reservoir engineering are governed by the heat flow (or other parabolic) equation and a condition on a moving boundary. Locating the boundary and determining the solution of the equation is known as Stefan's problem. A simple form of Stefan's problem is (I): (a) $u_{xx}=u_t$, $0<x<x(t)$; (b) $u_x(0,t)=-1$, $t>0$; (c) $u(x(t),t)=0$, $t\geqq 0$; (d) $t=x(t)+\int_0^{x(t)}u(x,t)dx$, $t>0$; (e) $x(0)=0$. Existence and uniqueness of the solution of this nonlinear system has been proved recently by several authors, but evaluation of the solution is not yet in a definitive state. An analogous system of difference equations is (II): (a) $\Delta_x^2 u_{i,n+1}=(u_{i,n+1}-u_{in})/k_n$, $i=1,\cdots,n$; (b) $(u_{1,n+1}-u_{0,n+1})/k=-1$; (c) $u_{n+1,n+1}=0$; (d) $k_n=(n+1+\sum_{i=1}^{n}u_{i,n+1})k-\sum_{i=0}^{n-1}k_i$. In this paper a method for solving the nonlinear system (II) by iteration on the time step k_n is proved to converge. It is then proved that the solution and boundary of (II) converge to those of (I) as $k\to 0$. Stability with respect to small errors is proved. The method of proof depends on a maximum principle for parabolic difference equations which is of interest in itself. (Received July 18, 1955.)

776. A. C. Downing, Jr.: *A finite difference net applicable to multi-region problems*. Preliminary report.

In treating multi-region problems by finite difference methods, one of the most awkward questions in setting up a difference scheme which approximates the differential problem is the handling of interior corner points. With lattice systems customarily used, this problem becomes increasingly difficult as the number of space dimensions increases. A very simple grid scheme which overcomes the difficulties associated with corner points is presented in this paper. This grid has the property that all interfaces and boundaries are located midway between adjacent planes of the lattice of the grid points. As an application, in order to demonstrate the simplicity and the usefulness of this grid scheme, the multi-region diffusion problem in n dimensions is considered. In particular, it is shown that all the eigenvalues of the finite difference problem are real under conditions which correspond to the most general conditions, known to the author, which are sufficient to prove the realness of the eigenvalues of the corresponding differential problem. (Received July 18, 1955.)

777. G. E. Forsythe: *Asymptotic lower bounds for all eigenvalues of a convex membrane.*

Let R be a convex plane region bounded by a piecewise analytic curve C with strictly convex corners. Let the eigenvalues $0<\lambda^{(1)}<\lambda^{(2)}\leqq\lambda^{(3)}\leqq\cdots$ and eigenfunctions $u^{(k)}$ be such that $\Delta u^{(k)}=u_{xx}^{(k)}+u_{yy}^{(k)}=-\lambda^{(k)}u^{(k)}$ in R with $u^{(k)}=0$ on C. (Subscripts denote derivatives.) Let a self-adjoint finite difference approximation to the above problem be introduced as in G. E. Forsythe, Pacific Journal of Mathematics vol. 5 (1955). Let $0<\lambda_h^{(1)}<\lambda_h^{(2)}\leqq\lambda_h^{(3)}\leqq\cdots$ be its eigenvalues. Lemma: For any $k=1, 2, \cdots$ as $h\to 0$ one has $\lambda_h^{(k)}/\lambda^{(k)}\leqq 1-A_kh^2+o(h^2)$, where $A_k>0$. Estimates for A_k are given in terms of integrals of $u_{xx}^{(k)}u_{xx}^{(k)}+u_{yy}^{(k)}u_{yy}^{(k)}$ over R. Theorem: With the same hypotheses, for each $k=1, 2, \cdots$ there exists $h_0=h_0(k)$ such that $\lambda_h^{(k)}<\lambda^{(k)}$ for all $h\leqq h_0$. The lemma is proved by finding the stationary values of the Rayleigh quotient for the difference problem over the subspace spanned by the net functions induced by $u^{(1)},\cdots,u^{(k)}$. (Received July 13, 1955.)

778. H. M. Gurk: *Five person extreme games.*

An n-person game G is *extreme* provided there do not exist games G_1 and G_2 such that $v(S) = av_1(S) + (1-a)v_2(S)$ for all $S \subset (1, \cdots, n)$ where $0 < a < 1$ and where v, v_1, and v_2 are the characteristic functions of the three games. von Neumann and Morgenstern have pointed out that all n-person simple games are extreme (*Theory of games and economic behavior*, Princeton, 1947, Chap. 10), but the converse is true only for $n \leqq 4$. *Theorem* 1: G is an extreme, constant-sum 5-person game if and only if $v(S) \neq 0, 1$ for $S \subset (1, \cdots, 5)$ implies that there exists a chain of coalitions $T_1, \cdots, T_{2r}$ such that (i) $T_1 = T_{2r} = S$, (ii) $T_j \cdot T_{j+1} = \phi$, and (iii) $v(T_j) + v(T_{j+1}) = 1$ for $j = 1, \cdots, 2r-1$. From this theorem, there are 14 different types of games (excluding permutations of players), 8 of which are nonsimple. Counting permutations there are 294 5-person extreme constant-sum games altogether as compared with 8 4-person ones. For a 5-person game G, let $W^m = \{S \subset (1, \cdots, 5) \mid v(S) = 1 \text{ and } v(T) < 1 \text{ for } T \subsetneq S\}$, and let $W_I^m = \{S \in W^m \mid v(S-T) < 1 - v(T)$ for $T \neq \phi$, $T \subsetneq S$. *Theorem* 2: Let G be a 5-person constant sum extreme game. Let $V_S = \{\alpha \mid \alpha$ is an imputation such that $\sum_S \alpha_i = 1$ and $\sum_T \alpha_i \geqq v(T)$ for $T \subset S\}$ for $S \in W^m$. Then (1) if $S \in W_I^m$, V_S is a solution of G, and (2) if $W_I^m = \phi$, then $V = \bigcup_{S \in W^m} V_S$ is a solution of G. (Received June 17, 1955.)

779*t*. P. C. Hammer, O. J. Marlowe, A. H. Stroud: *Numerical integration over simplexes and cones.*

Hammer and Wymore have recently made a first step in the direction of a systematic exploration of numerical integration in n-dimensional spaces for $n \geqq 2$. They have laid down the general principles of using affine transformations and cartesian products to extend the applicability of every specific formula developed. Specific formulas and methods they have given for numerical integrations over certain types of symmetrical regions. In this paper we develop methods of numerical integration over simplexes and cones and give specific numerical formulas for triangles, tetrahedra, and solid circular cones. The specific theory permits determination of formulas for n-simplexes which are exact for polynomials of at most kth degree in n variables. (Received July 5, 1955.)

780. T. W. Hildebrandt: *Self-adjointness in one-group, multiregion diffusion problems.* Preliminary report.

Consider the differential equation $\nabla^2 u_i + D_i u_i = \lambda\theta_i u_i$ holding in each of the N connected subregions S_i of the connected region S; D_i and $\theta_i \geqq 0$ are constant in each S_i. On the interfaces Γ_{ij} between two adjacent regions, $u_i = \alpha_{ij}u_j + \beta_{ij}\partial u_j/\partial n$, $\partial u_i/\partial n = \gamma_{ij}u_j + \delta_{ij}\partial u_j/\partial n$; $\Delta_{ij} = (\alpha_{ij}\delta_{ij} - \beta_{ij}\gamma_{ij}) \neq 0$. On the outer boundaries Γ_{i0}, similar conditions hold, with $\Delta_{i0} = 0$. Then a sufficient condition that the problem be self-adjoint in the sense that all eigenvalues are real, and eigenfunctions corresponding to distinct eigenvalues are orthogonal, is that $\Delta_{ij} > 0$ and $\Delta_{ij}\Delta_{jk}\Delta_{ki} = 1$, for all $0 < i, j, k \leqq N$. (Received July 14, 1955.)

781*t*. J. R. Isbell: *Absolute games.*

A cooperative theory is proposed for general games with continuous but not linear transfer of utility. (See von Neumann and Morgenstern, *Theory of games and economic behavior*; and several papers of Nash. The former theory assumes linear transfer and the latter is incomplete. The present theory contradicts both.) The Neumann-Morgenstern utility axioms are extended to a categorical system in which the utility space of a person is a line segment. By normalizing to the segment [0, 1) one has interpersonally comparable absolute utility. (von Neumann and Morgenstern refuse to use interpersonal comparison; Nash postulates and exploits its nonexistence.) Solutions of general two-player games are obtained as in Raiffa, *Arbitration schemes for generalized two-person games*, Annals of Mathematics Studies, no. 28, using the unique arbitration scheme which preserves ratios of base utilities. Then the n-person game is treated in terms of its two-player quotient games (as in Neumann-Morgenstern. A *player* is a coalition of persons). A family of examples shows that a dummy player possessing capital reserves can affect the outcome of a game, and explores the variation of the effect with the utility exchange rate. (Received June 29, 1955.)

782. J. R. Isbell: *A linear value estimate for weapons systems.*

A pair of weapons systems, friendly and enemy, is here described by a pair of matrices. For m friendly weapon types and n enemy types there are an m by n matrix A and an n by m matrix B; the typical entry a_{ij} measures the firepower of one i-type friendly unit against j-type enemy units. A possible interpretation is that a_{ij} is the number of j-type enemies which one i-type friendly unit can expect to destroy in one hour under certain standard conditions. The entries b_{ij} of B similarly measure enemy firepower (or friendly vulnerability). The proposed value is to satisfy $\lambda x_i = \max a_{ij}y_j$, $\lambda y_j = \max b_{ji}x_i$, with some proportionality factor λ. Thus the values are to be proportional to the maximum possible destructiveness of the units in terms of these same values. Provided A and B are non-negative and neither has a vanishing row, there exist non-negative nonzero values. If all a_{ij} and b_{ji} are positive then so are all x_i and y_j; and λ is unique. Except for a nondense set of (non-negative) matrices, the values are unique up to a scalar multiple. The values fail to be essentially unique, for strictly positive matrices, if and only if there exists a set of values such that the tactic of maximizing destruction necessarily splits the battle into two unconnected parts. (Received July 14, 1955.)

783. Joachim Kaiser, Jr.: *The effect of bandpass limitations on analogue computer solutions.*

Simple systems of differential equations often solved by the use of electronic analogue computers can be described by expressions of the form $\ddot{y}+2\zeta\omega\dot{y}+\omega^2y=f(t)$. The bandpass characteristics of the operational amplifiers and integrators used to solve these systems normally raise the order of the differential equations giving rise to undesired exponential terms in the solution. The damping ratio ζ is affected by these exponential terms and a set of curves is derived showing damping ratio errors vs frequency ω. Computer users can consult these curves and obtain an estimate of the accuracies to be expected in the solutions of many problems, particularly in the missile flight simulation field. When electromechanical servos are used in the simulation of missile flight systems the errors introduced by the bandpass of the servos can seriously restrict the use of the computing equipment. Curves are derived showing the bandpass requirements of the computing servos as a function of frequencies and damping ratios encountered in the missile system for computing servos with simple second order transfer functions. (Received July 11, 1955.)

784. L. R. Langdon: *Approximation of functions for machine computation.* Preliminary report.

Polynomial and Padé function methods for the economical approximation of the arc tangent, arcsin, log, exponential, x^n (n not an integer) and the gas tables are discussed. Means of minimizing the error due to the truncation of series and the truncation of coefficients are outlined. (Received July 11, 1955.)

785. Imanuel Marx: *On the "principal axis transformation" for complex quadratic forms.*

To the quadratic form $Q=\sum_{j,k=1}^{n}\alpha_{jk}\xi_j\xi_k$, $\alpha_{jk}=\alpha_{kj}$ complex numbers, ξ_j complex variables, corresponds the complex symmetric transformation A with matrix (α_{jk}). An n-dimensional vector space S over the complex numbers, non-isotropic relative to a complex symmetric scalar product, is mapped into itself by A, and S is a direct sum of mutually orthogonal, indecomposable, non-isotropic subspaces invariant relative to A. Hence there exists a complex orthogonal matrix (τ_{jk}), $(\tau_{jk})(\tau_{lk})=(\delta_{jl})$, such that $(\tau_{jl})(\alpha_{jk})(\tau_{jk})^{-1}$ is a simple canonical form. Correspondingly, Q is transformed by $\eta_s=\sum_{k=1}^{n}\tau_{sk}\xi_k$, $\sum_{s=1}^{n}\eta_s^2=\sum_{k=1}^{n}\xi_k^2$, to a "principal axis" form, which displays, in addition to the characteristic values, parameters independent of the characteristic equation. (This work continues a program initiated by C. L. Dolph and described in Comm. Pure Appl. Math. vol. 7 (1954) p. 622.) (Received July 18, 1955.)

786. M. E. Muller: *On discrete operators connected with the Dirichlet problem.*

An approach is suggested to determine optimum operator approximations and admissible networks for finite difference techniques. A detailed study is made for the Dirichlet problem. The approach looks promising for the higher dimensional problems which will depend for their solution on the use of high speed electronic computing equipment. For the two-dimensional Dirichlet problem this approach adds further justification for the networks already in use. Contrary to popular belief, it is shown that when the dimensionality of the problem is higher than two, it is not always possible to obtain better finite difference operators by increasing the number of neighboring points that the operators will utilize. Attention is then given to showing that Monte Carlo estimates for the N-dimensional Dirichlet problem can be obtained by using random walks on the closed neighbor simplex networks that are introduced. The results of the paper may also be applied to relaxation methods. (Received June 22, 1955.)

787. G. Y. Rainich: *Determination of electric and magnetic vectors in terms of stresses in general relativity theory.*

It has been proved before that the electromagnetic stress-energy tensor S when given only at a point does not determine uniquely the field tensor F (whose components interpreted three-dimensionally are the components of the electric and magnetic force vectors); more precisely, there is a one-parameter set of F's that give the same S. It has also been shown, however, that when S is given in a portion of space-time containing discrete charged particles, F *can* be calculated. The case of continuously distributed charged matter (represented by a tensor M) has not been considered so far. In the present paper expressions are derived which determine F uniquely in terms of S, its divergence, and M. The importance of this lies in the fact that it permits one to arrive at the conclusion: the curvature of space-time determines both matter and electromagnetic field, since it has been shown before that the curvature tensor (in the nonsingular case) can be split uniquely into S and M. The paper also contains a new proof of this last proposition. (Received July 18, 1955.)

788. D. E. Richmond: *On the existence of limit cycles.*

The existence of a limit cycle for $x''+f(x, x')x'+g(x)=0$ ($'=d/dt$) is proved by applying Bendixson's theorem to an unusually simple ring-shaped region in the xx' plane, under the following conditions on the continuous functions f and g: (I) $f(x, x')>0$ if $|x|>a$, $f(0, 0)<0$, $f(x, x')\geqq -k$ if $|x|<a$; (II) $g(x)$ increases monotonically and $g(0)=0$; (III) The existence of a positive function ϕ such that $f(x, \phi(x))\geqq -g(x)/\phi(x)-\phi'(x)$ if $x\leqq -b<-a$. Each of these conditions may be relaxed. Thus in (I), $f(x, x')$ may be negative ($\geqq -m$) for $x>a$ with properly chosen m (in some cases, arbitrarily large). (Received July 15, 1955.)

789. A. E. Roach: *The effect of particle-size distribution on wear rate.*

An energy-balance analysis of wear rate requires the introduction of a term containing the particle-size distribution. This paper shows that the term is relatively independent of the size-distribution function and can be expressed as a simple function of the mean particle size provided the mean is properly defined. (Received July 11, 1955.)

790. Seymour Sherman: *On Segal's postulates for general quantum mechanics.*

In [*Postulates for general quantum mechanics*, Ann. of Math. vol. 48 (1947) pp. 930–948] I. E. Segal has presented "a set of postulates for a physical system" and deduced from these "the main general features of the quantum theory of stationary states." An important way in which he differed from his precursors, P. Jordan, J. von Neumann, and E. P. Wigner, was by *not requiring a distributive postulate*, which according to them was a postulate lacking physical meaning. The only models he presented for his system were, in essence, uniformly closed, self-adjoint algebras of bounded operators on real or complex hilbert space. Natural questions that arise are (1) Are there any other models for Segal systems? (2) Are Segal systems necessarily distributive? (3) Does an analogue to the Stone-Weierstrass Theorem hold for Segal systems? (4) Are lattice-ordered Segal systems necessarily commutative? These questions are answered by presenting a class of systems of observables which satisfy Segal's postulates yet do not form algebras since the distributive law fails to hold. For these examples at least one analogue of the Stone-Weierstrass Theorem fails to hold. An example is given of a system of observables which is a lattice in its natural order but is not commutative. With the aid of a remark of Albert it is shown that M_3^8, the set of 3×3 matrices whose elements are Cayley numbers, constitutes a Segal system. It is shown that Postulate II, 4 of Segal's paper is redundant. (Received May 25, 1955.)

791. Patricia J. Wells: *The integral equation for the prolate spheroidal antenna.*

The Albert and Synge integral equation [Quarterly of Applied Mathematics vol. 6 (1948)], satisfied by the current on a surface of revolution, is solved for the case of the prolate spheroid. The solution is found by means of the expansion of an angularly symmetric dipole field in terms of the prolate spheroidal wave functions of order one. Comparison of this expansion with a known expansion in terms of functions of order zero yields recursion relations between the spheroidal wave functions of orders one and zero. (Received July 18, 1955.)

GEOMETRY

792t. Louis Auslander: *On the prolongation of exterior differential equations*.

Let Σ be a system of exterior differential equations in a coordinate neighborhood U and let x^i, $i=1, \cdots, g$, be independent variables and z^α, $\alpha=1, \cdots, n-g$, be dependent variables. Assume that over the point $x \in U$ there exist involutive prolongations of all orders. Then a set of prolongations, one of each order, uniquely determines a set of formal power series $z^\alpha = a_0^\alpha + \sum a_i^\alpha x^i + (1/2!) \sum a_{ij}^\alpha x^i x^j + \cdots$ with the property that if each of the above series converge the surface thus determined will be an involutive integral surface for Σ. (Received June 17, 1955.)

793. K. D. Fryer (p) and Israel Halperin: *On the coordinatisation theorem of J. von Neumann*.

A precise statement of the von Neumann theorem is as follows. Let L be a complemented modular lattice possessing a homogeneous basis $a_1, \cdots, a_n$, $n \geq 4$, and let L_{12} be the set of all inverses of a_2 with respect to $a_1 + a_2$. Then (i): addition and multiplication can be defined for the elements of L_{12} is such a way that L_{12} becomes a regular ring $\mathfrak{R}$ and the sublattice $L(a_1)$, consisting of all $x \leq a_1$, is isomorphic to the lattice of all left principal ideals of $\mathfrak{R}$, and (ii): L is lattice-isomorphic to (coordinatized by) the lattice of all left principal ideals of $\mathfrak{R}_n$. Here $\mathfrak{R}_n$ consists of all $n \times n$ matrices with elements in $\mathfrak{R}$ and, as shown by von Neumann, is regular along with $\mathfrak{R}$. In the present paper, a direct proof is given for (ii)′ (equivalent to (ii)): L is lattice-isomorphic to (coordinatized by) the lattice of $\mathfrak{R}$-left modules of finite span in the space V of vectors $(\alpha^1, \cdots, \alpha^n)$, all α^i in $\mathfrak{R}$. (Received July 1, 1955.)

794. David Gale: *On convex polyhedra*.

Let V be the set of vertices of a convex polyhedron P. The vertices belonging to a subset S of V are called *neighbors* if S is the intersection of V with a supporting hyperplane to P. If every m vertices of V are neighbors then P is called *m-neighborly*. Suppose P is k-dimensional and m-neighborly, $m \leq k$. If $k < 2m$, one easily shows that P must be a k-simplex. It is shown, on the other hand, that if $k \geq 2m$, then P may have any number of vertices. As a special case this shows that there exist polyhedra in 4-space with arbitrarily many vertices every two of which are connected by an edge. (Received July 7, 1955.)

795. Michael Goldberg: *Central tesselations*.

Regular tesselations, discussed in many books and papers, are periodic arrangements of congruent regions covering the infinite plane. Heesch [Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen N.S. vol. 1 (1935) pp. 115–117] and Voderberg [Jber. Deutschen Math. Verein. vol. 46 (1936) pp. 229–231; vol. 47 (1937) pp. 159–160] gave examples of tesselations which are not regular. The present paper presents several types of nonregular tesselations which are characterized by the possession of a unique center. They include sector, spiral, concentric polygon, vortex, dodecagonal and parallel core tesselations. Modifications of regular honeycombs to central honeycombs are given as examples of extension to three dimensions. (Received July 11, 1955.)

796t. S. I. Goldberg: *Note on semi-symmetric connections in Hermitian manifolds*.

Consider a complex-analytic manifold M^n of n complex dimensions endowed with a (general) Hermitian metric. The author defines formally (as analogues of the Kaehler case) manifolds of "constant" curvature and "constant" holomorphic curvature. More generally, Einstein spaces are introduced with the various contractions of the curvature tensor. If the contracted torsion tensor vanishes ($S^i_{ri}=0$), the manifold is Kaehlerian. If the scalar curvature is not constant ($S^i_{ri}\neq 0$), it is shown that the connection is semi-symmetric and that an Einstein space can be transformed conformally into a Kaehler manifold. Moreover, if M^n is compact and $\Delta(\log|\lambda|)\geq 0(\leq 0)$ for some scalar field λ, M^n is torsion free and the conformal map is a similitude. It follows that a compact manifold of "constant" (nonvanishing) holomorphic curvature is torsion free and is, in fact, a Hodge manifold. (Received July 21, 1955.)

797. C. C. Hsiung: *Some characterisations of closed umbilical hypersurfaces in Riemannian space of constant curvature.*

Let V^n be an orientable hypersurface, with a closed boundary V^{n-1} of dimension $n-1$ ($n\geq 2$), differentiably of class C^3 imbedded in an $(n+1)$-dimensional Riemannian space R^{n+1} of constant curvature; $\kappa_1, \cdots, \kappa_n$ the n principal curvatures at a point P of V^n; and M_i the ith mean curvature of V^n at P defined by $C_{n,i}M_i=\sum\kappa_1\kappa_2\cdots\kappa_i$ ($i=1,\cdots,n$), where the expression on the right side is the ith elementary symmetric function of $\kappa_1,\cdots,\kappa_n$, and $C_{n,i}$ denotes the number of combinations of n different things taken i at a time. Let dA be the area element of V^n at P, and p the scalar product of the unit normal vector of V^n at P and the position vector of P with respect to any orthogonal frame in the space R^{n+1}. The purpose of this paper is first to show that the integral $\int_{V^n}(M_{n-1}+M_np)dA$ can be expressed as an integral over the boundary V^{n-1} and then from this integral formula and another one obtained by the author [Bull. Amer. Math. Soc. Abstract 61-2-331] to deduce some simple conditions for a closed V^n to be umbilical. For a Euclidean space R^{n+1} these results were obtained in a previous paper by the author [Math. Scand. vol. 2 (1954) pp. 286–294]. (Received July 11, 1955.)

798. W. R. Hutcherson: *A note on nonequivalent cyclic homographies.*

J. Dessart [*Sur les surfaces représentant l'involution engendré par un homography de cinque du plan*, Memoires de la Société Royale des Sciences de Liège no. 17 (1931) pp. 1–23] and other writers have used the homography $x_1':x_2':x_3'=x_1:Ex_2:E^\alpha x_3$ where $E^p=1$ (p is prime) and α(positive integer) $\leq p$. Of the p^2 homographies, there are p sets, each containing $p-1$ equivalent (each generates the same p set of points) homographies. For example, the two equivalent homographies $(x_1, E^{p-1}x_2, Ex_3)$ and $(x_1, Ex_2, E^{p-1}x_3)$ are found in only one of the p sets. If x_2 and x_3 are interchanged and one excludes those collineations which relate to perfect points [W. R. Hutcherson and J. C. Morelock, *Concerning a pattern for perfect points*, Bull. Amer. Math. Soc. Abstract 60-6-554] it is discovered that there are exactly $(p-1)/2$ distinct nonequivalent sets of homographies for each prime number p. (Received June 9, 1955.)

799*t*. Harold W. Kuhn: *On certain convex polyhedra.*

Let T_n be the set of tours (i.e., cyclic arrangements of $1,\cdots,n$) represented as n by n permutation matrices $t=(t_{ij})$. Let C_n denote the convex hull of T_n in n^2-dimensional Euclidean space. The polyhedron C_n spans the (n^2-3n+1)-dimensional linear variety of all $x=(x_{ij})$ with $x_{ii}=0$ and $\sum_j x_{ij}=\sum_i x_{ij}=1$ for all i and j. Every nonzero matrix $b=(b_{ij})$ defines a half-space $\sum b_{ij}x_{ij}\geq\beta$ supporting C_n by the rule: $\beta=$ minimum of $\sum b_{ij}t_{ij}$ for $t=(t_{ij})\in T_n$. All faces of C_n (i.e., (n^2-3n)-dimensional intersections of C_n with supporting hyperplanes) can be obtained from non-negative, integral b. For $n<5$, matrices of zeros and ones suffice. For $n=5$, the following two classes: $x_{12}+x_{13}+x_{22}+x_{35}+x_{31}+x_{34}+x_{43}+x_{53}\geqq 1$ (60 distinct faces with renumbering of $1, \cdots, 5$) and $x_{13}+2x_{14}+x_{15}+x_{22}+x_{25}+x_{31}+x_{33}+x_{41}+x_{43}+x_{45}+2x_{52}\geqq 2$ (120 faces) together with the 210 faces given by Heller [Bull. Amer. Math. Soc. Abstract 59-6-664] describe C_5 in an irredundant manner. (Received July 13, 1955.)

800*t*. B. Z. Linfield: *Integral and matric geometry.*

With functions as coordinates and integrals as distances, any orbit, any velocity or acceleration of a particle in its orbit, any surface, any space, any direction in a space, is determined by one scalar equation. Each set of points, be it a curve, a surface, a space, or a point like a centroid, is characterized by one scalar equation $f=f(s, u)$, where each value of u determines one point in the set, and where the number of components in u determines the dimension of this f-space $f(s, u)$. The square of the distance between two points $f_1=f(s, u_1)$ and $f_2=f(s, u_2)$ is $\int(f_1-f_2)^2ds$. The limits of integration are here from $-\pi$ to π. Every covariant X in $f(s, u)$ is a matric function, which satisfies a polynomial equation with invariant coefficients; and is linearly dependent on basic covariants, which satisfy polynomial equations with numerical coefficients. Every invariant F in $f(s, u)$ is a scalar function whose value at every point is independent of the choice of u-coordinates. For a Euclidean plane or space $f=f(s, u)$ a certain covariant $Z(f)=0$. (Received July 5, 1955.)

801*t*. B. Z. Linfield: *Covariant functions of invariants in f-space.*

Every invariant F in $f(s, u)$ has a covariant $Z(F)$ such that the leading coefficient the equation $Z(F)$ satisfies is the Laplacian $L(F)$. It has a covariant $Y(F)$ which satisfies $Y^2(F)=Y(F)=Y(G)$, for any invariant $G=G(F)$. Any two invariants (F, G) have an invariant $T(F, G)=T(G, F)$, and two conjugate covariants $Z(F, G)$, $Z(G, F)$ which satisfy the same quadratic equation $X^2=T(F, G)X$. When G depends functionally on F, its covariant $Z(G)$ is linearly dependent on $Z(F)$ and $Y(F)$, the covariants $Z(F, G)$, $Z(G, F)$ $Y(G)$ are linearly dependent on $Y(F)$, the invariant $L(G)$ is linearly dependent on $L(F)$ and $T(F, F)=k^2$, and $T(G, G)=k^2(dG/dF)^2$, $Z(F, F)=k^2Y(F)$. For any surface $f(s, u)$, $\dim u=2$, in 3-dimensional space; and any n-dimensional surface $f(s, u)$, $\dim u=n$, in $(n+1)$-dimensional space, the covariant $Z(f)$ is factorable, i e. $Z(f)=P(u)q(s, u)$, where covariant $P(u)$ depends functionally on u only, and invariant $q(s, u)$ is the direction of the normal to $f(s, u)$ and satisfies $\int q^2ds=1=\int f(s, u)ds$. For covariant $P(u)$ there exist two covariant functions $C(V, X)$, $D(X, U)$, of any covariant X, such that $C(V, P)=D(P, U)=0$. When $\dim u=2$, then $Z^2(f)=L(f)Z(f)-q^2|P|I$. (Received July 5, 1955.)

802*t*. B. Z. Linfield: *Covariant functions of covariants in f-space.*

Any covariant vector U, any contravariant vector V, and any covariant X, have three covariants $\theta(X)=\theta(V, X)$, $\theta=A, C, E$; and two covariants $\theta(X)=\theta(X, U)$, $\theta=B, D$. These five covariants $\theta(X)$ satisfy like X polynomial equations with invariant coefficients, are linear in X, U, V: $\theta(X+Y)=\theta(X)+\theta(Y)$, etc., and $\theta(XY)$ satisfy simple functional equations. For any surface $f(s, u)$, where $Z(f)=P(u)q(s, u)$, the covariant $Z(q)$ is linearly dependent on orthonormal directions $q, w_1, w_2, \cdots$ with coefficients $-P^2$ and $\theta(P)$, i.e. $Z(q)=-qP^2+\sum_1 w_1B(P, U_1)/k_1$, where $\int w_1w_2ds=0=\int q(s, u)ds$, etc. For any sphere, or circular cylinder, we get $Z(q)=-qP^2$. For the sphere, the determinant $|Z(q)|=(q|P|)^2\neq 0$. There are similar covariant equations to characterize other surfaces. Practically every scalar, vector, or matric equation in function coordinates takes the place of many scalar, vector, or matric equations not in functions coordinates. They are therefore more economical than tensor equations, and much easier to use and understand. (Received July 5, 1955.)

803*t*. Albert Nijenhuis: *Scalar fields whose second covariant derivatives are equal to a given Riemannian metric tensor.*

The problem, proposed by E. T. Parker (Bull. Amer. Math. Soc. Research Problem 61-3-19), is when there exists in a Riemannian space (not necessarily positive definite) a scalar function s such that $\nabla_b\nabla_a s = g_{ba}$. Putting $v^a = g^{ab}\partial_b s$ one finds that for each s there is a vector field v^a satisfying $\nabla_b v^a - \delta^a_b = 0$, i.e. v^a is invariant under the Cartan connection of "rollings" of the tangent space. Every vector $v \in T_p$ (tangent space at p) invariant under the corresponding holonomy group $H(M, p)$ gives rise to such a vector field. Once v^a is formed, s is solved from the equations $\partial_a s = g_{ab}v^b$, which are then (locally) completely integrable. For the local existence of s one only needs vectors $v \in T_p$ invariant under the local holonomy group $H^*(p)$. If M is compact or complete no vector fields v^a, and therefore no scalar fields s of the proper kind, exist, except when M is Euclidean space with a Euclidean metric. General reference: A. Nijenhuis, *On the holonomy groups of linear connections* [Proc. K. Akad. Wetensch. A vol. 56 (1953) pp. 233–249, vol. 57 (1954) pp. 17–25]. (Received July 1, 1955.)

804*t*. R. Z. Norman: *On the convex polyhedra of the symmetric traveling salesman problem.*

In the symmetric traveling salesman problem, a tour (cyclic arrangement of $1, \cdots, n$) need not be distinguished from its opposite (the same arrangement in opposite order). Both tours can be represented by a triangular array of elements $t = (t_{ij})$ of an $n \times n$ matrix for which $i<j$, in which t_{ij} is 0 or 1 and in which for all j, $\sum_{i<j} t_{ij} + \sum_{i>j} t_{ji} = 2$. The tours are thus points in an $((n^2-3n)/2)$-dimensional manifold of an $((n^2-n)/2)$-dimensional Euclidean space. Let C_n denote the convex hull of these points. A face of C_n, an intersection of C_n with a supporting hyperplane, can be described by means of the corresponding supporting half space. For $n=5$, as is already known, there are two classes of these face-defining supporting half-spaces, I: $x_{ij} \geqq 0$ and II: $x_{ij} \leqq 1$, each having 10 half-spaces. For $n=6$ there are 15 half-spaces in each of class I and class II, 10 in class III: $x_{ij}+x_{ik}+x_{jk} \leqq 2$, and 60 in class IV: $x_{ij}+x_{jk}+x_{ik}+x_{il}+x_{km}+x_{km} \leqq 4$. For $n=7$ there are 21 in each of the first two classes, 35 in Class III, 840 in class IV, and 1260 in class V: $x_{ij}+x_{jk}+x_{kl}+x_{km}+x_{ln}+x_{mn}+x_{mp}+x_{nq} \leqq 5$. These supporting half-spaces were found independently by Dantzig and Johnson. The proof that the faces corresponding to these sets of half-spaces give an irredundant description of the convex sets C_n is in part the same as Kuhn's proof of the corresponding statement for the convex of all tours (see Abstract 799), and follows in part from a graph-theoretic argument. (Received July 18, 1955.)

805*t*. T. K. Pan: *On families of curves in a surface.*

Associate a surface S in Euclidean space with a congruence of unit vectors λ which make a constant angle with the surface normals. A curve in S is a line of λ-projectile if along it the component vectors in S of the derived vectors of the λ-congruence are tangent. A curve in S is a line of λ-orthoprojectile if along it the component vectors in S of the derived vectors of the λ-congruence are orthogonal. A line of λ-projectile coincides with a line of curvature of S if the component vectors in S of the λ-congruence are either orthogonal to or parallel along the line of λ-projectile. A line of λ-orthoprojectile coincides with an asymptotic line of S if the component vectors in S of the

λ-congruence are either tangent to or parallel along the line of λ-orthoprojectile. A plane geodesic or a geodesic line of curvature is a line of projectile, a geodesic asymptotic line, or a straight line in S is a line of orthoprojectile. (Received July 18, 1955.)

806. C. M. Petty and Daniel Waterman (p): *An extremal theorem for N-simplexes.*

Comparing the sum $R_0+R_1+R_2$ of the distances from the vertices of a triangle to an arbitrary point P with the area Δ of the triangle, L. Fejes Toth gives the relation $R_0+R_1+R_2 \geq 2\cdot 3^{1/4}\Delta^{1/2}$, equality holding if and only if the triangle is equilateral and P is its centroid. This result is generalized to n dimensions. Theorem: If R_i $(i=0, \cdots, n)$ are the distances from an arbitrary point P to the vertices of a nondegenerate simplex in E^n of volume Δ, then $R_0+ \cdots +R_n \geq (n+1)^{(n-1)/2n}n^{1/2}(n!)^{1/n}\Delta^{1/n}$ where the equality holds if and only if the simplex is regular and P is its centroid. The proof of the theorem is elementary, utilizing a characterization of orthocentric simplexes developed there. The theorem is also formulated as an inequality concerning determinants. (Received July 18, 1955.)

807. I. M. Singer: *Transitive holonomy groups.* II.

Let C^m be the manifold of complex projective space of m homogeneous variables, and let S^{2m-3} be the space of rays through the origin of the tangent space at a point of C^m. If m is odd, the holonomy group of any affine connection on C^m acts transitively on the sphere S^{2m-3}. Similarly let Q^m be the manifold of quaternion projective space of m homogeneous variables, with S^{4m-1} the rays through the origin. If m is not divisible by twenty-four, then the holonomy group of any affine connection acts transitively on S^{4m-1}. These theorems are proved by using the fact that S^{2m-1} is a bundle over C^m [S^{4m+3} is a bundle over Q^m] and computing the characteristic map of the bundle $p^{-1}(B)$ where p is the projection map of S^{2m-1} onto C^m [S^{4m-1} onto Q^m], B is the bundle of bases over C^m [Q^m], and $p^{-1}(B)$ is the induced bundle. (Received July 18, 1955.)

LOGIC AND FOUNDATIONS

808. Kurt Bing: *On simplifying propositional formulas.*

A necessary and sufficient condition, not depending on developed normal formulas, is obtained for a given propositional formula to be implied by the irredundant conjunctions of negated and unnegated propositional letters first investigated by Quine (Amer. Math. Monthly vol. 59 (1952) pp. 521–531; Bol. Soc. Mat. Mexicana vol. 10 (1953) pp. 64–70). Among the consequences is an extension of the known conditions under which a disjunction of conjunctions is reduced to simplest normal form by deletion of those conjunctions which contain other conjunctions as parts. (Received July 18, 1955.)

809*t*. Ben Dushnik: *Mixed sums of order types.* Preliminary report.

Let A and B be two simply ordered sets, and let C be the union of the (unordered) sets A and B. Define a simple ordering on C in any manner compatible with the sole requirement that the elements of A and of B retain their original orderings. If α and β are the order types of A and B, and if $\Gamma: (\gamma', \gamma'', \cdots)$ is the set of all possible order types which C may have when ordered as above, then Γ will be called the mixed sum of α and β. Hausdorff's "natural" sums of ordinals is an instance of (an element of) a mixed sum; the present paper indicates possible uses of mixed sums. (Received July 19, 1955.)

810. P. C. Hammer: *The foundations of general topology.*

Let M be a set and f a function mapping a class D of subsets of M onto a class R of subsets of M. A set X in M is *f-closed* if and only if $X\supset Y$, $Y\in D$ implies $X\supset fY$. To each f there corresponds a *limit function* f_1 defined as a subclass D_1 of D such that the intersection of Y and f_1Y is empty for $Y\in D_1$ and such that the class of f_1-closed sets coincide with the class of f-closed sets. With each f there are associated two uniquely determined functions g and h defined for all subsets of M. The function g is an inclusion preserving enlargement function and the function h is a closure function. Using these functions a critical analysis of the structure of general topology is made, revealing the necessity of a new definition of limit point and continuity and extending general theorems on connectedness to more general forms of connectedness. (Received July 5, 1955.)

811*t*. A. C. Sugar: *Axiomatic systems for classical and relativistic timeless dynamics.* Preliminary report.

One of the most pronouncedly counterintuitive concepts in relativistic mechanics is the notion of time. Intuitively the notion of time stems from our concepts of motion and can be defined in terms of our notions of distance and uniform motion. In order to clarify our concept of time in general, and in the particular circumstance of relativistic dynamics, the author formalizes mechanics in timeless form. Acceleration is expressed in the familiar form in which we take the product of velocity expressed as a point function and its derivative, namely, vdv/ds. The second law expressed heuristically becomes then $F=mvdv/ds$. Although the axioms will parallel those of a previous paper (Journal of Rational Mechanics and Analysis, April 1953) by the author and J. C. C. McKinsey, one has the important exception that force and velocity are functions of position instead of time. Clearly there exists a class of realizations of these axioms in which the above function "v" may designate a "generalized" or pseudo velocity in terms of which one may define a pseudo time. Hence dynamics formalized in this fashion embraces new generalizations of classical and relativistic mechanics. (Received July 13, 1955.)

STATISTICS AND PROBABILITY

812*t*. D. G. Austin: *Differentiation properties of Markoff functions where the terminal state is stable.*

The author considers a Markoff matrix $\{p_{ij}(t)\}$ under hypothesis on $\{p_{ij}(t)\}$ used in the author's note (Proc. Nat. Acad. Sci. U.S.A. vol. 41, pp. 224–226). It is shown that if the terminal state is stable $(Dp_{jj}(0)\neq-\infty)$ then each of the functions $p_{ij}(t)$ $(i=1, 2, \cdots)$ has a continuous first derivative satisfying the system of differential equations: $Dp_{ij}(ts)=\sum_{k=1}^{\infty} p_{ik}(t)Dp_{kj}(s)$, $t\geq 0$, $s>0$. (Received July 19, 1955.)

813*t*. H. A. Bender: *Bivariate distribution.*

If one expresses the integral for bivariate distribution in polar form, assuming the origin to be at the mean of the two variates, and supposes $\rho=a\sec\theta$, or $\rho=a\csc\theta$, then this integral simplifies to $(e^{-a^2/2}/2\pi)\int_0^t(e^{-a^2t^2/2}/(t^2+1))dt=\int_0^t f(a, t)dt=F(a, t)$. The integral $\int_0^t f(a, t)dt$ can play a role for bivariate distribution very similar to the role played by the probability integral $\int_0^t\phi(t)dt$. In addition to the above application, one has the following interesting relations: (I) $F(a, t)=2F(a, 1)-\int_0^a\phi(t)dt\int_a^\infty\theta(t)dt-F(at, 1/t)$. (II) $F(a, \infty)=2F(a, 1)-\int_0^a\phi(t)dt\int_a^\infty\theta(t)dt$. (III) $2\int_0^\infty f(a, t)dt=\int_a^\infty\theta(t)dt$

$=1/2-\int_0^t p(t)dt$. A table of values for $F(a, t)$ has been computed for $a=1/2, 1, 3/2, 2,$ and $5/2$. This table is rather brief since, for example, to four decimal places $F(1, 2.8) = F(1, \infty) = 0.0793$. With such a table it is possible to find the probability $a_1\sigma_x < x < a_2\sigma_x$ and $b_1\sigma_y < y < b_2\sigma_y$ for any given value of the correlation between x and y and for certain values of a. (Received May 13, 1955.)

814t. J. L. Brenner and G. H. Weiss: *On a general form of renewal theory*. Preliminary report.

Ordinary renewal theory can be regarded as a study of a one-dimensional unidirection random walk, with steps of varying length. This is generalized to the multidimensional case. In particular the author assumes an n-dimensional random walk in which the walker may take a step of varying length in the direction of any axis. A vector of probability densities, $[\alpha_j(x)]$, for the length x, of a step in the jth direction is assumed known, as well as a constant irreducible matrix $P[=(p_{ij})]$, of conditional transition probabilities. The element p_{ij} is the probability that a step in the jth direction will be followed by one in the ith direction. A vector integral equation is found to describe this process. A uniqueness and existence theorem is found for this equation. Asymptotic results for the projection along any axis are given, in terms of principal diagonal minors of the matrix $I-P$, where I is the unit matrix. The results are shown to be consistent with certain theorems on determinants with weakly dominant main diagonals. (Received July 20, 1955.)

815. D. A. Darling (p) and Mark Kac: *On occupation times for Markoff processes.*

Let X_n, $n=0, 1, \cdots$, be a Markoff chain taking on values in an abstract measure space $(\Omega, \mathcal{F})$ with transition probability $P(x, E)=\Pr\{X_{k+1}\in E \mid X_k=x\}$, $x\in\Omega$, $E\in\mathcal{F}$. Suppose that $V(x)$, $x\in\Omega$, is real-valued, non-negative and measurable, and that $\sum_{n=0}^{\infty} P^{(n+1)}(x, E)s^n = k(s)Q(E)+k_1(x, s, E)$ where $P^{(k)}(x, E)$ is the k-step transition probability, $k(s)\to\infty$, $s\to 1$, $0<\int V(x)Q(dx)<\infty$, $(1/k(s))\int V(x)k_1(x, s, dx)\to 0$, $s\to 1$. Then the necessary and sufficient condition that there exist constants a_n such that $(1/a_n)\sum_{k=1}^{n} V(X_k)$ has a nondegenerate limiting distribution is that $k(s)$ is $(1-s)^{-\alpha} \cdot L(1/(1-s))$, $0\leq\alpha<1$, where $L(x)$ is a slowly varying function. We can then choose $a_n=k(1-1/n)$ and the limiting distribution is necessarily a Mittag-Leffler-Pollard distribution of index α. Exactly the same theorems result, *mutatis mutandis*, for a process $X(t)$ with continuous time. These results give immediately, and generalize, known theorems on duration times, renewal theory, changes of sign, etc. for sums of independent random variables. (Received July 19, 1955.)

816t. Aryeh Dvoretzky, Jack Kiefer, and Jacob Wolfowitz: *Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator.*

The asymptotic minimax character of the sample distribution function (d.f.) for estimating an unknown d.f. F (in the class C of all continuous univariate d.f.'s, or of all univariate d.f.'s) is proved for a wide variety of weight functions. The weight functions include nondecreasing functions of the normalized maximum deviation $n^{1/2}\delta_n$ between estimated and true d.f., weight functions obtained as integrals, over the line, of nondecreasing functions of the deviation at individual points on the line, and many other weight functions for which the risk function of the sample d.f. need not even be constant over C. For the multinomial estimation problem, the classical estimator is shown to be asymptotically minimax for a wide variety of weight functions. By-products of the investigation are a general discussion of minimization of symmetric convex or monotone functionals of symmetric random processes, and a proof that, as $n\to\infty$, the moment generating function of $n^{1/2}\delta_n$ (for $F\in C$) converges to that of the limiting d.f. of $n^{1/2}\delta_n$. (Received May 26, 1955.)

817*t*. Jack Kiefer and Jacob Wolfowitz: *On the characteristics of the general queueing process, with applications to random walk.*

This paper continues the study of the general queueing process (many servers, arbitrary distributions of service time and time between successive arrivals) initiated by the authors in Trans. Amer. Math. Soc. vol. 78 (1955) pp. 1–18, the notation of which paper is used here. A typical result in the case $\rho<1$ where a limiting distribution function (d.f.) exists (the case $\rho\geqq 1$ is trivial) is that $\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}(w_{i1})^k=m_k$ with probability one, where m_k $(k>0)$ is the kth moment of the limiting d.f. of the waiting time w_{i1}, with analogous results for the mean queue length and mean busy period. Necessary and sufficient conditions for the finiteness of various moments are given, with the following random walk result as a special case: Let $x_1, x_2, \cdots$, be independent and identically distributed random variables with $-\infty<Ex_1<0$. Let $S_0=0$ and $S_n=\sum_1^n x_i$. Let $v=\sup_{n\geqq 0} S_n$. Then for any positive k, $Ev^k<\infty$ if and only if $E(x_1^+)^{k+1}<\infty$. (Received May 26, 1955.)

818*t*. Seymour Sherman: *A theorem on convex sets with statistical applications.*

Let $\mathcal{C}_0$ be the closed convex cone generated in the uniform norm (that is, $\|f\|=\sup\{|f(x)|\}$) by the characteristic functions of symmetric, compact, convex sets in real n space, $\mathcal{E}$. Let $\mathcal{C}_1$ be the closed convex cone generated in the L_1 norm (i.e. $\|f\|_1=\int|f(x)|\,dx$) by the characteristic functions of symmetric, compact, convex sets in $\mathcal{E}$. Consider the norm $\|\ \|_2$ given by $\|f\|_2=\max\{\|f\|,\|f\|_1\}$. Let $\mathcal{C}_2$ be the closed convex cone generated in the $\|\ \|_2$ norm by the characteristic functions of symmetric, compact, convex sets in $\mathcal{E}$. Note that for E, K compact, convex and symmetric, the convolution of χ_K and χ_E, the characteristic functions of K and E respectively, evaluated at y becomes $\chi_K*\chi_E(y)=\int\chi_K(x)\chi_E(y-x)dx$. THEOREM 1. $\mathcal{C}_1*\mathcal{C}_1\subset\mathcal{C}_1$ *and* $\mathcal{C}_2*\mathcal{C}_2\subset\mathcal{C}_2$. THEOREM 2. *Let E be a convex set in $\mathcal{E}$, symmetric about the origin. Let f be a non-negative function such that* (i) $f(x)=f(-x)$ *for each* $x\in\mathcal{E}$, (ii) $\{x\,|\,f(x)\geqq u\}=K_u$ *is convex for each* u, $0\leqq u<\infty$, *and* (iii) $\int_E f(x)dx<\infty$, *then* $\int_E f(x+y)dx\in\mathcal{C}_2$. This is a little stronger than a theorem of Anderson [*The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities*, Proc. Amer. Math. Soc. vol. 6 (1955) p. 170] and in fact the proof uses ideas in his proof but squeezes additional conclusions from the argument. The paper also gives a generalization to the multivariate case (with slight weakening of the symmetry assumptions in the hypothesis even in the univariate case) of results of Z. W. Birnbaum [*On random variables with comparable peakedness*, Ann. Math. Statist. vol. 19 (1948) pp. 176–181]. (Received July 8, 1955.)

819. F. L. Spitzer: *On the maximum of partial sums of random variables.*

Let $\{X_k\}$ be a sequence of identically distributed random variables and $\{S_k\}$ the sequence of their successive partial sums. S_k^+ is taken to equal S_k if $S_k\geqq 0$ and zero otherwise. The characteristic functions of the random variables S_k^+ are called $\psi_k(\lambda)$

and those of the random variables $\max_{1\leq k\leq n} S_k^+$ are called $\phi_k(\lambda)$ for $k\geq 1$, while $\phi_0(\lambda)=1$. Then it follows from certain very general combinatorial identities for partial sums of real numbers that the following relation holds. For $|t|<1$, $\sum_0^\infty \phi_n(\lambda)t^n = \exp \sum_1^\infty (\psi_k(\lambda)/k)t^k$. This result generalizes apparently dissimilar identities discovered by E. Sparre Andersen and M. Kac. (Received July 19, 1955.)

820*t*. F. L. Spitzer: *Some limit theorems for random variables.*

Let $X_1, X_2, \cdots$ be identically distributed independent random variables with distribution $F(x)$, and $S_n = X_1 + \cdots + X_n$, $a_n = \Pr\{S_n>0\}$, $p_n = \Pr\{S_1>0, S_2>0, \cdots, S_n>0\}$ for $n\geq 1$ and $p_0=1$. E. Sparre Andersen (Math. Scand. vol. 2 (1954) pp. 195–223) proved that $\sum_0^\infty p_n t^n = \exp(\sum_1^\infty (a_n/n)t^n)$ for $|t|<1$. The next three theorems follow from his result. I: $\Pr\{\sup_{n\geq 1} S_n>0\} = 1-\exp(-\sum_1^\infty a_n/n)$. II: $\Pr\{S_n>0 \text{ i.o.}\} = \Pr\{\limsup_{n\to\infty} S_n = +\infty\} = 1$ or 0 according as $\sum_1^\infty a_n/n = \infty$ or $<\infty$. III: $\sum_1^\infty (1/n)\Pr\{|S_n/n-m|>\epsilon\}<\infty$ for every $\epsilon>0$ if and only if $\int_{-\infty}^{\infty} x\,dF(x) = m$. (Received July 15, 1955.)

TOPOLOGY

821*t*. F. W. Anderson: *Ordered systems of regular open sets.*

Define the transitive binary relation $\ll$ on the family $R(X)$ of all regular open subsets of the completely regular space X by $U\ll V$ in case U and the complement of V are completely separated in X. Shirota (Osaka Math. J. vol. 4 (1952) pp. 121–132) proved that if X is locally compact and $B\subset R(X)$ consists of those regular open sets with compact closure, then the system $(B, \ll)$ characterizes X. Using Shirota's result it is proved that the system $(R(X), \ll)$ characterizes the Stone-Čech compactification βX. In view of Tukey's generalization of Urysohn's Lemma (*Convergence and uniformity in topology*, Princeton, 1940, p. 53) this provides a completely "internal" construction of βX. It should be noted that, in general, $(R(X), \ll)$ fails to characterize the Q-space νX (cf. Hewitt, Trans. Amer. Math. Soc. vol. 64 (1948) p. 88). Finally, it is observed that an arbitrary basis $B\subset R(X)$ need not characterize βX, but if X is metrizable and B contains all bounded U in $R(X)$, then $(B, \ll)$ characterizes X. (Received July 18, 1955.)

822. R. W. Bagley: *A note on topologies on* $2^{\mathfrak{R}}$.

The following topologies on $2^{\mathfrak{R}}$ are considered; the sequential order topology, τ_1; the order topology, τ_2; the topology of pointwise convergence, τ_3 (here we consider $2^{\mathfrak{R}}$ as the cartesian product "$\mathfrak{R}$" times of $\{0, 1\}$ with the discrete topology on $\{0,1\}$); and the interval topology, τ_4 (the closed intervals are taken as a sub-basis for the closed sets). It is shown that when $\mathfrak{R}$ is countable $\tau_1=\tau_2=\tau_3=\tau_4$. However, when $\mathfrak{R}$ is uncountable $\tau_1<\tau_2<\tau_3=\tau_4$, where $\tau\leq\tau'$ means that every set open under τ' is also open under τ. A partial solution to problem 76 (Birkhoff, *Lattice theory*) is given (see also Northam, Proc. Amer. Math. Soc. (1953) pp. 824–827). (Received June 3, 1955.)

823*t*. B. J. Ball: *A note on the separability of an ordered space.*

It is shown that a connected ordered space S is separable provided it satisfies Souslin's condition (there does not exist an uncountable collection of mutually exclusive open subsets of S) and there is a countable family F of continuous functions of S into itself such that each point p of S is a limit point of $\{f(p) \mid f\in F\}$. If S is not required to satisfy Souslin's condition, the existence of such a family F does not imply that S is separable; however, if no element of F has a fixed point or if the elements of F can be arranged in a sequence $\{f_n\}$ such that for each point p of S, $\{f_n(p)\}\to p$, then S must satisfy Souslin's condition and hence must be separable. (Received July 19, 1955.)

824t. R. H. Bing: *Approximating wild surfaces with polyhedral ones.*

Suppose S is a continuum in E^3 topologically equivalent to the surface of a sphere. It may be wild and its complement topologically different from the complement of a polyhedral surface. However, it is shown that for each positive number ϵ, there is a polyhedral 2-sphere S' in E^3 and a homeomorphism h of S onto S' such that if $s \in S$, $\rho(s, h(s)) < \epsilon$. The method used is to take a ruling of E^3 of small mesh, consider how S intersects the boundaries of this ruling, and then construct the polyhedral surface S' near S so that it intersects the boundaries of the ruling in a fashion approximating the way in which S intersects the boundaries. Point set methods are used in the proof. (Received July 18, 1955.)

825. R. L. Blair: *A note on lattices of continuous functions.* II.

If X is a T_1-space and R a chain with its order topology, let $C(X, R)$ ($C^*(X, R)$) be the lattice of all continuous (bounded) functions from X to R. If $\alpha, \beta \in R$, then $\alpha \ll \beta$ if (i) $\alpha \leqq \beta$ and (ii) either $\alpha < \beta$ or β is minimal in R; dually for $\alpha \gg \beta$. Consider the following conditions on a sublattice B of $C(X, R)$: C_1. Each prime ideal of B is associated (cf. Kaplansky, Bull. Amer. Math. Soc. vol. 53 (1947) p. 618) with some $x \in X$. C_2. For $f, g \in B$ and disjoint closed sets F, G of X there is an $h \in B$ with $h(x) \ll f(x)$, $h(y) \gg g(y)$ for all $x \in F$, $y \in G$. C_3. Same as C_2 except F, G are required only to be distinct points of X. C_4. For each prime ideal P of B, if P is associated with x, y, then $f(x) = f(y)$ for all $f \in B$. If B satisfies C_1 then B is *characterizing* (*c-characterizing*, *w-characterizing*) if B also satisfies C_2 (C_3, C_4). Generalizing Kaplansky's Theorem 1 (loc. cit.) one has: If B is a characterizing sublattice of $C(Y, R)$ then Y is a continuous (1-1 continuous, homeomorphic) image of X if and only if B is isomorphic to a w-characterizing (c-characterizing, characterizing) sublattice of $C(X, R)$. For X completely regular and R = reals, constructions of βX (Stone-Čech compactification) are given in terms of $C(X, R)$ and $C^*(X, R)$. For X normal, characterizations of compactness in terms of $C(X, R)$ are also obtained. (Received July 18, 1955.)

826t. C. E. Burgess: *A note on the separation of connected sets by finite sets.*

An elementary proof is presented to show that if n is an integer greater than two and every set consisting of n points separates the connected topological space M, then some set consisting of $n-1$ points separates M. This theorem was first proved for plane continuum by J. R. Kline [*Closed connected sets which are disconnected by the removal of a finite number of points*, Proc. Nat. Acad. Sci. U.S.A. vol. 9 (1923) pp. 7–12], and G. T. Whyburn has obtained a stronger result for locally compact metric continua [*Local separating points of continua*, Monatshefte für Mathematik und Physik vol. 36 (1929) pp. 305–314]. (Received July 18, 1955.)

827. C. E. Capel (p) and W. L. Strother: *Traces for multi-valued functions.*

Let $S(X)$ denote the space of non-null, closed subsets of a compact, Hausdorff space X. If $\mathcal{A} \subset S(X)$ then a continuous function $s: \mathcal{A} \to X$ is a *selection* on $\mathcal{A}$ if $s(A) \in A$, for A in $\mathcal{A}$. E. Michael (*Topologies on spaces of subsets*, Trans. Amer. Math. Soc. vol. 71 (1951) pp. 152–182) has shown there is a selection on $S(X)$ if X is linearly ordered in a continuous sense. This idea is extended to finding selections on subsets of

$S(X)$ when X has a continuous partial order (for definition see L. E. Ward, Jr., *Partially ordered topological spaces*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 144–161). It can then be shown, for a suitable space X such as an n-cell or Tychonoff cube, that a continuous multi-valued function $F: X \to X$, with restrictions on the image of a point such as point-convex, has a trace and hence a fixed point. (Received July 8, 1955.)

828t. W. H. Cockcroft: *The cohomology groups of certain fibre spaces.*

Let E be a fibre space, with fibre F a space of type $K(\pi, n)$, base space B simply connected, and characteristic class (minus transgression of fundamental class ι^n of fibre) $k^{n+1} \in H^{n+1}(B; \pi)$. Standard spectral sequence arguments yield the following results. For a field A of coefficients, $H^i(E; A) \approx H^i(B; A)$, $0 \leqq i \leqq n-1$; $H^n(E; A)$ is a group extension, of the subgroup $\{\rho \in \operatorname{Hom}(\pi, A) \mid \rho k^{n+1} = 0\} \subset \operatorname{Hom}(\pi, A)$, by $H^n(B; A)$; $H^{n+1}(E; A) \approx H^{n+1}(B; A)/\{\rho k^{n+1} \mid \rho \in \operatorname{Hom}(\pi, A)\}$; there is a composition series $H^{n+2}(E; A) \supset D_1 \supset D_2 \supset 0$, where $H^{n+2}(E; A)/D_1 \approx \{Sq^2 \rho \iota^n \in H^{n+2}(F; A) \mid \rho \in \operatorname{Hom}(\pi, A),\ Sq^2 \rho k^{n+1} = 0\}$, $D_1/D_2 \approx \{\sum b^2 \otimes \rho \iota^n \in H^2(B; A) \otimes H^n(F; A) \mid b^2 \in H^2(B; A),\ \rho \in \operatorname{Hom}(\pi, A),\ \sum b^2 \cup \rho k^{n+1} = 0\}$, and $D_2 \approx H^{n+2}(B; A)$. In case B is of type $K(\pi', n')$, $n > n' > 1$, (and thus E has two nonvanishing homotopy groups π, π') these results extend those obtainable from the relevant Eilenberg-MacLane theorem (Theorem IV, Ann. of Math. vol. 51 (1950) pp. 514–533). (Received June 9, 1955.)

829. A. H. Copeland, Jr.: *Binary operations on sets of mapping classes.*

Let $\pi(X; Y)$ be the set of homotopy classes of maps $f: X \to Y$, where X, Y are topological spaces, topological pairs, etc. If X is a space of Lusternik-Schnirelmann category 2, $x_0 \in X$, then for all spaces Y and $y_0 \in Y$ the set $\pi(X, x_0; Y, y_0)$ has a binary operation such that the class of the constant map is a (2-sided) identity and a map $g: Y, y_0 \to Y', y_0'$ induces a natural homomorphism $g_\#: \pi(X, x_0; Y, y_0) \to \pi(X, x_0; Y, y_0)$. The converse to this statement is proved. A similar study is made for the case Y fixed and X arbitrary. Here, the existence of a binary operation is related to Y being an H-space. As applications, a new proof that a contractible fibre is an H-space is found, and it is proved that if X is a suspended space, $x_1, \cdots, x_r \in H^n(X; G)$, $y \in H^q(G, n; H)$ then the Eilenberg-MacLane product $[x_1, \cdots, x_r; y] = 0 \in H^q(X; H)$. (Received July 5, 1955.)

830. H. S. M. Coxeter: *Abstract definitions for the plane crystallographic groups.*

The 17 two-dimensional space-groups (i.e., the 17 ways of repeating a flat design, as on wallpaper) were discovered empirically by the Moors in their decoration of the Alhambra. The first mathematical treatment was by Fricke and Klein (*Vorlesungen über die Theorie der automorphen Funktionen*, Leipzig, 1897). The groups were rediscovered by Pólya and Niggli (1924) and were shown by Nowacki (Schweiz. Min. Petr. Mitt. vol. 34 (1954) pp. 160–168) to be abstractly distinct. The present paper gives an abstract definition for each group in terms of two, three, or four generators. The two cases in which a single relation suffices have a connection with topology: $XY = YX$, generated by two translations, and $P^2 = Q^2$, generated by two parallel glide-reflections, are the fundamental groups of the torus and the Klein bottle, respectively. (Received July 15, 1955.)

831. M. L. Curtis: *The covering homotopy theorem.*

Let E and B be spaces and $p:E\rightarrow B$ be a map. A test space C is constructed such that if (E, B, p) satisfies the covering homotopy theorem for the space C, then it satisfies the covering homotopy theorem with respect to any topological space. For (E, B, p) a locally trivial fiber space, Huebsch has proved that one has the covering homotopy theorem for paracompact spaces. So, for example, if (E, B, p) is a locally trivial fiber space and E and B are metric, then one has the covering homotopy theorem for arbitrary spaces. (Received July 5, 1955.)

832. M. L. Curtis and R. K. Lashof (p): *Homotopy equivalence of fiber bundles.*

For a fiber bundle with fiber F, an associated fiber space with an H-space M as fiber is constructed. M consists of all maps of F into F which have homotopy inverses. These associated spaces play the same role for fiber homotopy equivalence that associated principal bundles play for bundle equivalence. Bundles are also classified according to fiber homotopy equivalence by a map of the loops of the base into the H-space M. It is known that any two tangent sphere bundles of a differentiable manifold are fiber homotopy equivalent, and the above theory is applied to this situation to show that some characteristic classes are independent of the differential structure. (Received July 13, 1955.)

833. W. F. Davison: *An equivalence relation for compact Hausdorff varieties.*

The basic structure is a compact Hausdorff space X and a class F of functions from X into a set Y, with the compatibility condition: for all f, $g\in F$, if E in X is closed then $g^{-1}f(E)$ is closed. A subset $g(X)$ of Y is topologized by defining G in $g(X)$ to be closed if and only if $g^{-1}(G)$ is closed; this topology is compact Hausdorff and g is closed and continuous. If $f(X)\subset g(X)$ then f is closed and continuous. If Φ is the unique uniformity for X then for $U\in\Phi$ and $x\in g(X)$, $gUg^{-1}(x)$ is a neighborhood of x. The collection of relations $\{V(U): U\in\Phi\}$, where $V(U)=gUg^{-1}gUg^{-1}$, is a basis for the unique uniformity for $g(X)$. Define $f\sim g$ if and only if for every $U\in\Phi$ there is a homeomorphism h: $X\rightarrow X$ such that $fh\subset V(U)g$. Then $\sim$ is an equivalence relation and $f\sim g$ implies $f(X)=g(X)$. If Y is topologized so that every compact set is closed then the continuous functions are compatible and the Y topology relativized to $g(X)$ agrees with the above topology. If X is Peanean and Y metric, then continuous functions are $\sim$-equivalent if and only if they are Frechét equivalent. (Received July 18, 1955.)

834. D. O. Ellis: *The topology of finitary approximation.*

A topology previously employed (R. Vaidyanathaswami, *Treatise on set topology* I, Madras, Indian Mathematical Society, 1947, p. 154) in the space of mappings of an arbitrary set into the unit interval is generalized to the case of an arbitrary metric space as range and is compared with the topologies induced by pointwise and uniform convergence. The methods are metric and topolattice-theoretic (Robert Bagley and David Ellis, *On the topolattice and permutation group of an infinite set*, Mathematica Japonicae vol. 3 (1954) pp. 63–70). (Received June 29, 1955.)

835. M. K. Fort, Jr.: *Extensions of mappings into n-cubes.*

Let X be an m-dimensional separable metric space, and let h be a mapping on a closed subset K of X into the closed n-cube I^n, $m\geq n>0$. It is proved that h can be extended to a mapping f on X into I^n such that $\dim f^{-1}(y)\cap(X-K)\leq m-n$ for each

$y \in I^n$. It is also shown that if $m \geqq n > 0$ and X is a compact metric space, then dim $X \leqq m$ if and only if there exists a mapping f on X into I^n such that $\dim f^{-1}(y) \leqq m-n$ for each $y \in I^n$. For the case $m=n$, this last result was obtained by M. L. Curtis and G. S. Young in *A theorem on dimension*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 159–161. (Received June 6, 1955.)

836. R. E. Fullerton: *The Carathéodory mapping theorem for two-dimensional manifolds.*

Let M be a compact two-dimensional manifold with boundary and G an open subset of M with a finite number of boundary components. Ends and prime ends of G can be defined in a manner similar to that used for simply connected plane regions. Certain ordered segments of ends and prime ends are defined and by rectifying these segments it is shown that there exists a mapping of $\overline{G}$ onto a subset $K \subset M$ which is a two manifold with boundary. This mapping is a piecewise linear homeomorphism on the interior of $\overline{G}$ and defines a homeomorphism between segments of ends and prime ends of G and boundary components of K where the order topology is used on the segments. (Received July 18, 1955.)

837*t*. C. D. Gorman: *A note on recurrent flows.*

The following theorem is proved which answers a question by Nemyckiĭ concerning a characterization of pseudominimal sets: if f is a continuous flow in a metric space R, consider the statements: (I) f is recurrent; (II) there exists a sequence $t_1 < t_2 < \cdots \to \infty$ of real numbers such that $\lim_{n\to\infty} f(A, t_n) = A$ for every closed subset A of R', (III) there exists a sequence $t_1 < t_2 < \cdots \to \infty$ of real numbers such that $\limsup_{n\to\infty} f(A, t_n) \subset A$ for every closed subset A of R; then (1) I implies II; II implies III, (2) if R is compact, then I, II, III are equivalent. (Received June 22, 1955.)

838. Alexandre Grothendieck: *On Stein manifolds.*

Let X be a connected Stein manifold of dimension $n \geqq 2$, and Y any Stein manifold. A holomorphic function on a connected neighborhood $\mathcal{U}$ of ∞ in X, with values in Y, extends holomorphically over X (this result is well known if Y is the complex field C). Hence the analogous result for holomorphic sections of an analytic fibre bundle E over X, with fibre Y, holds true if we know that E is Stein, for instance, if E is *principal* with *linear* complex structure group Y. Hence, if such a bundle is trivial on a neighborhood of ∞, it is trivial throughout. Interesting information on the fundamental group of X, including $H_1(\mathcal{U}, Z) \to H_1(X, Z)$ is onto, is obtained by taking Y as a discrete linear group. Another example: $Y = C^*$ (multiplicative group of C), then the preceding result becomes $H^1_*(X, 0_{C^*})$ (cohomology with compact supports, coefficients in the bundle of germs of nonvanishing holomorphic functions), which classically means: $H^2_*(X, Z) = 0$, that is $H_{2n-2}(X, Z) = 0$ for $n \geqq 3$, $H^2_*(X, Z) = H_2(X, Z)$ without torsion if $n = 2$ (Z = integers). (The corresponding result for Betti numbers is known.) (Received May 2, 1955.)

839. Mario Gutiérrez-Burzaco: *Uniform homotopy groups.*

Let X, Y be uniform spaces, and $I = [0, 1]$. A uniformly continuous map (abbreviated u-map) h: $X \times I \to Y$ is called a uniform homotopy (u-homotopy). Consider an arcwise connected subset $A \subset Y$ and a base point $y_0 \in A$. Let I^n be the unit n-cube in some Euclidean space, and let $\dot{I}^n$ be its boundary. For $0 < p < n$ let $I^{n,p} = I^n - \{x \in I^n \mid 0 < x_i < 1,\ 1 \leqq i \leqq p;\ x_j = 1/2,\ p < j \leqq n\}$. The set of classes of u-homotopic u-maps $f: (I^{n,p}, \dot{I}^n) \to (A, y_0)$ forms a group denoted by $\pi_{n,p}(A)$. Let $I_-^{n,p} = \{(x_1, \cdots, x_n)$

$|(x_1, \cdots, x_n+1) \in I^{n,p}\}$. Using $I^n \cup I^{n,p}$ as antecedent, one defines similarly the group $\pi_{n,p}(Y, A)$; for $0<p<n$ these are "intermediate" between the usual relative homotopy group $\pi_n(Y, A)$ and the absolute h. g. $\pi_n(Y)$. The following homomorphisms are then defined among the above four types of groups: $\partial: \pi_{n+1,p+1}(Y, A) \to \pi_n(A)$, $i: \pi_n(A) \to \pi_n(Y)$, $j: \pi_n(Y) \to \pi_{n,p}(Y)$, $k: \pi_{n,p}(A) \to \pi_{n,p}(Y, A)$, $l: \pi_n(A) \to \pi_{n,p}(A)$. For each positive $n-p$ a branched sequence can be formed thus: upper branches: $\cdots, \partial, i, j, \partial, \cdots$; lower branches: $\cdots, \partial, l, k, \partial, \cdots$. The following "exactness" properties hold: (i) $\partial = i^{-1}(0) \cap l^{-1}(0)$, (ii) $\partial^{-1}(0) = j + k$ (sum of complexes), (iii) $i = j^{-1}[k]$, (iv) $l = k^{-1}[j]$, (v) $j^{-1}(0) = i[l^{-1}(0)]$, (vi) $k^{-1}(0) = l[i^{-1}(0)]$, where ∂ stands for the image of ∂, and similarly for the other homomorphisms. The proof uses an extension theorem for u-homotopies defined on *uniform* absolute neighborhood retract (u-ANR) subsets of u-ANR metric spaces. (Received June 30, 1955.)

840. Frank Harary: *The number of dissimilar supergraphs of a linear graph.*

Two subgraphs of a linear graph are *similar* if there exists an automorphism sending one onto the other. A (p, q) *graph* is one with p vertices and q lines. A formula is obtained for the number of dissimilar occurrences of a given (a, b) graph G as a subgraph of all (p, q) graphs, $a \leq p$, $b \leq q$, i.e., for the number of dissimilar (p, q) supergraphs of G. The enumeration methods are those of Pólya, Acta Math. vol. 68 (1937) pp. 145–254. This result is then applied to obtain formulas for the number of dissimilar cycles and the number of dissimilar complete subgraphs (cliques) among all (p, q) graphs. The formula for the number of rooted graphs obtained in Trans. Amer. Math. Soc. vol. 78 (1955) pp. 445–463 is a special case of the number of dissimilar cliques. This note complements *The number of dissimilar line-subgraphs of a given graph*, Pacific Journal of Mathematics (to appear) in which the number of dissimilar (p, k) subgraphs of a given (p, q) graph is obtained. *Unsolved problem.* What is the number of dissimilar Hamilton lines in an n-cube? And in general, how many dissimilar times does a fixed graph H occur as a subgraph of a fixed graph G? (Acknowledgment: This work was supported by the Bell Telephone Laboratories, Summer 1955.) (Received July 14, 1955.)

841. L. J. Heider: *Directed limits on rings of continuous functions.*

Let βX be the collection of all maximal proper dual ideals α of subsets of set X. Real valued function f on X has a directed limit at α with value $\bar{f}(\alpha)$ provided a number $\bar{f}(\alpha)$ exists such that, for every $\epsilon > 0$, $|\bar{f}(\alpha) - f(x)| < \epsilon$ for all x in some subset of X in α. Every real bounded function on X has a unique directed limit at every α. Hausdorff quotient spaces of βX as under the Wallman topology together with the $\bar{f}$ determine X_T, βX_T, υX_T, $C^*(X_T)$, $C(X_T)$, $C(\beta X_T)$ and $C(\upsilon X_T)$ for every completely regular topology T on set X. Here βX_T and υX_T denote the Čech compactification and the Hewitt Q-space associated with X_T. Maximal ideals of $C(X_T)$ consist of elements f with zero sets $Z(f)$ in some α contained in a given maximal collection of α identified on $C^*(X_T)$. The real maximal ideals of $C(X_T)$ consist of elements f with $\bar{f}(\alpha) = 0$ on a given α such that $\bar{f}(\alpha)$ exists for all f in $C(X_T)$. Alternatively, they consist of elements with $\bar{f}(\alpha) = 0$ on a given α such that $\bar{f}(\alpha) = 0$ for an f in $C^*(X_T)$ if and only if $Z(f)$ is nonempty. (Received July 13, 1955.)

842. Erik Hemmingsen: *Some regular homeomorphisms in three space.*

Let M be a continuum which is the closure of an open set in E^3 and let f be a regular homeomorphism of M onto itself, i.e., one whose iterates form an equicontinuous family. If there is a point of M whose orbit closure (under the iterates of f) is connected, then the orbit closures of all the non-fixed points of M must be either simple closed curves or the product of two simple closed curves. (Received June 20, 1955.)

843*t*. V. L. Klee, Jr.: *Some finite-dimensional affine topological spaces.*

In Euclidean m-space E^m with $m \geq 2$, the following enlargements of the usual topology are considered: (T_1) A subset X of E^m is T_1-open if and only if each point of X is radially interior to X. (T_2) A subset X of E^m is T_2-open if and only if it can be expressed in the form $X = G \cup H$, where G is an open subset of E^m and each point h of H is radially interior to some closed subset of $G \cup \{h\}$. For each i, (E^m, T_i) is an affine topological space in the sense of Fréchet, though not a linear topological space in the sense of Tychonoff. It is proved that (E^m, T_1) is not regular, that (E^m, T_2) is completely regular but not normal, and that for each i, (E^m, T_i) is separable but lacks the Lindelöf property and fails to satisfy the first axiom of countability. (Received June 15, 1955.)

844*t*. S. D. Liao: *Periodic transformations and fixed point theorems.* I. *Cup products and special cohomology.*

The main aim of this paper is to introduce certain relationships which connect (i) the special cohomology modulo p arising from a transformation of prime period p acting on a space, and (ii) the multiplicative structure modulo p of the cohomology of the space and the fixed point set. Let T be a periodic transformation on a compact Hausdorff space X of period p with fixed point set F. Let X_0 be a closed subset of X, invariant under T. Certain relations, called "special relatedness," given by T between a (finite) set Δ of cohomology classes over the pair $(F, X_0 \cap F)$ and a cohomology class α_q over the pair (X, X_0) are considered. Let X_0' be also a closed subset of X, invariant under T. If a set Δ_1 of cohomology classes over $(F, X_0 \cap F)$ is specially related to a cohomology class α_q over (X, X_0) and if a set Δ_2 of cohomology classes over $(F, X_0' \cap F)$ is specially related to a cohomology class β_r over (X, X_0'), then special relatedness exists between $\Delta_1 \cup * \Delta_2$ and $\lambda \alpha_q \cup \beta_r$ in most cases, where $\Delta_1 \cup * \Delta_2$ consists of certain linear combinations of cohomology classes $\gamma_1 \cup \gamma_2$ over $(F, (X_0 \cup X_0') \cap F)$ with γ_1 in Δ_1, γ_2 in Δ_2, and λ is ± 1 or $\pm (1-T)^{p-1}$. The result is also generalized to relations concerning local special cohomology groups ${}^\rho H^i(X_0 | X, (X_0 \cap F) | F; Z_p)$ etc. and local cup products $H^i(X_0, Z_p) \cup H^j(X_0 | X, Z_p) \rightarrow H^{i+j}(X_0 | X, Z_p)$ etc. (Received July 25, 1955.)

845*t*. S. D. Liao: *Periodic transformations and fixed point theorems.* II. *Manifolds.* Preliminary report.

Let M^n be an n-dimensional compact connected, orientable manifold, and let X_0 be a closed subset of M^n. The Poincaré-Alexander duality in M^n can be inversed in terms of local cup or cap products between the cohomology or homology over X_0 and those in the neighborhoods of X_0 in M^n. The results in the previous paper (*Periodic transformations and fixed point theorems.* I) are applied to study the homology properties of fixed point sets of periodic transformations acting on manifolds. The main result of this paper is the following "torsion abelian group" theorem: The fixed point set of a torsion abelian group of differentiable transformations of class C^∞ acting on a sphere is a homology sphere over rationals. Other results are also established. For instance: Let T be a periodic transformation on an n-sphere S^n of prime period p.

Let M^q, N^r be respectively q-dimensional and r-dimensional compact connected orientable submanifolds in S^n with $q+r=n-1$. Suppose that M^q and N^r are disjoint, invariant under T, and linked modulo p. Then, if M^q contains no fixed points of S^n, N^r contains all the fixed points of $\dot{S}^n$. (Received July 25, 1955.)

846. G. R. Livesay: *Involutions on a manifold.*

Let $T:M\rightarrow M$ be a simplicial mapping of the triangulated compact manifold M onto itself, with no fixed points. The Smith-Richardson homology theory shows that there exists a "fundamental domain," $|C_n|$, of T satisfying (a) C_n+TC_n is the fundamental n-cycle on M, (b) ∂C_n is a T invariant $n-1$ cycle not T-homologous to zero, (c) C_n is $n-1$ connected, (d) if A and B are closed subcomplexes of M such that $A\cup B=M$, and no component of A contains an involution pair, then $\partial C_n\subset B$. If, in addition, M is unicoherent, and T_1 and T_2 are isometric involutions, then it follows that the boundaries of the corresponding fundamental domains must intersect. This result implies the following conjecture of M. H. Steinhaus and B. Knaster [Colloquium Mathematicum 1, 1947–1948]: Let $f:S_2\rightarrow E_1$ be continuous, p_1, p_2, p_3 any three points of S_2 ($=$2-sphere). Then there exists a rotation r of S_2 such that $f(rp_1)=f(rp_2)=f(rp_3)$. The manifold M, in this application, is $SO(3)$. T_1 and T_2 are the involutions obtained by right multiplication by r_2 and r_3, where r_i is defined by $r_ip_i=p_1$, $r_i^2=$identity. (Received May 2, 1955.)

847t. E. A. Michael: *On the completion of a uniformly LC^n or ANR metric space.*

Methods developed by J. Dugundji show that the results in §1 of S. Eilenberg and R. L. Wilder [Amer. J. Math. vol. 64 (1942) pp. 613–622] are valid without separability restrictions. This implies, in particular, that if Y is a dense subset of a metric space Z, and if Y is uniformly LC^n, then so is Z; in fact, if every continuous image of a k-sphere ($k\leq n$) in Y of diameter $<\delta(\epsilon)$ is contractible over a subset of Y of diameter $<\epsilon$, then the same is true for Z with $\delta(\epsilon)$ replaced by $\delta(\epsilon/2)$. The same methods yield analogous results for spaces γ which are uniformly ANR$_{\text{metric}}$ (i.e. to every $\epsilon>0$ there corresponds a $\delta(\epsilon)>0$ such that every partial realization of a simplicial CW-complex into Y of mesh $<\delta(\epsilon)$ can be extended to a full realization of mesh $<\epsilon$). Combined with Theorems 1 and 1′ of J. Dugundji and E. Michael [To appear in Proc. Amer. Math. Soc.], this yields the following corollary. If a metrizable space Y is ANR$_{\text{metric}}$, or AR$_{\text{metric}}$, or LC^n, or LC^n and C^n, then so is the completion of Y with some admissible metric ρ; moreover, ρ can be chosen to be uniformly finer than any given admissible metric. (Received July 18, 1955.)

848t. E. A. Michael: *On two extension theorems of Hanner and Dowker.*

The corollary in the preceding abstract yields the following sharpening of two theorems of O. Hanner and C. H. Dowker, concerning the extension of continuous functions whose domain is normal or collectionwise normal: If X is normal (resp. collectionwise normal), if $A\subset X$ is closed, and if Y is a metric (resp. separable metric) ANR$_{\text{metric}}$ with completion $\overline{Y}$, then every continuous $f:A\rightarrow Y$ can be extended to a continuous $g:U\rightarrow\overline{Y}$ for some open $U\subset A$; if moreover Y is an AR$_{\text{metric}}$, one can take $U=X$. The n-dimensional analogues of the theorems of Hanner and Dowker (E. Michael, Bull. Amer. Math. Soc. Abstract 61-5-631) can be similarly sharpened; one then obtains the above result with ANR$_{\text{metric}}$ replaced by LC^n, and AR$_{\text{metric}}$ replaced by C^n, provided one also assumes that every closed (in X) subset of $X-A$ has Lebesgue dimension $\leq n+1$. (Received July 18, 1955.)

849*t*. Deane Montgomery, Hans Samelson, and Leo Zippin: *Singular points of a compact transformation group.*

Let G be a compact Lie group which acts on an n-dimensional manifold M. Let s be the maximum of the dimensions of the orbits of G in M; it is known that the union of all orbits of dimension s is an open set X. It is proved now that the singular set, the union of all orbits of dimension less than s, has no interior points and is of dimension at most $n-2$; it follows that the set X is connected. In more detail it is shown that the subset of the singular set, made up of orbits of any fixed dimension $s'(<s)$, has dimension at most $n-1-(s-s')$. Finally it is shown that the orbit space M^* (the orbits under G considered as points) has dimension at most $n-s$. The proofs utilize the homology characterization of dimension and Gleason's construction of a cross section of an invariant set, if all the stability groups are conjugate to each other. (Received July 15, 1955.)

850*t*. P. S. Mostert and A. L. Shields: *Differentiability in semigroups on manifolds with boundary.*

Let S be a compact manifold with connected boundary B. Let S be a topological semigroup with identity, such that B is a Lie group. A. D. Wallace has raised the question (Bull. Amer. Math. Society Research Problem 61-1-10) whether the multiplication may be assumed differentiable throughout all of S. This is in general false. In fact, it is true if and only if the quotient semigroup S/B is the unit interval $[0, 1]$ under ordinary multiplication. In general, one has differentiability in a neighborhood of any point x such that x^2B contains no idempotents. (Received June 13, 1955.)

851*t*. P. S. Mostert and A. L. Shields: *On abelian semigroups which are topologically manifolds with boundary.*

Let S be a compact manifold with connected boundary B. Suppose S is an abelian topological semigroup with identity, such that B is a sub-semigroup. Then B is a toroidal group, and S is topologically and algebraically of the form: $D\times B$. Here D is one of the following: I. If S is orientable, then D is a semi-group on the unit disc with boundary which is a circle group. II. If S is non-orientable, then D is a semigroup on the Moebius strip with boundary circle a circle group. (Received June 13, 1955.)

852. P. S. Mostert (p) and A. L. Shields: *On the existence of one-parameter subsemigroups in topological semigroups.*

Define an *adequate local semigroup* to be a Hausdorf space S and a neighborhood W in S with the following properties: there is a continuous, associative multiplication defined for elements of $\overline{W}$ with products in S, there is an element $1\in W$ such that $x1=1x=x$ for all $x\in\overline{W}$, and the set $H(1)=[x\in\overline{W}$: there is a $y\in\overline{W}$ with $xy=yx=1]$ is a topological group contained in W. (In particular, a compact semigroup with identity is an adequate local semigroup.) *Theorem*: Let S be an adequate local semigroup with $\overline{W}$ compact. Suppose, moreover, that $\overline{W}$ contains no idempotents except 1 and that $H(1)$ is a Lie group and is not an open set. Then there exists a continuous function f: $[0, 1]$ to $\overline{W}$, such that $f(0)=1$, $f(a+b)=f(a)f(b)$ for $0\leq a+b\leq 1$, f is 1-1, and $f(a)$ is not in $H(1)$ for $a\neq 0$. (Received June 13, 1955.)

853. P. S. Mostert and A. L. Shields (p): *On the structure of semigroups which are topologically manifolds with boundary.*

Let S be a compact manifold with connected boundary B. Let S be a topological semigroup with identity, 1, such that B is a Lie group. Pierre Conner has shown that K, the minimal two-sided ideal of S, must be a group. Let e be the idempotent in K. Then there is an arc J in S from e to 1 such that: (1) J is a subsemigroup of the type described in a previous abstract on the unit interval, (2) J is in the center of S, (3) $S=JB$, (4) if $J_0=J-\{e\}$, and $S_0=S-K$, then J_0B, $J_0\times B$, and S_0 are all isomorphic and homeomorphic under the natural map. This completely determines the topological and algebraic structure of S. (Received June 13, 1955.)

854t. P. S. Mostert and A. L. Shields: *On topological semigroups on the unit interval.*

Let $A=[a, b]$ be a closed interval with a continuous, associative multiplication such that: $ax=xa=a$ and $bx=xb=x$ for all x (b is an identity and a is a zero). If A contains no other idempotents, then A is topologically and algebraically isomorphic to one of the following: I. $[0, 1]$ under ordinary multiplication; II. $[1/2, 1]$ with the multiplication $x*y=\max\ [1/2, xy]$, where xy denotes the ordinary product, and $*$ denotes the semigroup product. In the general case, let E be the set of idempotents of A. Then E is closed and contains a and b. Its complement consists of disjoint open intervals (c, d). The closure of each such interval is a subsemigroup of the form I or II. Finally, if we order A in the usual way from left to right, and if x, y are not in the same complementary interval of E, then $x*y=\min\ (x, y)$. (Monroe Faucett, *Compact semigroups irreducibly connected between two idempotents*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 741–747 has shown that A must be abelian, and that if A contains no idempotents other than a and b, and has no nilpotent elements, then A is I.) (Received June 13, 1955.)

855. Mary E. Rudin: *A separable normal non-paracompact space.*

It has been shown that every normal Hausdorff space with a countable base is paracompact. The purpose of this paper is to show that there is a separable, normal Hausdorff space that is not paracompact. (Received July 11, 1955.)

856. W. L. Strother (p) and C. E. Capel: *Fixed point property for the space of subsets of an absolute retract.*

Wojdyslawski [*Retracts absolus et hyperspaces des continus*, Fund. Math. vol. 32 (1939) pp. 184–192] asked "If X is a CAR* (=retract of a cube) is the space of closed subsets $S(X)$ a CAR*?" The question has been answered in the affirmative [W. L. Strother, *Fixed points, fixed sets, and M-retracts*, Duke Math. J., to appear] for the special case in which X is a Peano continuum. This result is used here to prove that if X is a CAR* then $S(X)$ has the fixed point property. (Received July 22, 1955)

857. Emery Thomas: *A generalisation of the Pontrjagin square cohomology operation.*

Let K be a finite regular cell complex and n an even integer. For each prime p we define a function $\mathfrak{P}_p: H^n(K; \bmod p^r) \to H^{pn}(K; \bmod p^{r+1})$ $(r \geqq 1)$, such that $\mathfrak{P}_2$ is the Pontrjagin square cohomology operation. Each function $\mathfrak{P}_p$ is topologically invariant, and hence is a cohomology operation. Using the complex projective space of $2p$ real dimensions, we construct an example of two complexes which known cohomology invariants fail to distinguish as to homotopy type, but which the operation $\mathfrak{P}_p$ does distinguish. The operation $\mathfrak{P}_p$ is then generalized to an operation $\mathfrak{P}_t$ ($t=0, 1, \cdots$) such that $\mathfrak{P}_t: H^n(K; \Pi) \rightarrow H^{tn}(K; \Gamma_t(\Pi))$, where Π is a finitely generated abelian group and $\Gamma_t(\Pi)$ is a subgroup of the ring $\Gamma(\Pi)$ defined by Eilenberg and MacLane [Ann. of Math. vol. 60 (1954) p. 107]. The sequence of functions $\mathfrak{P} = [\mathfrak{P}_t]_{t=0}^{\infty}$ then satisfies the same set of relations as do the generators of the ring $\Gamma(\Pi)$: namely, if $u \in H^n(K; \Pi)$, (i) $\mathfrak{P}_0(u) = 1$, the unit of the cohomology ring of K with integer coefficients; $\mathfrak{P}_1(u) = u$; (ii) $\mathfrak{P}_r(u) \cup \mathfrak{P}_s(u) = C_{r+s,r}\ \mathfrak{P}_{r+s}(u)$; (iii) $\mathfrak{P}_t(u+v) = \sum_{r+s=t} \mathfrak{P}_r(u) \cup \mathfrak{P}_s(v)$, for $u, v \in H^n(K; \Pi)$. (Received July 7, 1955.)

858*t*. A. D. Wallace: *One-dimensional homogeneous clans are groups.*

The theorem of the title is proved.

859. C. T. Yang: *Mappings from spheres to euclidean spaces.*

The following theorem is proved: A mapping from an (mn)-sphere to the euclidean m-space maps the end points of some n mutually orthogonal diameters into a single point. A Borsuk-Ulam's theorem (Fund. Math. (1933)), a Dyson's theorem (Ann. of Math. (2) (1951)) and some results of the author (Ann. of Math. (2) (1954) and Bull. Amer. Math. Soc. Abstract 60-6-547) are all included here as special cases. The proof of this theorem needs a generalized Borsuk-Ulam's theorem and a generalized Kakutani-Yamabe-Yujobô's theorem, established by the author (Ann. of Math. (2) (1954)), and the following lemma. Let X be a compact subset of a real projective space P such that the homomorphism $H_{m+n}(X) \rightarrow H_{m+n}(P)$, induced by the inclusion mapping, is not trivial, where homology groups are Čech groups with integers mod 2 as coefficients. Then for any mapping f of X into the euclidean m-space E there is a point y of E such that the homomorphism $H_n(f^{-1}(y)) \rightarrow H_n(P)$ is not trivial. (Received July 15, 1955.)

860. J. A. Zilber: *Type-complexes.* Preliminary report.

Let $\{\pi_n\}$ denote a sequence of groups, $n=1, 2, \cdots$, abelian for $n>1$, together with an operation of π_1 on each π_n, $n>1$. For any complete semi-simplicial (css) complex K, let $|K|$ be the topological space obtained from a family $\{\Delta_\sigma\}$ of disjoint euclidean simplexes, $\sigma \in K$, $\dim \Delta_\sigma = \dim \sigma$, with the identification induced by the barycentric maps $\alpha: \Delta_{\sigma\alpha} \rightarrow \Delta_\sigma$. An algebraic construction is described which assigns to any system $\{\pi_n\}$ a family $\mathfrak{K}\{\pi_n\}$ of css complexes, called type-complexes, and an equivalence relation in $\mathfrak{K}$. The study of homotopy type of polyhedra and homotopy classes of maps is equivalent with the study of type-complexes and simplicial maps thereof, in consequence of the following facts. 1. For any space X, every minimal complex at $x \in X$ is isomorphic with every member of a uniquely determined equivalence class in $\mathfrak{K}\{\pi_n(X, x)\}$. 2. For any abstract system $\{\pi_n\}$, any complex $K \in \mathfrak{K}\{\pi_n\}$ is isomorphic with the minimal complex of the topological space $|K|$, and the system $\{\pi_n\}$ is isomorphic with the system $\{\pi_n(|K|, e)\}$, e the 0-simplex of K. 3. For any css complex L, if M is a minimal complex of $|L|$, then $|M|$ and $|L|$ are of the same homotopy type. 4. Any continuous map of $|L|$ into $|K|$, $K \in \mathfrak{K}$, is homotopic to a simplicial map. (Received July 18, 1955.)

J. W. T. Youngs,
Associate Secretary

REPORT OF THE TREASURER

The Treasurer is this year presenting to the membership an abridged statement of the Society's financial position, set up in a semi-informal narrative style. It is believed that most members of the Society will find such a summary preferable to the more formal, but nevertheless abridged, report formerly printed in the Bulletin.

A copy of the complete Treasurer's Report as submitted to the Trustees and the Council will be sent to any member requesting it from the Treasurer at the Providence office. Moreover, the Treasurer will be happy to answer any questions members may wish to put to him concerning the Society's financial affairs.

In general, it may be said that the Society's finances continue satisfactory, although the margin between income and expense is very narrow—about 95% of our income was spent during the year under review.

Returns on invested funds this year have been at the rate of 4.8% computed on book value, after deduction of custodial expense. This is a small increase as compared with fiscal 1954.

I

A DESCRIPTION OF THE FINANCIAL POSITION OF THE SOCIETY AS OF MAY 31, 1955

The Society had Cash on deposit		
In the Rhode Island Hospital Trust Co.	$ 47,178.84	
In various interest-bearing savings accounts	67,058.40	
In petty cash and drawing accounts in Providence and New Haven	1,050.00	$115,287.24
It had reserves invested until needed in Government bonds		54,605.26
There was owing to it		
By the United States Government	$ 21,992.52	
By members, subscribers and others	9,169.70	
By an insurance company, to settle a claim	12.00	31,174.22
It had in stamps and in the postage meter		519.55
And had temporarily advanced to the Investment Account		1,187.99
Making a total of CURRENT ASSETS of		$202,774.26
The Society also held investment securities valued at		319,039.83
(The market value, May 31, 1955 was $415,901.01)		
TOTAL ASSETS, therefore, were		$521,814.09

Offsetting these assets, the Society			
Owed members, subscribers and vendors..	$3,047.70		
Held contributions designated for special purposes........................	1,153.29		
Had credited in advance uncollected income of..........................	38.91		
Making a total of Current Liabilities, money it might have to pay out on short notice, of........... ...		$ 4,239.90	
Held funds received from various special sources to support particular projects, such as the Summer Institute, the Register of Mathematicians, etc.........		38,843.01	
Had advanced for recovery from future sales for various Society publications—Colloquium and Survey volumes, Birkhoff papers, Russian translations, etc....		35,072.88	
And held in its General Fund, the result of 67 years of prudent operations, the sum of..................		124,618.47	
Thus accounting for all the CURRENT FUNDS......			$202,774.26
The Invested Funds represent the following:			
(1) The Endowment Fund, largely the gift of members about thirty years ago..................		$ 94,000.00	
(2) The Library Proceeds Fund, derived from the sale of the Society's Library in 1950...........		66,000.00	
(3) The Prize Funds—Bôcher, Cole, Moore.......		6,575.00	
(4) The Mathematical Reviews Fund, a gift of the Rockefeller Foundation to make possible the establishment of the Reviews in 1940..........		80,000.00	
(5) Reserves established by the Trustees to protect the life memberships and life subscriptions formerly available, and as a "hedge" against investment losses................................		42,725.37	
(6) Other funds, derived mainly from bequests to the Society by members which the Trustees were either required to invest or which they have invested at their option—the income being used for the general purposes of the Society............		28,551.47	
A total of invested funds of.................... ..		$317,851.84	
And this, together with the cash due the current funds (see above) of....		1,187.99	
Accounts for the total holding of investment securities of..........			$319,039.83
TOTAL LIABILITIES, therefore, were...			$521,814.09

II

AN ACCOUNT OF THE FINANCIAL TRANSACTIONS OF THE SOCIETY DURING THE FISCAL YEAR 1954–1955

The Society has two types of receipts—funds for special purposes and projects, and the General Fund, from which are met the general operating expenses of the organization, including the publication of the Bulletin, the Proceedings and the Transactions. Income from sales of and subscriptions to these journals is placed in the General Fund, but in practice is allocated to the expenses of the journals themselves, as though such income were in fact special. It is so treated in the following presentation.

To meet its GENERAL obligations, the Society RECEIVED:

From dues and contributions of individual members	$57,629.25		
Less enlisted men's special discount	507.00	$57,122.25	
From dues of institutional members		21,075.00	
From investments and trusts		21,386.87	
From contracts, in payment of indirect costs		14,419.62	
From miscellaneous sources		2,136.60	
Total General Receipts			$116,140.34

These funds were EXPENDED

For general administrative expenses (Includes maintenance of financial records and accounts, mailing lists and other membership records, office supplies and equipment, telephone, stationery and printing, programs and other expenses of meetings)		$59,098.63	
To meet deficits in Society publications:			
Bulletin: $16,366.20 − $6,047.66 (income)	$10,318.54		
Proceedings: $22,182.14 − $6,308.71 (income)	15,873.43		
Transactions: $23,929.63 − $19,296.16 (income)	4,633.47		
Mathematical Reviews: $77,865.40 − $75,748.66 (income)	2,116.74	32,942.18	
In subsidies to non-Society publications		7,889.81	
To repay the balance of the debt incurred in moving Society offices to Providence		7,884.17	
To cover the Society's share of the cost of certain joint projects with other organizations: the Policy Committee, the Combined List of Members, the Employment Register		1,209.47	
For miscellaneous expenses		1,050.96	
Total General Expenses			$110,075.22

Leaving an EXCESS of INCOME over EXPENSES of.................. $ 6,065.12

Which has been retained in the General Fund for future contingencies.

The nature of the Special Funds is generally indicated in the statement of Liabilities given above in Part I.

Respectfully submitted,

ALBERT E. MEDER, JR.
Treasurer

August 8, 1955

THE RESEARCH CONFERENCE ON THE THEORY OF NUMBERS IN PASADENA

A *Research Conference on the Theory of Numbers* was held at the California Institute of Technology from June 22 to June 24, 1955 in conjunction with the thirty-sixth Annual Meeting of the Pacific Division of the American Association for the Advancement of Science. It was supported by a grant from the National Science Foundation to the California Institute of Technology. The members of the organizing committee were Sarvadaman Chowla, University of Colorado, Hans Rademacher, University of Pennsylvania, J. B. Rosser, Cornell University, J. T. Tate, Harvard University, Morgan Ward, California Institute of Technology, A. L. Whiteman (chairman), University of Southern California. Professor H. F. Bohnenblust of the California Institute of Technology was the local director of arrangements. The Conference was open to all interested mathematicians, and the total attendance was approximately seventy-five.

Through the courtesy of Caltech and the A.A.A.S. hotel accommodations, lounges, recreational and dining facilities were made available to the Conference participants. A banquet was held at Taix French Restaurant on Thursday evening, June 23. On Friday afternoon a group photograph of those attending the meeting was taken in front of Throop Hall.

The program of the Conference consisted of twenty-four invited addresses. Fifteen of the speakers received subsidization from the National Science Foundation. There were six sessions presided over respectively by Professors J. B. Rosser, Morgan Ward, Sarvadaman Chowla, Emil Artin, Richard Brauer, and D. H. Lehmer.

Bound sets of abstracts of most of the papers on the program were circulated to all participants at the beginning of the Conference. It was felt that the value of the Conference was thereby greatly enhanced. The papers were presented in the following order with Dr. Joseph Lehner's paper being read by title.

Ernst G. Straus: *The arithmetic of analytic functions.*

Olga Taussky Todd: *Matrix methods in algebraic number theory.*

Morgan Ward: *Divisibility sequences.*

Emma Lehmer: *On the location of Gauss' sums.*

Albert Leon Whiteman: *A sum connected with the partition function.*

Tom M. Apostol: *The approximate functional equation of Hecke's Dirichlet series.*

Richard Bellman: *The generalised theta-functions of Hecke, Siegel and Maass.*

Joseph Lehner: *Partial-fraction decompositions and expansions of zero.*

Lowell Schoenfeld: *On the order of the zeta function near the line* $\sigma=1$.

Atle Selberg: *Discontinuous groups and harmonic analysis with applications to Dirichlet series and modular forms.*

D. H. Lehmer: *New results about Ramanujan's tau function.*

P. T. Bateman: *General properties of partition functions.*

Richard Brauer: *Number-theoretical investigations on groups of finite order.*

J. T. Tate: *Cohomology and class-field theory.*

N. C. Ankeny: *Universal zeta functions.*

H. S. Vandiver: *Fermat's last theorem.*

Gordon Pall: *Simultaneous representation by adjoint quadratic forms.*

Marshall Hall, Jr.: *The minima of binary quadratic forms.*

Harvey Cohn: *Accessibility of algebraic numbers with rounded norms.*

Emil Artin: *The classical finite simple groups and their orders.*

J. Barkley Rosser: *Some new extensions of Brun's method.*

Sarvadaman Chowla: *Remarks on Bernoulli numbers.*

Ivan Niven: *Normal numbers.*

Alfred T. Brauer: *The Schnirelman density of the sum of two sequences of which one has positive density.*

The Conference members expressed regret that Professor Paul Erdős, who had been chosen to be the first speaker at the Conference, was unable to attend. The Conference members strongly supported the request being made by the American Mathematical Society that Professor Erdős be permitted to reenter the United States of America.

A resolution of thanks and appreciation to the California Institute of Technology was read by Professor D. H. Lehmer and unanimously approved.

By fostering open discussions the Conference proved to be a valuable means of disseminating information and ideas about fields in which vigorous advances are being made. The inspiration provided by the informal contacts between the students and the masters will unquestionably serve as an impetus for future important research.

ALBERT LEON WHITEMAN

BOOK REVIEWS

Functionals of finite Riemann surfaces. By M. M. Schiffer and D. C. Spencer. Princeton University Press, 1954. 10+451 pp. $8.00.

In the terminology of the authors a finite Riemann surface is a Riemann surface of finite genus with a finite number of nondegenerate boundary components. Such a surface has a double, obtained by reflection across the boundary, and one of the main features of the book is the systematic use of this symmetrization process. The authors even introduce a disconnected double of a closed Riemann surface, and for nonorientable Riemann surfaces the double is a two-sheeted orientable covering surface.

The first three chapters contain a development of the classical theory. The topological classification is taken for granted, but the existence theorems are carefully proved by the method of orthogonal projection in de Rham's and Kodaira's version. The treatment of Abelian differentials includes the Riemann-Roch theorem. A somewhat disturbing omission in this introductory part is the lack of practically all references to covering surfaces, in spite of their use later on.

The more advanced theory begins with the fourth chapter. The main interest, throughout the book, is attached to the so-called *domain functionals*, represented by Green's and Neumann's function, harmonic measure, the period matrix, and similar quantities. The first task is to compare different functionals of the same domain, and this is accomplished by deriving all functionals from a common source. The simplest way is to start from normalized integrals of the third kind.

We introduce some of the notations used by the authors. For a closed surface $\mathcal{J}$, let $\Omega_{qq_0}(p)$ be the integral with logarithmic poles at q, q_0 and single-valued real part. Then

$$\frac{\partial^2\Omega_{qq_0}(p)}{\partial p\partial q} = -\frac{1}{[z(p) - z(q)]^2} + \text{regular terms}$$

is a bilinear differential on $\mathcal{J}$ (independent of the auxiliary point q_0). Suppose now that $\mathcal{J}$ is the double of $\mathcal{M}$, and let $\bar{q}$ denote the symmetric point of q. The authors introduce

$$L(p, q) = -\frac{1}{\pi}\frac{\partial^2\Omega_{q\bar{q}}(p)}{\partial p\partial q}$$

and prove on one hand the symmetry relations

$$L(p, q) = L(q, p), \qquad L(p, \bar{q}) = (L(q, \bar{p}))^{-},$$

on the other hand the reproduction formulas

$$(df, L(p, \bar{q})dz) = -f'(q), \qquad (df, L(p, q)dz) = 0.$$

Here $df \in M$, where M is the Hilbert space of square integrable differentials on $\mathfrak{M}$, and the inner products are extended over $\mathfrak{M}$. These formulas show that $-L(p, \bar{q})$ is the Bergman kernel function for the space M. The Green's and Neumann's functions are expressed by

$$G(p, q) = \tfrac{1}{2}\{\Omega_{q\tilde{q}}(p) - \Omega_{q\bar{q}}(\bar{p})\},$$

$$N(p, q, q_0) = \tfrac{1}{2}\{\Omega_{qq_0}(p) + \Omega_{\bar{q}\bar{q}_0}(\bar{p}) + \Omega_{qq_0}(\bar{p}) + \Omega_{\bar{q}\bar{q}_0}(p)\}.$$

Finally, the Abelian differentials of the first kind that correspond to a Betti basis $\{K_\mu\}$ can be defined as

$$Z_\mu'(q) = -\frac{2}{\pi}\int_{K_\mu} \frac{\partial^2 G(p, q)}{\partial p \partial q}\, dp.$$

This definition implies

$$(dw, dZ_\mu) = -\int_{K_\mu} dw.$$

The next chapter is concerned with the situation that arises when a surface $\mathfrak{M}$ is imbedded in another surface $\mathfrak{R}$. Let the corresponding kernels be $L(p, q)$ and $\mathcal{L}(p, q)$ respectively. The difference

$$l(p, q) = L(p, q) - \mathcal{L}(p, q)$$

is then regular on $\mathfrak{M}$ and can be shown to satisfy a number of important identities. It is easy to generalize to the case of an $\mathfrak{M}$ which is not necessarily connected.

The consideration of $l(p, q)$ leads not only to identities, but also to inequalities which express that certain quadratic forms are positive definite. These inequalities characterize the imbedding so completely that it becomes possible to decide whether a locally given bilinear differential can or cannot be extended to all of $\mathfrak{M}$. With the help of this result the authors derive a very strong theorem which concerns the possibility of imbedding one Riemann surface in another, in terms that generalize the well-known schlichtness conditions of Grunsky. It is regrettable that the otherwise lucid reasoning becomes quite obscure at this crucial point. Certainly, the reader expects to be told, in no uncertain terms, how the imbedding is constructed. Instead, the authors merely hint that a certain relation may be used to extend the mapping "by continuation." It seems likely that a proof can be carried out along such lines, but the impression remains that an essential and perhaps difficult part of the argument has been omitted.

The theory of subdomains is continued in the sixth chapter which by its novelty and depth is one of the most fascinating sections of the book. Let M and R be the spaces of square integrable differentials over $\mathfrak{M}$ and $\mathfrak{R}$, $\mathfrak{M}\subset\mathfrak{R}$. Given $f'\in M$, the inner product $(g', f')_{\mathfrak{M}}$ defines a linear function of $g'\in\mathfrak{R}$, and consequently there exists a $Tf'\in\mathfrak{R}$ with the property that

$$(g', f')_{\mathfrak{M}} = (g', Tf')_{\mathfrak{R}},$$

for all $g'\in\mathfrak{R}$. The linear mapping T is considered as a canonical mapping of M into R.

Actually, the authors proceed somewhat differently and define directly two linear operators.

$$T_f(q) = (\mathcal{L}(q, p)dp, f'dp)_{\mathfrak{M}}, \qquad \bar{T}_f(q) = (f'dp, \mathcal{L}(p, \bar{q})dp)_{\mathfrak{M}}.$$

Here $\bar{T}_f$ corresponds to the operator that we described above, while T_f is a generalized Hilbert transform which has to be evaluated as a principal value. More generally, T and $\bar{T}$ may be defined for differentials that are analytic in $\mathfrak{M}$ and $\mathfrak{R}-\mathfrak{M}$ but may be discontinuous on the boundary of $\mathfrak{M}$. Then $\bar{T}$ is an isometry, and $T_{f_1}\perp\bar{T}_{f_2}$ with respect to $\mathfrak{R}$. The algebra generated by T and $\bar{T}$ satisfies the relations

$$T^2 = I + \bar{T}, \qquad \bar{T}^2 = -\bar{T}, \qquad T\bar{T} = \bar{T}T = 0.$$

The first of these is far from evident.

For many purposes it is sufficient to consider operations in the domain $\mathfrak{M}$ only. It is proved that the corresponding restrictions t, $\bar{t}$ of T, $\bar{T}$ are completely continuous. Thus one obtains an interesting spectral theory which yields powerful generalizations of the methods of Poincaré and Fredholm in classical potential theory.

The last part of the book, except for an appendix on higher dimensional Kähler manifolds, is devoted to variational techniques. Here again, the main problem is to find the variations of the domain functionals. The technique, which goes back to Hadamard and has been perfected by Schiffer, is described in great detail, and different types of variations are introduced which involve cutting holes, attaching handles, cross-caps and disks. The connections with quadratic differentials are explained and utilized.

An important problem, which has hardly been touched before, is the construction of variations which preserve the conformal type. Once more, the reader will find that the authors are reluctant to detail proofs that cannot be readily expressed through analytic formulas. For instance, they use the natural device of introducing, as conformal modules, the real periods of a differential dZ (with given imaginary periods) and the integrals

$$\int_{p_1}^{p_i} dZ$$

between the zeros p_i of Z'. In the situation that arises through a variation they assert, as a triviality, that these quantities determine the Riemann surface. The reviewer agrees that a precise proof is not difficult, but it will necessarily involve considerations that the text does not even touch upon. Apart from such minor inconsistencies the proofs seem satisfactory and the results are very far-reaching.

Numerous applications are given, and it is shown, in particular, that the period matrix depends differentiably on the moduli. There is no attempt, however, to introduce a complex structure on the space of Riemann surfaces.

The last chapter is rather loosely connected with the rest of the book. It gives a concise and readable presentation of the Hodge theory of harmonic differentials on Kählerian manifolds, together with its extension to manifolds with boundary.

The authors must be congratulated on having got out of their hands and before the eyes of the mathematical public a volume that must have been very difficult to edit. It contains a plethora of ideas, each interesting in its own right, and on the whole they have been tied together in a successful manner. At a first reading the wealth of formulas is almost forbidding, especially since the authors have not been very fortunate in their choice of notations. However, the patient reader will be richly rewarded and will become aware of many challenging problems that remain to be solved.

The publishers have gone out of their way to give the formulas an attractive appearance, and the proofreading is excellent.

LARS V. AHLFORS

Topological dynamics. By W. H. Gottschalk and G. A. Hedlund. American Mathematical Society Colloquium Publications, vol. 36. Providence, American Mathematical Society, 1955. 8+148 pp. $5.10.

The authors begin by explaining that by *topological dynamics* they mean "the study of transformation groups with respect to those topological properties whose prototype occurred in classical dynamics." Thus, they say, *topological* has reference to the mathematical content of the subject and *dynamics* to its historical origin.

Part I (*The theory*) defines transformation groups and studies their recursion properties; Part II (*The models*) applies the results to some classically important special cases.

A transformation group is a topological space X, a topological group T, and a continuous mapping $(x, t)\to xt$ from $X\times T$ into X such that if e is the identity element of T, then $xe=x$ for all x in X, and if x and t are arbitrary elements of T, then $(xs)t=x(st)$ for all x in X. Section 1 introduces the basic concepts associated with transformation groups (e.g., isomorphism, transformation subgroup, direct product, invariance, orbit), and establishes the connections among them. The main purpose of section 2 is to conclude, under various sets of appropriate hypotheses, that the orbit-closures constitute a partition of X. A subset A of the group T is called *syndetic* (or, rather, left syndetic) if there exists a compact set K such that $AK=T$. A typical theorem states that if X is compact and minimal under T, and if S is a syndetic invariant subgroup of T, then the orbit-closures under S constitute a star-closed decomposition of X. Explanation of terms: minimality means that the orbit under T of every point of X is dense in X; a decomposition is a partition of X into compact sets; and a partition is star-closed if, for every closed set E in X, the union of all those sets of the partition that intersect E is a closed set.

Section 3 defines various generalizations of periodicity. The period of the group T at a point x in X is the set P of those elements t for which $xt=x$; the group T is said to be periodic at x if P is a syndetic subset of T. Sample theorem: if X is a Hausdorff space, and if T is locally compact and separable, then a necessary and sufficient condition that T be periodic at a point x is that the orbit xT be compact. The broadest generalization of periodicity is called recursion; its special cases include almost periodicity, regular almost periodicity, and recurrence. The group T is said to be almost periodic at x if, corresponding to every neighborhood U of x, there exists a (left) syndetic subset A of T such that $xA\subset U$. If, in this definition, "syndetic subset" is replaced by "syndetic invariant subgroup," the resulting concept is called regular almost periodicity; if, on the other hand, "syndetic subset" is replaced by "extensive subset," the resulting concept is called recurrence. Explanation of terms: a subset of T is extensive if it intersects every replete semigroup of T; a subset of T is replete if it contains some (bilateral) translate of each compact subset of T. The authors treat all these concepts (and several others) simultaneously by the device of defining recursion in terms of the neutral phrase "admissible set." Thus, for example, the special concept of recurrence is obtained from the general concept of recursion by interpreting "admissible" to mean "extensive." As far as possible the theorems concerning recursion are proved in the general setting. The principal result (the inheritance theorem) asserts that certain subgroups of T are recursive if and only if T is recursive.

Section 4 specializes recursion to almost periodicity. The inheritance theorem asserts here that if T is locally compact and if S is a closed syndetic invariant subgroup of T, then a necessary and sufficient condition that S be almost periodic at a point x is that T be almost periodic at x. The main purpose of the section is to study the relations of almost periodicity to various equicontinuity conditions, and, toward the end, to prove the existence of a generalized mean for certain almost periodic functions with values in a uniform space. Sections 5 and 7 do for regular almost periodicity and for recurrence, respectively, what section 4 does for almost periodicity; section 6 supplies the necessary auxiliary facts about replete semigroups. To ensure the existence of sufficiently many replete semigroups, the authors restrict the treatment of recurrence to generative groups, i.e., to abelian groups generated by a compact neighborhood of the identity.

Section 8 proves that periodicity or recurrence properties of a generative group are equivalent to incompressibility properties. Call a subset M of X compressible (this is not the authors' terminology) if there exists a replete semigroup P in T such that $MP \subset M$ and such that the set $M-MP$ is of second category; in the contrary case call M incompressible. Sample theorem: a necessary and sufficient condition that every subset of X be incompressible is that the set of those points at which T is not periodic be a set of the first category.

Sections 9 and 10 study transitivity and asymptoticity, respectively, and the relations of these concepts to various forms of recursion. The group T is transitive at x if the orbit of x is dense in X; related concepts are regional transitivity (each non-empty open set intersects some transform of every non-empty open set), universal transitivity (every orbit in X is X), etc., etc. To define asymptoticity, suppose that X is a compact metric space and that f is a homeomorphism from X onto X. A point x of X is said to be (positively) asymptotic to a closed f-invariant subset B of X if the distance from the nth f-transform of x to B tends to 0 as n tends to $+\infty$.

Part I concludes with section 11 on function spaces; this is essentially a standard treatment of the subject, with special emphasis on uniform spaces, and, correspondingly, on questions of equicontinuity and completeness.

Part II has three sections. Section 12 treats symbolic dynamics. This means, by definition, the combinatoric study of a particular transformation σ on a particular space X. The space X is the set of all sequences $\{x_n\}$, where $x_n \in S$ for $n=0, \pm 1, \pm 2, \cdots$, and where S is a finite set containing more than one element; the transformation σ (the shift) is defined by $(\sigma x)_n = x_{n+1}$. Section 13 culminates in the theorem concerning the (topological, not measure-theoretic) transitivity properties of the geodesic flow associated with a surface of constant negative curvature. Section 14 studies the recursion and transitivity properties of certain cylinder homeomorphisms. A cylinder homeomorphism ϕ is determined by a topological space Y, a homeomorphism θ from Y onto Y, and a continuous function f from Y into $\mathfrak{R}$ (= the real line). The homeomorphism ϕ acts on the product space $Y \times \mathfrak{R}$ by allowing θ to act on the first coordinate y and by translating the second coordinate by the amount $f(y)$.

The most striking virtue of the book is its organization. The authors' effort to arrange the exposition in an efficient order, and to group the results together around a few central topics, was completely successful; they deserve to be congratulated on a spectacular piece of workmanship. The results are stated at the level of greatest available generality, and the proofs are short and neat; there is no unnecessary verbiage. The authors have, also, a real flair for the "right" generalization; their definitions of periodicity and almost periodicity, for instance, are very elegant and even shed some light on the classical concepts of the same name. The same is true of their definition of a syndetic set, which specializes, in case the group is the real line, to Bohr's concept of a relatively dense set.

The chief fault of the book is its style. The presentation is in the brutal Landau manner, definition, theorem, proof, and remark following each other in relentless succession. The omission of unnecessary verbiage is carried to the extent that no motivation is given for the concepts and the theorems, and there is a paucity of illuminating examples. The striving for generality (which, for instance, has caused the authors to treat uniform spaces instead of metric spaces whenever possible) does not make for easy reading. The same is true of the striving for brevity; the shortest proof of a theorem is not always the most perspicuous one. There are too many definitions, especially in the first third of the book; the reader must at all times keep at his finger tips a disconcerting array of technical terminology. The learning of this terminology is made harder by the authors' frequent use of multiple statements, such as: "The term $\{$*asymptotic*$\}$ $\{$*doubly asymptotic*$\}$ means negatively $\{$or$\}$ $\{$and$\}$ positively asymptotic."

Conclusion: the book is a mine of information, but you sure have to dig for it.

PAUL R. HALMOS

Theorie der linearen Operatoren im Hilbert-Raum. By N. I. Achieser and I. M. Glasmann. Berlin, Akademie-Verlag, 1954. 13+369 pp. 28.00 DM.

The book under review is a translation from the Russian. The original version was published in 1950; a detailed review of it (by Mackey) appears in vol. 13 of Mathematical Reviews. There are by now well over a dozen books one of whose chief purposes is to introduce the reader to the concepts and methods of operator theory in Hilbert space; in the last five or six years they have been appearing at the rate of slightly more than one a year. In view of these facts, a detailed, discursive review of still another contribution to the expository literature of the subject does not seem necessary. What follows is a list of the standard topics that are treated, a brief description of some of the special topics that the authors chose to include, and an appraisal of the didactic value of the book.

Standard topics: definition of Hilbert space, subspaces, bases, linear functionals and their representation, bounded operators, projections, unitary operators, unbounded operators, spectrum, resolvent, graph, the spectral theorem for not necessarily bounded self-adjoint operators and for unitary operators, defect indices, Cayley transforms, extensions of symmetric operators. Comments: infinite-dimensionality is built into the definition of Hilbert space; separability is not part of the definition, but is usually assumed; weak convergence is treated from the sequential point of view only.

Special topics: completely continuous normal operators; Neumark's generalized extension theory for symmetric operators; Krein's generalized resolvents; differential operators. The spectral theory of completely continuous normal operators is treated in detail before the more general spectral representations are attacked; it is made to serve, quite effectively, as a psychological stepping stone. This occurs in a chapter in the main body of the book. The Neumark-Krein theory and differential operators appear in two appendices. The first appendix states and proves Neumark's theorem on positive operator measures (they are compressions of spectral measures) and Neumark's extension theorem for symmetric operators (they are compressions of self-adjoint operators). The same appendix presents Krein's representation theorem for the generalized resolvents of symmetric operators with defect index (1, 1) (in terms of analytic mappings of the upper half-plane into itself) and a study of quasi-self-adjoint extensions. (If A is symmetric with defect index (m, m), $m < \infty$, a quasi-self-adjoint extension is an operator B such that $A \subset B \subset A^*$ and such that the co-dimension of the domain of A in the domain of B is equal to m.) The second appendix studies self-adjoint extensions of differential operators; it concludes with some concrete examples (such as the differential equations satisfied by the Bessel functions).

A student would find the detailed reading of the book highly profitable. If the authors need an analytic fact, they prove it. They prove, in particular, the completeness of L_2, they prove Plancherel's theorem, and they prove Bochner's theorem on the representation of positive definite functions. (The latter is used to prove the spectral theorem for self-adjoint operators; similarly, the spectral theorem for unitary operators is made to follow from the solution of the trigonometric moment problem.) The book contains many refreshing comments. Examples: the distance from a vector y to the span of the linearly independent vectors $x_1, \cdots, x_n$ is the quotient of the Gramian of $x_1, \cdots, x_n, y$ by the Gramian of $x_1, \cdots, x_n$; a Jacobi matrix represents a completely continuous operator if and only if its coefficients tend to zero; if a one-to-one mapping of a Hilbert space onto itself preserves inner products, then it is linear (and therefore unitary). The book also contains many illuminating examples worked out in complete detail; they include multiplication and differentiation on function spaces and shifts on sequence spaces. The authors' treatment of cyclic (self-adjoint or unitary) operators provides a very good introduction to multiplicity theory (which they do not treat). The style throughout is unhurried, precise, and clear.

PAUL R. HALMOS

Quadratische Formen und orthogonale Gruppen. By Martin Eichler. (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 63.) Berlin, Göttingen, Heidelberg, Springer, 1952. 12+220 pp. 24.60 DM; bound, 27.60 DM.

This is a competent and highly original monograph on the algebraic and arithmetic theory of quadratic forms from the modern point of view of Witt's Crelle 176 paper (1937). Much of the work is done in an arbitrary classical product formula (c.p.f.) field—namely a finite algebraic extension of the rational field or a field of rational functions of one variable over a Galois field—subject only to the perpetual assumption that the characteristic is not 2. It contains the classical theory of equivalence of forms under linear transformations with coefficients in the ground field, a new arithmetical theory mainly due to the author, and a new treatment, with proofs, of a considerable part of the recent deep analytical work of Hecke and Siegel, all from a new point of view to be described soon. Obviously it is an extremely important and valuable contribution to the literature of its subject.

Like Witt, Eichler defines a *semisimple metric space* (here called simply "space") to be a structure of a finite dimensional vector space R over a field k with a commutative bilinear vector product of R, R into k, such that no nonzero vector of R is orthogonal to all elements of R. The problem of equivalence of two q. f. under linear transformations with coefficients in k is equivalent to the problem of isomorphism of the corresponding spaces. The *orthogonal group* $\mathfrak{O}$ of R is the group of all isometries—i.e. of all isomorphisms of the entire structure which leave elements of k fixed. The "orthogonal group" of analysis is a very special case. Eichler states in the Introduction that he considers the theory of q. f. as the geometry of the space R with respect to the group of operators $\mathfrak{O}$. He compares R and $\mathfrak{O}$ to the additive and multiplicative groups of a ring, and suggests that the theory of q. f. has the same relation to the theory of algebras as this orthogonal group has to the affine group.

When k is a c. p. f. field, Eichler proves that two spaces are isometric if and only if everywhere locally isometric and completely characterizes spaces by local invariants. His proof, using Witt's theory of space-types and the local-global theory of quaternion algebras, is much the same as Witt's (Crelle 176) and is technically simpler in some places; but it produces a much less unified impression because, not wanting to use the general theory of algebras, Eichler lists a set of formally described local invariants which look quite mysterious at the archimedean primes while Witt can simply list "the algebra S" as one big invariant; and for the same reason Eichler's proof finishes with a complicated construction instead of Witt's simple induction. This great theorem (proof for rational field sketched by Minkowski; general proof by Hasse, Crelle 152–153; idea of using algebras due to Artin) concludes the algebraic theory.

Eichler's framework for arithmetic is the set of lattices in a space and the group of similarity transformations (s. t.) of that space. Let $\mathfrak{o}$ be an integral domain in k, and R a space. A *lattice* $\mathfrak{J}$ is a finitely generated $\mathfrak{o}$-module contained in R and containing a k-basis for R. A *similarity transformation* Σ is a linear mapping of R onto itself such that the product of images of vectors equals a fixed constant, called the *norm* $n(\Sigma)$ of Σ, times product of original vectors. An s. t. is a *unit* if it is a mapping of $\mathfrak{J}$ onto itself and its norm is then a unit in $\mathfrak{o}$; it is an *automorphic unit* if it is an automorphism and its norm is then 1. In spaces of even dimension Eichler defines the slightly smaller group of the *proper* s. t. and in what follows "s. t." should usually be interpreted "proper s. t."

If $\mathfrak{o}$ has class number 1 then every lattice is a linear form module and there is a one-one correspondence between classes of isomorphic lattices and classes of integrally equivalent forms; but if the class number of $\mathfrak{o}$ is greater than 1 neither of these statements is so. For lattices it is no longer true that everywhere local isomorphism implies global isomorphism; also the invariant characterization of local lattices is difficult at those prime spots where 2 is not a unit, and hasn't been completely done when 2 is moreover not a prime element. For an excellent discussion of this latter problem see O'Meara, Amer. J. Math. vol. 77 (1955). Eichler says little about it but determines the structure, which turns out to be very simple, of all maximal local lattices—i.e. those which are contained in no larger lattice of the same norm in the same space.

Being interested in the ideal theory of s.t., Eichler studies not isomorphism classes but similarity classes of lattices. When $\mathfrak{o}$ is the ordinary integers these can as he points out be made to correspond to equivalence classes. Otherwise they are somewhat different and Eichler does not discuss the connection. A *local ideal* $\mathfrak{K}_\mathfrak{p}/\mathfrak{J}_\mathfrak{p}$ is the set of all s. t. which take the local lattice $\mathfrak{J}_\mathfrak{p}$ onto $\mathfrak{K}_\mathfrak{p}$. Two global lattices $\mathfrak{J}$ and $\mathfrak{K}$ are called *ideal related* if for every $\mathfrak{p}$ the $\mathfrak{p}$-adic completions $\mathfrak{J}_\mathfrak{p}$ and $\mathfrak{K}_\mathfrak{p}$ are similar; and the family, indexed by the set of all valuations, of the local ideals $\mathfrak{K}_\mathfrak{p}/\mathfrak{J}_\mathfrak{p}$ is called an *ideal*. Ideals have a theory much like the ideal theory of a division algebra. *Genera* are defined as follows: two lattices $\mathfrak{J}$ and $\mathfrak{K}$ are called *related* if there is a global s. t. Σ such that $\Sigma\mathfrak{J}$ and $\mathfrak{K}$ are everywhere locally isomorphic, and the equivalence classes of related lattices are called genera. When $\mathfrak{o}$ is the ordinary integers these genera correspond to the ordinary ones. There are the usual sort of finiteness theorems, whose proof in general requires Minkowski's theorem on linear forms (although the reduction theory is omitted from the book). For example the number of similarity classes in a genus is finite.

Just as in the older theory characterization of similarity classes by invariants is a big unsolved problem. Here Eichler makes a notable advance by deriving a new necessary condition for two related lattices to be similar. Namely, a space defines in a simple way a Clifford algebra (slightly different from the one used by Witt in Crelle 176). The s. t. correspond to certain inner automorphisms of this algebra. Using this correspondence Eichler defines for each s. t. an element of the multiplicative group of k modulo squares called its *spinor-norm*. (The formalism resembles that of spinors in quantum mechanics, though the actual structures are quite different.) Sometimes it is possible by studying local and global spinor-norms to prove that two related lattices cannot be similar. If this is not possible they are said to be in the same *spinor-genus*. In many cases—for example in all lattices of dimension more than 2 which contain a vector orthogonal to itself—the spinor-genera are the similarity classes. Eichler gives an example to prove that this is not always so.

This survey of the main features of Eichler's new branch of arithmetic has covered, though with many omissions, most of the first three of the five chapters together with the beginning of the fifth. The fourth chapter begins by studying numbers of ideals taking lattices of one similarity class into those of another, numbers of vectors of given square in a lattice, etc. Special cases give such things as known formulas for number of representations of an integer as sum of 4 squares. The rest of this chapter proves some of Hecke's recent results on theta functions (E. Hecke, *Analytische Arithmetik der positiven quadratischen Formen*, Kgl. Danske Videnskab. Selskab, Math.-fys Medd. XVII, 12). In this last part Eichler specializes to the rational field and uses forms instead of lattices. The part of the fifth chapter not already discussed studies the mass (German *Mass*) of a genus, concluding with a proof of certain formulas due to Siegel (Ann. of Math. vols. 36 and 37 (1935)). Here again the analytic work is restricted to the rational field. There are many original things in these last two chapters, but the reviewer lacks the exhaustive knowledge of analytic number theory necessary for critically comparing Eichler's methods with Hecke's and Siegel's.

The style of this book must be accepted, since it is not unusual in mathematical monographs; but the reviewer regrets that such an important book wasn't written in a more expansive way. In his opinion, it hasn't enough motivation, unifying explanation, and references clarifying the relation of the book's content to the whole literature. For example, in the theory of spaces the reviewer considers it an important mathematical fact that the "characters" have a unified description as invariants of an algebra class, and is surprised that Eichler gives no hint of this—not even a reference to Crelle 176. In many other places the reviewer got a similar impression of disunity, usually without knowing how to correct it. This is not only pedagoguery; careful striving for simplicity and unification has helped a lot in putting class field theory into its present elegant form and, as Eichler points out in his Introduction, the subject of quadratic forms needs similar improvement.

Except for hiding the footnotes in the back, Springer-Verlag continues its tradition of impeccable printing. The reviewer has never before seen upper case Fraktur I and J used in the same discussion (p. 105) and hopes he never sees it again.

G. WHAPLES

Higher transcendental functions. Vol. 3. Bateman Project Staff, Editor A. Erdélyi. New York, McGraw-Hill, 1955. 17+292 pp. $6.50.

The present publication is a continuation of the two volumes of the *Higher transcendental functions* completed under the Bateman Manuscript Project sponsored by the California Institute of Technology (cf. this Bulletin vol. 60 (1954) pp. 405–408). In the meantime also two volumes of the *Tables of integral transforms* have been published which are a helpful supplement to the other volumes. (For reviews see this Bulletin, vol. 60 (1954) pp. 491–493 and vol. 61 (1955) pp. 239–240.) The present book shows the same erudition and competency which were apparent in the previous ones. At the same time it is clear again how difficult it is to define natural boundaries and dividing lines in this field. Considering the present material this is particularly conspicuous for the chapter on number-theory and for some parts of the chapter on automorphic functions which deal with special functions very different in origin, aspect, and interconnections from those functions to which this term is usually applied. The book follows in general the same ordering principles and organization as the previous volumes; its usefulness is on the same level.

Chapter XIV deals with automorphic functions, XV with Lamé functions, XVI with Mathieu and similar functions. (In the meantime the important book of Meixner and Schäfke has been published on this subject.) Chapter XVII is an introduction to functions occurring in number-theory while Chapters XVIII and XIX deal with more scattered topics. Naturally the concept of "generating function" (Chapter XIX) is not sufficiently substantial in order to form the backbone to many rather heterogeneous subjects.

The overwhelming credit for the preparation of the book goes, of course, to the Editor, Professor Arthur Erdélyi, who was the Director of the Bateman Project Staff. Professors W. Magnus and T. M. Apostol have participated with success in the present work. The completion of the whole project by this volume is a good opportunity to stress again that this unique work will occupy for a long time to come an outstanding place in the modern literature on special functions occurring in mathematical physics. This well organized collection of important concepts and results will be a useful tool for those who deal with the application of special functions.

G. SZEGÖ

Die Lehre von den Kettenbruchen. 3d ed. Vol. I. *Elementare Kettenbruche.* By Oskar Perron. Stuttgart, Teubner, 1954. 6+194 pp. 29.40 DM.

Earlier editions of Die Lehre von den Kettenbruchen are divided into two parts, the first containing the arithmetic theory of continued fractions and the second containing the analytic theory. The author's plan for the third edition is to treat these two parts in separate volumes. Volume I is an enlargement and improvement of what was previously Part I. The chapter headings of Volume I are identical with those of Part I in the second edition (Teubner, Leipzig, 1929) and, in fact, most of the sections of Volume I are a verbatim reproduction of the corresponding sections of Part I. There has been some re-partitioning of material in the various sections, and six new sections have been added. The early introduction of matrix notation has improved the presentation of certain sections. Additional material on diophantine approximation is given (§14) and some references to recent investigations in this field are cited. A continued fraction proof is given for classical decomposition theorems on 2×2 matrices with determinant ±1 (§18), and a brief alternate proof of the Lagrange theorem on periodicity is given (§21). The continued fraction expansion for irrationals of the form $2^{-1}(1+(4G+1)^{1/2})$ is applied to the solution of the diophantine equation $x^2-xy-Gy^2=1$ (§30). The derivation of the regular continued fraction expansion for $e^{2/m}$ (§34) is accomplished easily without recourse to Lambert's function-theoretic expansion for tan z as in earlier editions. The chapter on semi-regular continued fractions includes material on the shortest and longest expansions for a given rational number (§39), on the approximation to a real number by the approximants of a semi-regular expansion of the number (§41), and on the nearest and "farthest" integer continued fractions (§43).

W. T. SCOTT

Grundlagen der analytischen Topologie. By G. Nöbeling. Berlin, Springer, 1954. 10+221 pp. 33 DM.; clothbound, 36.60 D.M.

This carefully written monograph has a good chance of becoming the definitive text on the subject which it treats.

The basic primitive concept dealt with is the topological (partially) ordered set V, the word "topological" conveying that there is a "closure operation" defined on the *elements* A of V, which is idempotent, monotone, and makes $A \leqq A^-$. Frequently the axioms are augmented to give abstract closure algebras, or even topological Boolean lattices.

The notion of an actual set of points has already been recognized as too specific to be a primitive concept for algebraic topology. In Professor Nöbeling's mature opinion, it is also not the best basis on which to build the ideas clustering about "point-set theory." For the purpose of making an orderly presentation of ideas, this is certainly true; but only a great deal of careful reflection can have persuaded the author to adopt closure systems (where there are no points) as his basic notion, and to assign a secondary position to the concept of topological space (that is, T-space). He recognizes the growing modern tendency of constructing topological spaces as vehicles for representing more abstract types of systems. Perhaps for such reasons (as well as those of logical economy) has he chosen to write a textbook in a manner which may at first seem pedagogically hazardous to some teachers. On the other hand, as he says, we can never estimate what concepts will be "important"; and we are now much richer in having such an essay available.

In contrast to some publications on topologized lattices, the author is not interested in a game of seeing how far you can get "without actually assuming a topological space." At frequent and appropriate points, some of the better known propositions of analytic topology, such as R. L. Moore's theorem on arc-wise connectedness, the Hahn-Mazurkiewicz characterization of a continuous curve, are presented in their original contexts.

Naturally, Stone's representation of Boolean rings, and Wallman's representation of topological Boolean lattices have prominent places; as do also their compactifications. (The work of Katětov, and of P. Samuel, is not included.) Filters and ultrafilters are the ideal vehicles of convergence for this treatment. Continuous real valued functions are replaced by the corresponding resolutions of the identity (confined to the dyadic rational indices), and complete regularity is defined in those terms. Considerable time is given to uniform structures and metrics.

Besides ["full"] compactness, $\aleph$-compactness, for each infinite cardinal $\aleph$, is studied. (Oddly, all alephs were inverted by the typesetter!) Paracompactness is mentioned, but later discoveries (A. H. Stone) about this concept apparently did not come to the author's attention in time.

There are exercises, not of the drill variety, occurring it would seem about one every five pages. The harder ones contain suggestions. There are more frequently occurring clearly marked examples. Physically (also) the book is well executed.

RICHARD ARENS

Kernel functions and elliptic differential equations in mathematical physics. By S. Bergman and M. Schiffer. New York, Academic Press, 1953. 14+432 pp. $8.00.

This book consists of two parts. Part A, entitled *Boundary value problems for partial differential equations of elliptic type*, discusses a number of general problems of classical mathematical physics which can be reduced to boundary value problems either for the Laplace equation or for the equation

$$(1) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - p(x, y)u = 0, \qquad p(x, y) > 0,$$

and its three-dimensional analogue, and the theory of these problems is developed from a more or less elementary and, occasionally, heuristic point of view. In part B (*Kernel function methods in the theory of boundary value problems*) equation (1) is subjected to detailed and rigorous analysis.

Part A consists of the following chapters: I. Theory of heat conduction; II. Fluid dynamics; III. Electro- and magnetostatics; IV. Elasticity. Chapter I starts with the derivation of the equation of conduction of heat in a nonhomogeneous medium. Steady heat flows are examined further. The homogeneous case leads to Laplace's equation, and the equation div (κ grad T) $=0$, which arises in the case of variable conductivity κ, is shown to be reducible to an equation of type (1) by a simple transformation. The Green function, the Neumann function, and the corresponding function for the boundary condition $\partial T/\partial \nu - \lambda T = 0$ (denoted by $R_\lambda(P, Q)$ and called Robin's function in the book) are introduced and are given the proper physical interpretation. Uniqueness and some descriptive properties are proved for these functions; questions of existence are postponed to part B. The minimum property of the Dirichlet integral is formulated.

The second chapter, dealing with fluid dynamics, is by far the largest chapter of the book. It consists of 24 paragraphs. After deriving the equations for general (§1) and stationary irrotational compressible flows (§2), the authors turn to the special case of three-dimensional incompressible flow, i.e. again to Laplace's equation. Here Neumann's function is studied also for infinite domains, and the notion of regularity at infinity of a harmonic function is discussed. The tensor of virtual mass of a body is linked to the power series expansion at infinity of certain velocity potentials, and the Diaz-Weinstein minimum property of the virtual mass is derived. Variational formulas for Neumann's function are established and applied to the study of free-boundary problems. The next 7 paragraphs are devoted to two-dimensional steady irrotational incompressible flows. The connection to analytic functions of a complex variable is made at the outset and is used to express Green's function in terms of a mapping function. The flow around an obstacle is treated next; the discussion includes a proof of the paradox of d'Alembert and of the formulas of Blasius and Kutta-Joukowski. At this point the complex kernel function is introduced as a certain combination of second derivatives of Green's function possessing a simple variational formula. In §19 and §20 axially symmetric motions of an incompressible fluid are discussed. As in the two-dimensional case the velocity potential and the stream function are related to each other by a system of first-order partial differential equations which can be considered as a generalization of the Cauchy-Riemann equations. The theory of such systems is studied from a general point of view. Finally (§21–24) the authors study two-dimensional compressible fluid flow. The relevant equation for the potential, nonlinear in the physical plane, is shown to become linear (and reducible to (1)) in the hodograph plane. Again this equation can be split up into a system of first-order equations. Particular solutions are obtained (a) by using the theory of l-monogenic functions (as developed by L. Bers et al.) and (b) by separation of variables. The first procedure foreshadows the use of integral operators in part B for the same purpose.

In chapter III the basic equations of electro- and magnetostatics are derived. Whereas in chapter II Neumann's function was fundamental, the significant quantities of the field are now expressed in terms of Green's function. To show an application of the general theory, the authors determine the magnetic field in the metal core of a transformer. An elegant treatment, based on a variational formula for Green's function, is given for the theory of forces and moments in a conductor system. For Laplace's equation the kernel function is introduced as the difference of Neumann's and Green's functions, and a number of its properties are derived.

Chapter IV starts with a derivation of the basic equations of elasticity and a discussion of the simplest boundary value problems. The rôles of the Green, the Neumann and the Robin function are now taken over by the corresponding tensor functions. The difference of the Neumann and the Green tensor is called the kernel tensor and is shown to have properties analogous to those of the kernel functions defined in chapters II and III. Some "general solutions" of the differential equations are indicated, and the theory of thin plates is touched at.

The chapters of part B are: I. Properties of solutions; II. The

$$u(Q) = -\int_C K(P, Q)\frac{\partial u}{\partial \nu_P}\,ds_P = \int_C u\,\frac{\partial K(P, Q)}{\partial \nu_P}\,ds_P$$

hold for $u \in \Sigma$; they show that the knowledge of $K(P, Q)$ alone is sufficient for the solution of the first as well as of the second boundary value problem. The kernel function can also be characterized independently as follows: Of all functions $u \in \Sigma$ having the value 1 at a point $Q \in D$, the function $u(P) = K(P, Q)/K(Q, Q)$ has the least norm. The kernel function has the series expansion

$$K(P, Q) = \sum_{n=0}^{\infty} u_n(P)u_n(Q),$$

where $\{u_n(P)\}$ is an arbitrary complete orthonormal set of solutions of (1) (complete in Σ and in the sense of the above norm). Two methods for the construction of a complete system are given. For a fixed fundamental singularity $S(P, Q)$ the functions

$$g(P, Q) = G(P, Q) - S(P, Q), \qquad n(P, Q) = N(P, Q) - S(P, Q)$$

are investigated. It is shown that $l(P, Q) = g(P, Q) + n(P, Q)$ is continuously differentiable in $D + C$ with respect to both argument points. Finally, the kernel $K(P, Q)$ is developed into a series of iterated integrals of $S(P, Q)$.

Chapter III contains a study of the behavior of the fundamental functions and of the kernel function under infinitesimal and finite changes of the domain D. A number of variational formulas are given here.

The remaining three chapters are shorter. In chapter IV the existence of a solution of the first boundary value problem for equation (1) is proved by the method of orthogonal projections. The "projection" of an arbitrary function $v \in \Omega$ with the prescribed boundary values is made into Σ rather than Ω^0, as was done in the classical treatments of the existence problem. In chapter V the dependence of the solutions on the character of the boundary conditions and on the coefficient $p(x, y)$ are studied. The Robin function $R_\lambda(P, Q)$ is shown to decrease with increasing λ, so that in particular $N(P, Q) \geqq R_\lambda(P, Q) \geqq G(P, Q)$. Also, for fixed boundary values the solutions of (1) decrease with increasing $p(x, y)$. In chapter IV, besides $E\{u, v\}$ which in Σ can be written

$$E\{u, v\} = -\int_C u\frac{\partial v}{\partial \nu}\,ds,$$

two new metrics are introduced in Σ; they are defined, respectively, by

$$(u, v) = \int_C u(P)v(P)ds$$

and

$$[u, v] = \int_C \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds.$$

The Stekloff functions $\chi_n(P)$, defined as those solutions of (1) which on C are the normalized eigenfunctions of the integral equation

$$\chi(P) = \int_C N(P, Q)\chi(Q)ds_Q, \qquad P \in C,$$

are shown to be orthogonal in all three metrics considered. They yield interesting new representations for the various fundamental functions and for the kernel function. The two metrics based on (u, v) and $[u, v]$ remain also applicable when the assumption $p(x, y) > 0$ is dropped.

At the end there are a list of symbols, an extensive bibliography and two indexes.

The book is written in a constructive spirit and the style is very readable. In spite of a remark in the preface this reviewer can see no reason why at least the earlier parts of the book could not be used successfully as basis of an advanced course in mathematical physics.

PETER HENRICI

RESEARCH PROBLEMS

26. Barney Bissinger: *A functional iteration problem.*

For two real variables x, y, the equalities $2 \min (x, y) = x+y-|x-y|$ and $2 \max (x, y) = x+y+|x-y|$ are useful (Fine, Proc. Amer. Math. Soc. vol. 5 (1954) p. 247). An extension for n real variables is given by $f_2(x_1, x_2) = x_1+x_2+f_1(x_1-x_2)$, $f_n(x_1, x_2, x_3, \cdots, x_n) = f_{n-1}(f_2(x_1, x_2), x_3, \cdots, x_n)$, $n \geqq 3$. When $f_1(x) = |x|$, $2^{1-n}f_n = \max [x_j, 1 \leqq j \leqq n]$ and $\lim_{n\to\infty} 2^{1-n}f_n = \text{l.u.b.} [x_j, j = 1, 2, \cdots]$. Similar results hold for the minimum and g.l.b. by choosing $f_1(x) = -|x|$. Interest has been shown in similar work (advanced problem 4646, Amer. Math. Monthly vol. 62 (1955) p. 447). For what functions f_1 does $\lim_{n\to\infty} 2^{1-n}f_n$ exist? (Received July 15, 1955.)

27. Albert Wilansky and Karl Zeller. *Inverses of infinite matrices.*

May an infinite matrix have a left inverse and a right inverse but not have a two-sided inverse? Reference: Wilansky and Zeller, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 414–420. (Received July 15, 1955.)

28. E. A. Michael: *Paracompact spaces.*

Is the cartesian product of a paracompact space and a metric space paracompact, or even normal? What if both spaces have the Lindeloef property? (Reference: E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 831–838). (Received July 18, 1955.)

29. E. A. Michael: *Paracompact spaces.*

Must the image of a paracompact space, under a continuous, closed mapping, be paracompact? (The answer is (a) yes, if the inverse image of every point is compact, (b) yes, if "paracompact" is everywhere replaced by "normal," (c) no, if "paracompact" is everywhere replaced by "metric".) (Received August 12, 1955.)

ERRATA, VOLUME 61

Lee Lorch and Peter Szego, *A singular integral whose kernel involves a Bessel function*. Abstract 61-2-262. Line 4 of the abstract: for "$[f(1-)/3+f(1+)/3]$" read "$f(1-)/3+[f(1+)/3]$."

General index, volumes 51–60 (Part 2 of the May issue). Page 37, column 1, line 28 should read
Cohen, H. J., **55**, 713.
Cohen, H. **58**, 60; **59**, 367.

Mary-Elizabeth Hamstrom: *A note concerning the imbedding of upper semicontinuous collections of continua in continuous collections of continua*. Abstract 61-4-521. The result announced in the abstract is not correct. In a paper to appear in the American Journal of Mathematics, *A characterization of plane continuous curves which are filled up by continuous collections of continuous curves*, the author gives an example which shows that the above mentioned result is not true.

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